PAPER 1: A new representation of the Riemann Zeta function for Re(z)≥0

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Abstract:

In this paper, we define the C-transformation as:

[1] $C_n\{f\} = \sum_{k=1}^{n} \left[f(k) - \int f(n)\, dn \right]$

And the C-values as:

[2] $C\{f\} = \lim_{n \to \infty} \left[C_n\{f\} \right]$

And we obtain a new representation for ζ(z) applying the C-transformation to the function $f(x) = 1/x^z$ for $z \in C, Re(z) \geq 0, z \neq 1$.

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Nomenclature and conventions

a. ζ(z)= $\sum_{k=1}^{\infty} k^{-z}$ is the Riemann Zeta function
b. α=a=Re(z) is the real part of a complex number z
c. ß=b=Im(z) is the imaginary part of a complex number z

1. C-Transformation of f(x)

The C-transformation of an integrable function f(x) is defined by:

[3] $C\_n \{f(x)\} = ∑\_(k = 1)\^n▒〖f(k) - ∫▒f(n) \ dn〗$

And the C-values iss the limit, if it exists, of the C-transformation when $n \to \infty$:

[4] $C\{f(x)\} = lim┬(n \to \infty) 〖C\_n \{f(x)\}〗$

1.1. C-Transformation of $f(x) = 1/x$ for $x \in R$:

[5] $C\_n \{1/x\} = ∑\_(\mathrm{k} = 1)\^\mathrm{n}▒〖1/\mathrm{k} - ∫▒dn/n 〗$

and

[6] $\mathrm{C}\{1/x\} = lim\_(n \to \infty) 〖(∑\_(\mathrm{k} = 1)\^\mathrm{n}▒〖1/\mathrm{k} - \log(n))〗〗 = \gamma$

(γ = Euler-Mascheroni constant = 0.5772…)

1.2. C-Transformation of $f(x) = m$, for $m \in R$ constant:

[7] $C\_n \{m\} = ∑\_(\mathrm{k} = 1)\^\mathrm{n}▒〖m - ∫▒〖m \ dn〗〗$

[8] $C\_n \{\mathrm{m}\} = m * n - m * n = 0$

and the C-values of f(x) =m constant is:

[9] $C\{m\} = 0$

1.3. C-Transformation of $f(x) = \sin(x)$ for $x \in R$:

[10] $C\_n \{\sin(x)\} = ∑\_(\mathrm{k} = 1)\^\mathrm{n}▒〖\sin(\mathrm{k}) - ∫▒〖\sin(n) dn〗〗$

[11] $C\_n \{\sin(x)\} = 1/2(\sin(n) - \cot(1/2) \ \cos(n) + \cot(1/2) + \cos(n)$

And the C-values of $f(x) = \sin(x)$ are in the interval:

[12] $\mathrm{C}\{\sin(x)\} \in [1/2 \ (2 \cot(1/2) - 3), \ 3/2]$

One can also calculate that:

[13] $\mathrm{C}\{\cos(x)\} \in [1/2 \ (\cot(1/2) - 4),1/2 \ (〖2 - \cot〗 (1/2))]$

1.4. C-Transformation of $f(x) = \mathrm{e}\^(-\mathrm{x})$ for $x \in R$:

[14] $C\_n\{e^{(-x)}\} = \sum_{k=1}^{n} \left[e^{(-k)} - \int e^{(-n)}\, dn \right]$

[15] $C\_n\{\sin(x)\} = \sum_{k=1}^{n} \left[e^{(-k)} + e^{(-n)}/n \right]$

And the C-values of $f(x) = e^{(-x)}$ are:

[16] $C\{e^{(-x)}\} = 1/(e-1)$

1.5. C-Transformation of $f(x) = x^{(-s)}$ for $x, s \in R$, s>1:

[17] $C\_n\{1/x^s\} = \sum_{k=1}^{n} \left[1/k^s - \int dn/n^s \right]$

[18] $C\_n\{1/x^s\} = \sum_{k=1}^{n} \left[1/k^s - n^{(1-s)}/(1-s) \right]$

and the C-value of $f(x) = 1/x^s$ is the Riemann Zeta function for s>1:

[19] $C\{1/x^s\} = \lim_{n \to \infty} \left[\left(\sum_{k=1}^{n} \left[1/k^s - n^{(1-s)}/(1-s) \right) \right] \right] = \lim_{n \to \infty} \left[\left(\sum_{k=1}^{n} \left[1/k^s \right) - \lim_{n \to \infty} \left[\left(n^{(1-s)}/(1-s) \right) = \right] \right] \right] \zeta(s) - 0 = \zeta(s)$

1.6. C-Transformation of $f(z) = 1/x^z$ for $z \in C, Re(z) \geq 0, z \neq 1$

[20] $C\_n\{1/x^z\} = \sum_{k=1}^{n} \left[1/k^z - \int dn/n^z \right]$

We will use Euler's identity:

[21] $e^x = \cos(x) + i * \sin(x)$

To calculate [20] for z=α+ßi:

[22] $k^{(-z)} = k^{(-\alpha)} \, [\cos(ß * \ln(k)) - i(\sin(ß * \ln(k))]$

And:

