On some mathematical connections concerning the three-dimensional pure quantum gravity with negative cosmological constant, the Selberg zeta-function, the ten-dimensional anomaly cancellations, the vanishing of cosmological constant, and some sectors of String Theory and Number Theory.

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Abstract

This paper is a review of some interesting results that has been obtained in the study of the quantum gravity partition functions in three-dimensions, in the Selberg zeta function, in the vanishing of cosmological constant and in the ten-dimensional anomaly cancellations. In the **Section 1**, we have described some equations concerning the pure three-dimensional quantum gravity with a negative cosmological constant and the pure three-dimensional supergravity partition functions. In the **Section 2**, we have described some equations concerning the Selberg Super-trace formula for Super-Riemann surfaces, some analytic properties of Selberg Super zeta-functions and multiloop contributions for the fermionic strings. In the **Section 3**, we have described some equations concerning the ten-dimensional anomaly cancellations and the vanishing of cosmological constant. In the **Section 4**, we have described some equations concerning p-adic strings, p-adic and adelic zeta functions and zeta strings. In conclusion, in the **Section 5**, we have described the possible and very interesting mathematical connections obtained between some equations regarding the various sections and some sectors of Number Theory (Riemann zeta functions, Ramanujan modular equations, etc…) and some interesting mathematical applications concerning the Selberg super-zeta functions and some equations regarding the **Section 1**.

1. On some equations concerning the pure three-dimensional quantum gravity with a negative cosmological constant. [1] [2]

This section is devoted to describing some explicit computations concerning the three-dimensional pure quantum gravity with negative cosmological constant. The classical action can be written

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R+\frac{2}{\ell^2}\right). \quad (1.1)$$

We describe the sum of known contributions to the partition function from classical geometries that can be computed exactly, including quantum corrections. Furthermore, we describe pure three-dimensional supergravity partition functions.

The automorphism group of AdS_3 in $SO(3,1)$, which is the same as $SL(2,C)/Z_2$. We may write the metric on a dense open subset of AdS_3 as

$$ds^2 = \frac{|dz|^2 + du^2}{u^2}, \quad u>0, \quad z \in C. \quad (1.2)$$

If we identify the real one-parameter subgroup $\mathrm{diag}\left(e^b, e^{-b}\right)$ of $SL(2,C)$ as the group of time translations, the AdS_3 metric (1.2) can be put in the form

$$ds^2 = \cosh^2 r dt^2 + dr^2 + \sinh^2 r d\phi^2, \quad (1.3)$$

with $-\infty < t < \infty$, $0 \le r < \infty$, and $0 \le \phi \le 2\pi$.

We write $Z_{c,d}(\tau)$ for the contribution to the partition function of the manifold $M_{c,d}$. Because the manifolds $M_{c,d}$ are all diffeomorphic to each other, the functions $Z_{c,d}(\tau)$ can all be expressed in terms of any one of them, say $Z_{0,1}(\tau)$, by a modular transformation. The formula is simply

$$Z_{c,d}(\tau) = Z_{0,1}((a\tau+b)/(c\tau+d)), \quad (1.4)$$

where a and b are any integers such that $ad-bc=1$. The partition function, or rather the sum of known contributions to it, is

$$Z(\tau) = \sum_{c,d} Z_{c,d}(\tau) = \sum_{c,d} Z_{0,1}((a\tau+b)/(c\tau+d)). \quad (1.5)$$

This formula shows that the key point is to evaluate $Z_{0,1}(\tau)$. In the most naive semiclassical approximation, $Z_{0,1}(\tau)$ is just $\exp(-I)$, where I is the classical action. Hence, we can write also the following expression

$$Z_{0,1}(\tau) = \exp\left[-\left(\frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R+\frac{2}{\ell^2}\right)\right)\right]. \quad (1.5b)$$

