Riemann Hypothesis

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1 Abstract

The Riemann Zeta function is defined as the Analytic Continuation of the Dirichlet series

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \ Re(s) > 1$

The Riemann Zeta function is holomorphic in the complex plane except for a simple pole at $s = 1$

The non trivial zeroes(i.e those not at negative even integers) of the

Riemann Zeta function lie in the critical strip

$0 \leq Re(s) \leq 1$

Riemann's Xi function is defined as[4, p.1],

$\epsilon(s) \ = \ s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2$

The zero of $(s-1)$ *cancels the pole of* $\zeta(s)$ *, and the real zeroes of* $s\,\zeta(s)$ *are cancelled*

by the simple poles of $\Gamma(s/2)$ *which never vanishes.*

$Thus,\ \epsilon(s)\ is\ an\ entire\ function\ whose\ zeroes\ are\ the\ non\ trivial\ zeroes\ of\ \zeta(s)(see[3,p.2])$

$Further,\ \epsilon(s)\ satisfies\ the\ functional\ equation$

$\epsilon(1-s)\ =\ \epsilon(s)$

2 Statement of the Riemann Hypothesis

The Riemann Hypothesis states that all the non trivial zeroes of the Riemann Zeta function lie on the critical line with real part equal to 1/2

3 Proof

The Riemann Xi function [1, p.47] is defined as

$\epsilon(s) = \epsilon(0)\prod_{\rho}(1-\frac{s}{\rho})$ ... (1)

$where\ \rho\ ranges\ over\ all\ the\ roots\ \rho\ of\ \epsilon(\rho)=0.$

$If\ we\ combine\ the\ factors$

$(1-\frac{s}{\rho})\ and\ (1-\frac{s}{1-\rho}),\ then\ \epsilon(s)\ is\ Absolutely\ convergent\ infinite\ product\ .$

$Also,\ \epsilon(0)=1/2$

$From,\ [1,\ p.42,\ Section\ 2.5]$

$\prod_{\rho}(1-\frac{s}{\rho})=\prod_{Im(\rho)>0}(1-\frac{s(1-s)}{\rho(1-\rho)})=\prod_{Im(\rho)>0}(1-\frac{s}{\rho})(1-\frac{s}{1-\rho}).$ ... (2)

$Since\ from\ (1)\ \epsilon(\rho)=0.$

$For,\ 0<Re(\rho)<1,\ \mid\epsilon(\bar{\rho})\mid=\mid\epsilon(\rho)\mid.(Proof\ in\ Appendix)$

$\epsilon(\rho) = 0 \Rightarrow \mid \epsilon(\rho) \mid = 0$

$\Rightarrow \mid \epsilon(\bar{\rho}) \mid = 0.$

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$From (1),$

$\epsilon(s) = \epsilon(0) \prod_{\rho} (1 - \frac{s}{\rho})$

$Using\ (2)\ we\ have,$

$\epsilon(s) = \epsilon(0) \prod_{Im(\rho)>0} (1 - \frac{s}{\rho})(1 - \frac{s}{1-\rho}).$

$\epsilon(\bar{\rho}) = 0 \Rightarrow \epsilon(0) \prod_{Im(\rho)>0} (1 - \frac{\bar{\rho}}{\rho})(1 - \frac{\bar{\rho}}{1-\rho}) = 0.$

$Let, \rho = \sigma + it.$

$\Rightarrow \bar{\rho} = \sigma - it.$

$\epsilon(0) \prod_{t>0} (1 - \frac{(\sigma - it)}{\sigma + it})(1 - \frac{(\sigma - it)}{1 - \sigma - it}) = 0.$

$\Rightarrow \epsilon(0) \prod_{t>0} \frac{(\sigma + it - \sigma + it)(1 - 2\sigma - it + it)}{(\sigma + it)(1 - \sigma - it)} = 0.$

$\Rightarrow \epsilon(0) \prod_{t>0} \frac{2it(1 - 2\sigma)}{(\sigma + it)(1 - \sigma - it)} = 0.$

$Since\ \epsilon(s)\ is\ a\ Convergent\ infinite\ product,$

$Thus\ it\ is\ convergent\ infinite\ product[see\ 2\ , p.290].$

$The\ value\ of\ a\ convergent\ infinite\ product\ is\ zero\ if\ and\ only\ if$

$atleast\ one\ of\ the\ factors\ is\ equal\ to\ zero[see\ 2, p.287].$

$\Rightarrow \frac{2it_0(1 - 2\sigma)}{(\sigma + it_0)(1 - \sigma - it_0)} = 0,\ for\ some\ t_0 > 0\ (Since,\ t > 0).$

$Since, t_0 > 0$

$\Rightarrow 1 - 2\sigma = 0.$

$\Rightarrow \sigma = 1/2.$

$\Rightarrow Re(\rho) = 1/2.$

This proves the Riemann Hypothesis.

4 Appendix

Claim: $\mid \epsilon(\bar{\rho}) \mid = \mid \epsilon(\rho) \mid, \ 0 < Re(\rho) < 1.$

The Riemann's Xi function is defined in $[1, p.16]$ *as,*

$\epsilon(s) = 1/2 - s(1-s)/2 \int_1^\infty \psi(x)(x^{s/2} + x^{(1-s)/2}) dx/x, \ where, \psi(x) = \sum_{n=1}^\infty e^{-n^2 \pi x}.$

$\mid \epsilon(\bar{\rho}) \mid = \mid 1/2 - \bar{\rho}(1-\bar{\rho})/2 \int_1^\infty \psi(x)(x^{\bar{\rho}/2} + x^{(1-\bar{\rho})/2}) dx/x \mid$

$= \mid \overline{(1/2 - \rho(1-\rho)/2 \int_1^\infty \psi(x)(x^{\rho/2} + x^{(1-\rho)/2}) dx/x)} \mid .$

$= \mid \overline{\epsilon(\rho)} \mid = \mid \epsilon(\rho) \mid .$

5 References:-

1. H.M Edwards - Riemann's Zeta function- Academic Press (1974).

2. Stanislaw Saks, Antoni Zygmund Analytic Functions, 2nd Edition Hardcover(1965) .

3. A Monotonicity of Riemann's Xi function and a reformulation of the Riemann Hypothesis, Periodica Mathematica Hungarica - May 2010.

4. E. C. Titchmarsh, D. R. Heath-Brown - The theory of the Riemann Zeta function [2nd ed] Clarendon Press; Oxford University Press (1986).

5.Kevin Broughan - Equivalents of the Riemann Hypothesis : Arithmetic Equivalents Cambridge University Press (2017) .

6. Tom M. Apostol - Introduction to Analytical Number Theory (1976).

7. 4. Lars Ahlfors - Complex analysis [3 ed.] McGraw -Hill (1979).