[23] $\int dn/n^z = n^{(1-\alpha)} \, [\cos(ß * \ln(n)) - i \sin(ß * \ln(n))] * \left([(1-\alpha) + iß] / [(1-\alpha)^2 + ß^2] \right)$

One can now express the real and imaginary components of $C\_n\{f\}$ as:

[24] $Re(C\_n\{f\}) = \sum_{k=1}^{n} \left[k^{(-\alpha)} (\cos(ß * \ln(k))\right]) +$

$+1/([(1-\alpha)^2+ß^{2}])(n^{(1-\alpha)})$ [(1-α) * cos(ß*ln(n)) + ß * sin(ß*ln(n))])

[25] $Im(C_n\{f\}) = -\sum_{k=1}^{n} k^{-\alpha} (\sin(ß*\ln(k)))+$
$+ 1/([(1-\alpha)^2+ß^{2}])(n^{(1-\alpha)})$ [ß*cos(ß*ln(n)) - (1-α) * sin(ß*ln(n))])

One can calculate that, for *α=Re(z)>2*, and for any ϵ arbitrarily small, there is a value of n=N such that for n>N, $C_N\{f\}$ - $\zeta(z) < \epsilon$, as the following table shows:

α	ß	$C_N\{f\}$ *for N=500*	$\zeta(z)$	$\lvert C_N\{f\}-\zeta(z)\rvert$
2	0	1.644934068	1.654934067	$< 10^{-8}$
2	1	1.150355702 + 0.437530865 i	1.150355703+0.437530866 i	$< 10^{-8}$
3	0	1.202056903	1.202056903	$< 10^{-9}$

Table 1. Values of $C_n\{f(n)=k^{-z}\}$ *for α=Re(z)>1 for N=500*

The error $C_n\{f\}$ - $\zeta(z)$ grows significantly in the critical strip for 0≤α<1 as we can see in the following table:

A	ß	$C_n\{f\}$	$\zeta(z)$	$\lvert C_n\{f\}-\zeta(z)\rvert$
0.0	0	$C_N\{f\}$ *for N=500*	-0.5	0.5
0.2	2	0.399824505+0.322650799 i	0.360103 + 0.266246 i	> 0.05
0.7	0	-2.777900606	-2.7783884455	$> 10^{-4}$

Table 2. Values of $C_n\{f(n)=k^{-z}\}$ *for 0≤Re(z)<1 for N=500*

To understand better the value of the difference $C_n\{1/k^z\}$ - $\zeta(z)$, one can plot the difference for $\alpha \in [0,1)$ and $ß = 0$: (Similar exponential charts occur for all values of $\alpha \in [0,1)$ for any given value of ß)

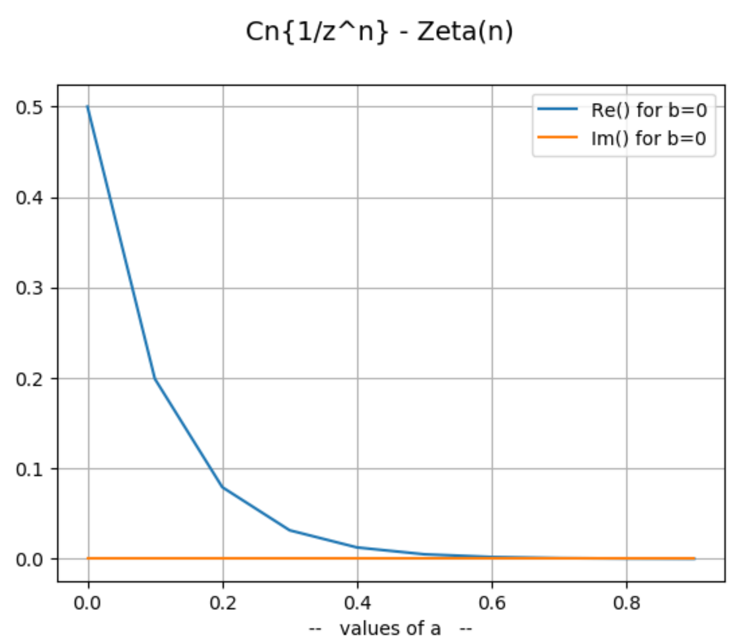

Figure 1 where a=α=Re(z) and b=ß=Im(z)

And plot the difference for variable values of ß $\in [0,1)$ and $\alpha = 0$: (Similar sine charts occur for all values of ß $\in [0,1)$ for any given value of α)

