TRIGONOMETRIC IDENTITIES

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ABSTRACT: I present the proof of Trigonometric Identities involving **Sin(x) and Cos(x)**.

1. INTRODUCTION

We are aware of the following identities;

$2\mathrm{Sin}x\mathrm{Cos}x = \mathrm{Sin}2x$

$2\mathrm{Cos}^2x = 1 + \mathrm{Cos}2x$

The question is have we ever been aware of the fact that the above two identities are special cases of some other identities? Well, they actually are, and i will try to show that.

2. NEW IDENTITIES

$$2^n\mathrm{Cos}^n x\mathrm{Sin}(n)x = \sum_{k=0}^{n}\binom{n}{k}\mathrm{Sin}2kx$$

$$2^n\mathrm{Cos}^n x\mathrm{Cos}(n)x = \sum_{k=0}^{n}\binom{n}{k}\mathrm{Cos}2kx$$

$$\mathrm{Sin}(n)x\sum_{k=0}^{n}\binom{n}{k}\mathrm{Cos}2kx = \mathrm{Cos}(n)x\sum_{k=0}^{n}\binom{n}{k}\mathrm{Sin}2kx$$

3. PROOF OF THE NEW IDENTITIES

Using binomial expansion, we see that;

$$(1 + e^{2ix})^n = \sum_{k=0}^{n}\binom{n}{k}e^{2ikx} \quad (1)$$

Now, let's try to manipulate LHS and RHS of (1);

LHS;

$$(1 + e^{2ix})^n = (e^{ix}(e^{-ix} + e^{ix}))^n$$

$$(1 + e^{2ix})^n = (2e^{ix}(\frac{e^{ix} + e^{-ix}}{2}))^n$$

We know that;

$$\mathrm{Cos}x = (\frac{e^{ix} + e^{-ix}}{2})$$

So,

$$(1+e^{2ix})^n = 2^n e^{inx}\cos^n x$$

We know that;

$e^{inx} = \text{Cos}nx + i\,\text{Sin}nx$

Therefore;

$(1+e^{2ix})^n = 2^n(\text{Cos}nx + i\text{Sin}(n)x)\text{Cos}^n x$

$(1+e^{2ix})^n = 2^n\text{Cos}^n x\text{Cos}nx + i(2^n\text{Cos}^n x\text{Sin}(n)x)$ (2)

RHS;

$\sum_{k=0}^{n}\binom{n}{k}e^{2ikx} = \sum_{k=0}^{n}\binom{n}{k}(\text{Cos}2kx + i\,\text{Sin}2kx)$

$\sum_{k=0}^{n}\binom{n}{k}e^{2ikx} = \sum_{k=0}^{n}\binom{n}{k}\text{Cos}2kx + i\sum_{k=0}^{n}\binom{n}{k}\text{Sin}2kx$ (3)

We see from (2) and (3) that;

$2^n\text{Cos}^n x\text{Cos}nx + i(2^n\text{Cos}^n x\text{Sin}(n)x) = \left(\sum_{k=0}^{n}\binom{n}{k}\text{Cos}2kx\right) + i\left(\sum_{k=0}^{n}\binom{n}{k}\text{Sin}2kx\right)$ (4)

Equating the real and imaginary parts of (4), we see that;

$2^n\text{Cos}^n x\text{Sin}(n)x = \sum_{k=0}^{n}\binom{n}{k}\text{Sin}2kx$ (5)

$2^n\text{Cos}^n x\text{Cos}(n)x = \sum_{k=0}^{n}\binom{n}{k}\text{Cos}2kx$ (6)

Dividing (6) by (5), we see that;

$\text{Sin}(n)x\sum_{k=0}^{n}\binom{n}{k}\text{Cos}2kx = \text{Cos}(n)x\sum_{k=0}^{n}\binom{n}{k}\text{Sin}2kx$ (7)

4. GENERALIZATION OF THE NEW IDENTITIES

- $2^n\text{Cos}^n x\text{Sin}(n+m)x = \sum_{k=0}^{n}\binom{n}{k}\text{Sin}(2k+m)x$
- $2^n\text{Cos}^n x\text{Cos}(n+m)x = \sum_{k=0}^{n}\binom{n}{k}\text{Cos}(2k+m)x$

- $\sin(n+m)x\sum_{k=0}^{n}\binom{n}{k}\cos(2k+m)x=\cos(n+m)x\sum_{k=0}^{n}\binom{n}{k}\sin(2k+m)x$

5. SOME OTHER NEW IDENTITIES

- $2^n\sin^n x\sin(n+m)x=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\sin(2k+m)x$ (n is even)
- $2^n\sin^n x\sin(n+m)x=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\cos(2k+m)x$ (n is odd)
- $2^n\sin^n x\cos(n+m)x=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\sin(2k+m)x$ (n is odd)
- $2^n\sin^n x\cos(n+m)x=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\cos(2k+m)x$ (n is even)