BINOMIAL THEOREM AND ARITHMETIC PROGRESSION

Abstract: I present a new binomial formula as well as the relationship between the 1st, 2nd, 3rd and 4th powers of an arithmetic progression.

NEW BINOMIAL THEOREM

If n is odd,

$$(a+b)^n = 2^{n-1}(a^n+b^n) - \sum_{k=1}^{\frac{n-1}{2}} \binom{n}{2k} (a+b)^{n-2k}(a-b)^{2k}$$

If n is even,

$$(a+b)^n = 2^{n-1}(a^n+b^n) - \sum_{k=1}^{\frac{n}{2}} \binom{n}{2k} (a+b)^{n-2k}(a-b)^{2k}$$

SUM OF TWO N POWERS

If n is odd, we can see that;

$$a^n + b^n = \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} (a+b)^{n-2k}(a-b)^{2k} \qquad \textbf{(1)}$$

If n is even, we can see that;

$$a^n + b^n = \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n}{2}} \binom{n}{2k} (a+b)^{n-2k}(a-b)^{2k} \qquad \textbf{(2)}$$

APPLICATION 1

If

$x - y = \alpha$ **(3)**
$x^4 + y^4 = \beta$ **(4)**

Can we find a solution to **x and y** without arriving at general quartic equation if **α and β** are given? Equation (2) above can answer this question because if n = 4,

$$x^4+y^4=\frac{1}{2^{4-1}}\sum_{k=0}^{\frac{4}{2}}\binom{4}{2k}(x+y)^{4-2k}(x-y)^{2k}$$

$$x^4+y^4=\frac{1}{8}\sum_{k=0}^{2}\binom{4}{2k}(x+y)^{4-2k}(x-y)^{2k}$$

$8(x^4+y^4)=(x+y)^4+6(x+y)^2(x-y)^2+(x-y)^4$ **(5)**

Putting (3) and (4) in (5), we get;

$$8\beta=(x+y)^4+6(x+y)^2(\alpha)^2+(\alpha)^4$$
$$(x+y)^4+6(x+y)^2(\alpha)^2+(\alpha)^4-8\beta=0$$

We can get the value of x+y using bi-quadratic formula

Let λ be the value of x + y such that;

$x+y=\lambda$ (6)

We can get the values of x and y by solving (3) and (6).

APPLICATION 2

If n is odd;

$$\text{Cos}nx=\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{2k}(-1)^k\text{Cos}^{n-2k}x\text{Sin}^{2k}x$$

If n is even;

$$\text{Cos}nx=\sum_{k=0}^{\frac{n}{2}}\binom{n}{2k}(-1)^k\text{Cos}^{n-2k}x\text{Sin}^{2k}x$$

RELATIONSHIP BETWEEN 1st, 2nd, 3rd AND 4th POWERS OF AN ARITHMETIC PROGRESSION

If

$$\alpha=\sum_{k=0}^{n-1}(a+kd)$$

$$\beta=\sum_{k=0}^{n-1}(a+kd)^2$$

$$\lambda=\sum_{k=0}^{n-1}(a+kd)^3$$

$$\gamma=\sum_{k=0}^{n-1}(a+kd)^4$$

Then

- $\lambda = \dfrac{\alpha(3\beta n - 2\alpha^2)}{n^2}$
- $\gamma = \left[\dfrac{12\beta - n(n^2 - 1)d^2}{12}\right]\left[\dfrac{15\beta + n(4n^2 - 1)d^2}{15n}\right] + \beta\left[\dfrac{3n^2 - 7}{20}\right]d^2$