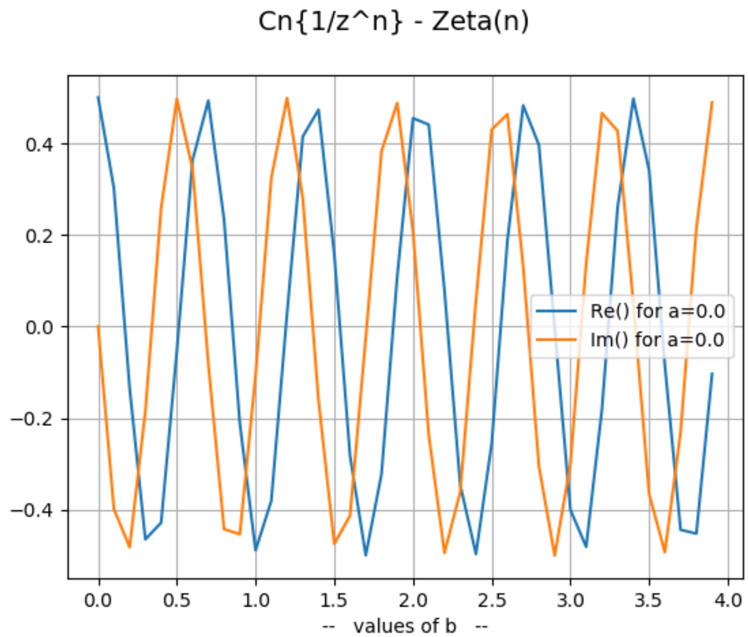

Figure 2 where a=α=Re(z) and b=ß=Im(z)

These charts lead to the following calculation of the difference $C\_n\,\{1/k\hat{}z\,\} - \zeta(z)$:

[26] $\mathrm{Re}[C\_n\,\{1/k\hat{}z\,\} - \zeta(z)] = 〚1/2\ \ n〛\ \hat{}(-\alpha) * \cos(ß * \ln(n)) + O(1/n)$

[27] $\mathrm{Im}[C\_n\,\{1/k\hat{}z\,\} - \zeta(z)] = 〚1/2\ \ n〛\ \hat{}(-\alpha) * \sin(ß * \ln(n)) + O(1/n)$

With *O(1/n)->0* when n->∞.

And one can finally write:

[28] $Re(C\_n\,\{f\}) = \sum\_(k=1)\hat{}n▒\ 〖k\hat{}(-\alpha)\ (\cos(ß * \ln(k)〗) +$

$+1/([(1-\alpha)\hat{}2 + ß\hat{}(2]))\ (n\hat{}■((1-\alpha)\ @)$ [(1-α) * cos(ß*ln(n)) + ß * sin(ß*ln(n))])

$+\ 〚1/2\ \ n〛\ \hat{}(-\alpha) * \cos(ß * \ln(n))$

[29] $Im(C\_n\,\{f\}) = -\sum\_■(k=1@)\hat{}n▒\ 〖k\hat{}(-\alpha)\ (\sin(ß * \ln(k)〗)+$

$+\ 1/([(1-\alpha)\hat{}2 + ß\hat{}(2]))\ (n\hat{}■((1-\alpha)\ @)$ [ß*cos(ß*ln(n)) - (1-α) * sin(ß*ln(n))])

$〚+1/2\ \ n〛\ \hat{}(-\alpha) * \sin(ß * \ln(n))$

and the C-value of $f(x) = 1/x\hat{}z$ for $z \in C, Re(z) \geq 0, z \neq 1$ is the Riemann Zeta function ζ(z).

1.7. A decomposition of ζ(z) based on the C-transformation of $f(x) = 1/x^z$ for $z \in C, Re(z) \geq 0, z \neq 1$

One can rewrite [28] and [29] creating the X(z) and Y(z) functions:

[30] $$X(z) = \left(\sum_{k=1}^{n} k^{-\alpha} \left(\cos(ß * \ln(k))\right) + 1/2\ n^{-\alpha} \cos(ß \ln(n))\right) + i * \left(\sum_{k=1}^{n} k^{-\alpha} \left(\sin(ß * \ln(k))\right) + 1/2\ n^{-\alpha} \sin(ß \ln(n))\right)$$

[31] $$Y(z) = \frac{n^{(1-\alpha)}}{(1-\alpha)^2 + ß^2} \left[\left((1-\alpha) * \cos(ß \ln(n)) + ß * \sin(ß \ln(n))\right) + i\left(ß * \cos(ß \ln(n)) - (1-\alpha) * \sin(ß \ln(n))\right)\right]$$

With:

[32] $$\zeta(z) = X(z) - Y(z)$$

The following table shows values for [32]:

z= 0 +j* 0
Zeta(z) = -0.5 + i* 0.0 X(z)-Y(z) = -0.5 +i* 0.0 ---> Error = 0.0 +i* 0.0
z= 0.2 +j* 2
Zeta(z) = 0.360102590022591 + i* -0.266246199765574 X(z)-Y(z) = 0.360102741838091 +i* -0.266246128959438 ---> Error= -1.5181550 e-7 +i* -7.080613 e-8
z= 0.4 +j* 0
Zeta(z) = -1.13479778386698 + i* 0.0 X(z)-Y(z) = -1.1347977871726 +i* 0.0 ---> Error= 3.305619 e-9 +i* 0.0

Table 13

The highest error for $\alpha \in [0,1)$, $ß \in [0,100]$, n=1000 is $8x10^{(-6)}$.

2. Conclusion

Using the defined C-transformation, one can write the Riemann Zeta function as the difference of two functions X(z) and Y(z) which will provide a new way of analyzing the zeros of the Zeta function.

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