The full action includes the Gibbons-Hawking boundary term, which has the opposite sign of the Einstein-Hilbert term (1.1). This extra term removes the divergence, and one arrives at a finite (negative) answer for the action of $M_{0,1}$:

$$I = -4\pi k \,\mathrm{Im}\,\tau \quad (1.6)$$

where $k = \ell/16G$. Hence, we can write also the following equation

$$\frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R+\frac{2}{\ell^2}\right)=-4\pi\frac{\ell}{16G}\,\mathrm{Im}\,\tau\,. \quad (1.6b)$$

Therefore, in this approximation, we have

$$Z_{0,1}(\tau)\cong|\bar{q}q|^{-k}\,. \quad (1.7)$$

Three-dimensional pure gravity with the Einstein-Hilbert action (1.1) is dual to a conformal field theory with central charge $c_L=c_R=3\ell/2G=24k$. For our purposes, we simply parametrize the theory in terms of $k=c_L/24=c_R/24$. Hence, we have the following connection:

$$I=\frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R+\frac{2}{\ell^2}\right)\Rightarrow c_L=c_R=3\ell/2G=24k\,. \quad (1.7b)$$

Let L_0 and $\tilde{L}_0$ be the Hamiltonians for left- and right-moving modes of the CFT. They are related to what we have called H and J by

$$H=L_0+\tilde{L}_0\,; \quad J=L_0-\tilde{L}_0\,. \quad (1.8)$$

The CFT ground state has $L_0=-c_L/24$, $\tilde{L}_0=-c_R/24$, or in the present context $L_0=\tilde{L}_0=-k$. Equivalently, this state has $H=-2k$, $J=0$. Its contribution to

$$Tr\exp(-2\pi(\mathrm{Im}\,\tau)H+2\pi i(\mathrm{Re}\,\tau)J)$$

is

$$\exp(4\pi k\,\mathrm{Im}\,\tau)=|\bar{q}q|^{-k}\,,$$

as in eqn. (1.7). If L_n and $\tilde{L}_n$ are the left- and right-moving modes of the Virasoro algebra, then a general such state is

$$\prod_{n=2}^{\infty}L_{-n}^{u_n}\prod_{m=2}^{\infty}\tilde{L}_{-m}^{v_m}|\Omega\rangle\,, \quad (1.9)$$

with non-negative integers u_n, v_m.

A state of this form is an eigenstate of L_0 and $\tilde{L}_0$ with $L_0=-k+\sum_{n=2}^{\infty}nu_n$, $\tilde{L}_0=-k+\sum_{m=2}^{\infty}mv_m$. The contribution of these states to the partition function is then

$$Z_{0,1}(\tau)=|\bar{q}q|^{-k}\frac{1}{\prod_{n=2}^{\infty}|1-q^n|^2}\,. \quad (1.10)$$

It is convenient to introduce the Dedekind η function, defined by

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,. \quad (1.11)$$

Eqn. (1.10) can then be rewritten

$$Z_{0,1}(\tau)=\frac{1}{|\eta(\tau)|^2}|\bar{q}q|^{-(k-1/24)}|1-q|^2 . \quad (1.12)$$

The known contributions to the partition function of pure gravity in a spacetime asymptotic to AdS_3 come from smooth geometries $M_{c,d}$, where c and d are a pair of relatively prime integers. Their contribution to the partition function, including the contribution from the Brown-Henneaux excitations, is

$$Z(\tau)=\sum_{c,d} Z_{0,1}(\gamma\tau), \quad (1.13)$$

where

$$\gamma\tau=\frac{a\tau+b}{c\tau+d} , \quad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,Z) \quad (1.14)$$

and

$$Z_{0,1}(\tau)=\left|q^{-k}\prod_{n=2}^{\infty}(1-q^n)^{-1}\right|^2=\frac{|\bar{q}q|^{-k+1/24}|1-q|^2}{|\eta(\tau)|^2} . \quad (1.15)$$

