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The Zeros of Riemann Xi Function are the non trivial zeros of the Riemann Zeta function.

The non trivial zeros of the Riemann Zeta function lie in the critical strip $0 < Re(\rho) < 1$

<u>Claim</u> :- $\xi(\rho) = 0 \,;\, 0 < Re(\rho) < 1 \Rightarrow Re(\rho) = \frac{1}{2}$.

<u>ETP</u>:- $Re(\rho) \neq \frac{1}{2} \Rightarrow \xi(\rho) \neq 0 \,.\, \forall \rho$

By contradiction $\xi(\rho_0) = 0$ for some non trivial zero ρ_0

$$\xi(s) = \xi(0) \prod \left(1 - \frac{s}{\rho}\right)$$

$$\xi(\rho) = 0 \Rightarrow 0 \leq Re(\rho) \leq 1 \quad —①$$

$$\xi(\rho_0) = 0 \Rightarrow \xi(\overline{\rho_0}) = 0$$

$$\Rightarrow \xi(0) \prod_{Im\,\rho > 0} \left(1 - \frac{\rho_0}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0 \quad —②$$

$Re(\rho) \neq \frac{1}{2}$ & $0 \leq Re(\rho) \leq 1$

$\Rightarrow 0 \leq Re(\rho) < \frac{1}{2}$ or $\frac{1}{2} < Re(\rho) \leq 1$.

Case 1 $\quad 0 \leq Re(\rho) < \frac{1}{2}$

$\Rightarrow 0 \leq Re(\rho_0) < \frac{1}{2}$

(2) $\Rightarrow \prod_{Im \rho > 0} \left(1 - \frac{\bar{\rho}_0}{\rho}\right)\left(1 - \frac{\bar{\rho}_0}{1-\rho}\right) = 0$

Suppose ρ_0 is a zero with multiplicity k.

$$\left(\frac{\rho_0 - \bar{\rho}_0}{\rho_0}\right)^K \left(\frac{1 - \rho_0 - \bar{\rho}_0}{1 - \rho_0}\right)^K \prod_{\substack{Im \rho > 0 \\ \rho \neq \rho_0}} \left(1 - \frac{\bar{\rho}_0}{\rho}\right)\left(1 - \frac{\bar{\rho}_0}{1-\rho}\right) = 0$$

$$\left[\frac{2i\, Im(\rho_0)\left(1 - 2\, Re(\rho_0)\right)}{\rho_0(1-\rho_0)}\right]^K \prod_{\substack{Im \rho > 0 \\ \rho \neq \rho_0}} \left(1 - \frac{\bar{\rho}_0}{\rho}\right)\left(1 - \frac{\bar{\rho}_0}{1-\rho}\right) = 0$$

$\because Im\, \rho_0 > 0$ & $\because 0 \leq Re(\rho_0) < \frac{1}{2}$

$$\therefore \prod_{\substack{Im \rho > 0 \\ \rho \neq \rho_0}} \left(1 - \frac{\bar{\rho}_0}{\rho}\right)\left(1 - \frac{\bar{\rho}_0}{1-\rho}\right) = 0 \quad — (*)$$

(*) is an Absolutely cgt. infinite product hence a cgt. infinite product. The value of a cgt. infinite product is 0 iff atleast one of the factors is 0

$$\left(1-\frac{\overline{\rho_0}}{\rho_1}\right)\left(1-\frac{\overline{\rho_0}}{1-\rho_1}\right)=0$$ for some $\rho_1 \in \mathbb{C}$ s.t $\text{Im}\,\rho_1 > 0$ & $\rho_1 \neq \rho_0$

$\Rightarrow 1-\frac{\overline{\rho_0}}{\rho_1}=0$ or $1-\frac{\overline{\rho_0}}{1-\rho_1}=0$

$\rho_1 = \overline{\rho_0}$ or $\overline{\rho_0} = 1-\rho_1$

$\text{Im}\,\rho_1 > 0$ or $\rho_0 = \sigma_0 + it_0$, $\rho_1 = \sigma_1 + it_1$

$\Rightarrow \text{Im}\,\overline{\rho_0} > 0$ or $\sigma_0 - it_0 = 1-\sigma_1 - it_1$

$-\text{Im}\,\rho_0 > 0$ or $\boxed{\sigma_0 = 1-\sigma_1}$ — (#)

$\text{Im}\,\rho_0 < 0$ or $t_0 = t_1$

Contradicts $\boxed{\text{Im}\,\rho_0 > 0}$

$\rho_1 \neq \rho_0 \Rightarrow \sigma_0 \neq \sigma_1$

(#) $\Rightarrow \sigma_0 = 1-\sigma_1$

$0 < \text{Re}\,\rho_0 < \frac{1}{2}$

$0 \leq 1-\sigma_1 < \frac{1}{2}$

$\frac{1}{2} < \sigma_1 \leq 1$

$\frac{1}{2} < \text{Re}\,\rho_1 \leq 1$

Contradicts $0 \leq \text{Re}(\rho) < \frac{1}{2}$

Case 2 $\frac{1}{2} < Re(\rho) \leq 1 \Rightarrow \frac{1}{2} < Re\,\rho_0 \leq 1$

$$② \Rightarrow \prod_{Im\,\rho > 0} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

Suppose ρ_0 is a zero with multiplicity 'm'.

$$\left(1 - \frac{\overline{\rho_0}}{\rho_0}\right)^K \left(1 - \frac{\overline{\rho_0}}{1-\rho_0}\right)^K \prod_{\substack{Im\,\rho > 0 \\ \rho \neq \rho_0}} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

$$\left[\frac{2i\,Im(\rho_0)(1 - 2Re(\rho_0))}{\rho_0(1-\rho_0)}\right]^K \prod_{\substack{Im\,\rho > 0 \\ \rho \neq \rho_0}} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

$\because$ $Im\,\rho_0 > 0$ & $\because$ $\left(\frac{1}{2} < Re(\rho_0) \leq 1\right)$

$$\therefore \prod_{\substack{Im\,\rho > 0 \\ \rho \neq \rho_0}} \left(1 - \frac{\overline{\rho_0}}{\rho}\right)\left(1 - \frac{\overline{\rho_0}}{1-\rho}\right) = 0$$

$$1 - \frac{\overline{\rho_0}}{\rho_2} = 0 \quad \text{or} \quad 1 - \frac{\overline{\rho_0}}{1-\rho_2} = 0$$

for some $\rho_2 \neq \rho_0 \in \mathbb{C}$, $Im\,\rho_2 > 0$

$\rho_2 = \overline{\rho_0}$ or $\overline{\rho_0} = 1 - \rho_2$

$\rho_0 = \sigma_0 + it_0$

$\rho_2 = \sigma_2 + it_2$

$\text{Im}\,\rho_2 > 0$

$\Rightarrow \text{Im}\,\overline{\rho_0} > 0$

$-\text{Im}\,\rho_0 > 0$

$\Rightarrow \text{Im}\,\rho_0 < 0$

$\longrightarrow \longleftarrow$

Contradiction

or $\sigma_0 - it_0 = 1 - \sigma_2 - it_2$

or $\sigma_0 = 1 - \sigma_2$

$\frac{1}{2} < \sigma_0 = \text{Re}\,\rho_0 \leq 1$

or $\frac{1}{2} < 1 - \sigma_2 \leq 1$

or $0 \leq \sigma_2 < \frac{1}{2}$

$0 \leq \text{Re}\,\rho_2 < \frac{1}{2}$

But $\frac{1}{2} < \text{Re}\,\rho \leq 1$

So we get a contradiction