The summation in (1.13) is over all relatively prime c and d with $c \geq 0$. Since $Z_{0,1}(\tau)$ is invariant under $\tau \to \tau+1$, the summand in (1.13) is independent of the choice of a and b in (1.14). This sum over c and d in (1.13) should be thought of as a sum over the coset $PSL(2,Z)/Z$, where Z is the subgroup of $PSL(2,Z)=SL(2,Z)/\{\pm 1\}$ that acts by $\tau \to \tau+n$, $n\in Z$. Given any function of τ, such as $Z_{0,1}(\tau)$, that is invariant under $\tau \to \tau+1$, one may form a sum such as (1.13), known as a Poincaré series. The function $\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2$ is modular-invariant. We can therefore write $Z(\tau)$ as a much simpler-looking Poincaré series,

$$Z(\tau)=\frac{1}{\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2}\sum_{c,d}\left(\sqrt{\operatorname{Im}\tau}|\bar{q}q|^{-k+1/24}|1-q|^2\right)_{\gamma}, \quad (1.16)$$

where $(...)|_{\gamma}$ is the transform of an expression $(...)$ by γ.
Now we see what we need to analyze the following sum

$$E(s,\kappa,\mu)=\sum_{c,d}\frac{y^s}{|c\tau+d|^{2s}}\exp\{2\pi\kappa \operatorname{Im}\gamma\tau+2\pi i\mu \operatorname{Re}\gamma\tau\}, \quad (1.17)$$

make the analytic continuation, and determine if the partition function Z is physically sensible.
We define $d=d'+nc$, where n is an integer and d' runs from 0 and $c-1$. We may separate out the sum over n in (1.17) to get

$$E(s,\kappa,\mu)=y^s e^{2\pi(\kappa y+i\mu x)}+\sum_{c>0}\sum_{d'\in Z/cZ}\sum_{n\in Z} f(c,d',n) \quad (1.18)$$

where

$$f(c,d',n)=\frac{y^s}{|c(\tau+n)+d'|^{2s}}\exp\left\{\frac{2\pi\kappa y}{|c(\tau+n)+d'|^2}+2\pi i\mu\left(\frac{a}{c}-\frac{cx+d}{c|c(\tau+n)+d'|^2}\right)\right\}. \quad (1.19)$$

Hence, we can write the following equation

$$E(s,\kappa,\mu)=y^s e^{2\pi(\kappa y+i\mu x)}+\sum_{c>0}\sum_{d'\in Z/cZ}\sum_{n\in Z}\times$$

$$\times\frac{y^s}{|c(\tau+n)+d'|^{2s}}\exp\left\{\frac{2\pi\kappa y}{|c(\tau+n)+d'|^2}+2\pi i\mu\left(\frac{a}{c}-\frac{cx+d}{c|c(\tau+n)+d'|^2}\right)\right\}. \quad (1.19b)$$

The first term in eqn. (1.18) comes from $c=0, d=1$.
The Poisson summation formula allows us to turn the sum over n into a sum over a Fourier conjugate variable $\hat{n}$

$$\sum_{n\in Z} f(c,d',n)=\sum_{\hat{n}\in Z}\hat{f}(c,d',\hat{n}) \quad (1.20)$$

where $\hat{f}(c,d',\hat{n})$ is the Fourier transform

$$\hat{f}(c,d',\hat{n})=\exp\left[2\pi i\left(\frac{\mu a-\hat{n}d'}{c}-\hat{n}x\right)\right]\int_{-\infty}^{\infty}dte^{2\pi i\hat{n}t}\left(\frac{y}{c^2(t^2+y^2)}\right)^s\exp\left\{\frac{2\pi(\kappa y-i\mu t)}{c^2(t^2+y^2)}\right\}. \quad (1.21)$$

We have written the integral in terms of a shifted integration variable $t=n+x+\frac{d'}{c}$. Upon Taylor expanding the exponential that appears in the integral and introducing $T=t/y$, we get

$$\hat{f}(c,d',\hat{n})=\sum_{m=0}^{\infty}c^{-2(s+m)}e^{2\pi i\left(\frac{\mu a-\hat{n}d'}{c}-\hat{n}x\right)}\frac{(2\pi)^m}{m!}y^{1-m-s}\int_{-\infty}^{\infty}dTe^{2\pi i\hat{n}Ty}\left(1+T^2\right)^{-m-s}(\kappa-i\mu T)^m. \quad (1.22)$$

For a given d', such an a exists if and only if d' lies in the set $(Z/cZ)^*$ of residue classes $\bmod c$ that are invertible multiplicatively. So, dropping the prime from d', we may write the sum over that variable as

$$S(-\hat{n},\mu;c)=\sum_{d\in(Z/cZ)^*}\exp\left\{2\pi i\left(\frac{-\hat{n}d+\mu d^{-1}}{c}\right)\right\}, \quad (1.23)$$

where $d^{-1}\in(Z/cZ)^*$ is the multiplicative inverse of d. This sum is known as a Kloosterman sum. Rearranging the sums in (1.18), we have also

$$E(s,\kappa,\mu)=y^s e^{2\pi(\kappa y+i\mu x)}+\sum_{\hat{n}}e^{-2\pi i\hat{n}x}E_{\hat{n}}(s,\kappa,\mu) \quad (1.24)$$

where

$$E_{\hat{n}}(s,\kappa,\mu)=\sum_{m=0}^{\infty}I_{m,\hat{n}}(s,\kappa,\mu)y^{1-m-s}\left(\sum_{c=1}^{\infty}c^{-2(m+s)}S(-\hat{n},\mu;c)\right). \quad (1.25)$$

Here we have defined the integral

$$I_{m,\hat{n}}(s,\kappa,\mu)=\frac{(2\pi)^m}{m!}\int_{-\infty}^{\infty}dTe^{2\pi i\hat{n}Ty}\left(1+T^2\right)^{-m-s}(\kappa-i\mu T)^m . \quad (1.26)$$

Hence, we have also the following equation

$$E_{\hat{n}}(s,\kappa,\mu)=\sum_{m=0}^{\infty}\frac{(2\pi)^m}{m!}\int_{-\infty}^{\infty}dTe^{2\pi i\hat{n}Ty}\left(1+T^2\right)^{-m-s}(\kappa-i\mu T)^m y^{1-m-s}\left(\sum_{c=1}^{\infty}c^{-2(m+s)}S(-\hat{n},\mu;c)\right). \quad (1.26b)$$

Note that (1.25) is independent of x, so that (1.24) has the form of a Fourier expansion in x with Fourier coefficients $E_{\hat{n}}(s,\kappa,\mu)$ given by (1.25). These Fourier coefficients are typically complicated functions of y, since the integral (1.26) depends on y.

Now we consider the Fourier mode which is constant in x, i.e. the $\hat{n}=0$ term in (1.24). In this case the integral (1.26) is independent of y, and may be evaluated explicitly. For $\mu=0$, the result can be expressed in terms of Γ functions

$$I_{m,0}(s,\kappa,0)=\kappa^m\frac{2^m\pi^{m+1/2}\Gamma(s+m-1/2)}{m!\Gamma(s+m)}, \quad (1.27)$$

while for $\mu=\pm1$, we require also hypergeometric functions

$$I_{m,0}(s,\kappa,\pm1)=\cos\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{m-1}{2}+s\right)}{m!\Gamma(m+s)}2F_1\left(\frac{m-1}{2}+s,-\frac{m}{2}+s;\frac{1}{2};\kappa^2\right)+$$
$$+m\kappa\sin\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+s\right)}{m!\Gamma(m+s)}2F_1\left(\frac{1-m}{2},\frac{m}{2}+s;\frac{3}{2};\kappa^2\right). \quad (1.28)$$

When $m=0$, this formula simplifies to

$$I_{0,0}(s,\kappa,\pm1)=\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}. \quad (1.29)$$

The sum over c may also be evaluated exactly. For $\mu=0$ this evaluation involves the Kloosterman sum $S(0,0;c)$; from the definition (1.23), one can see that $S(0,0;c)$ is equal to the Euler function $\phi(c)$, which is defined as the number of positive integers less than c that are relatively prime to c. The sum over c is a standard one

$$\sum_{c=1}^{\infty}c^{-2(m+s)}S(0,0;c)=\sum_{c>0}c^{-2(m+s)}\phi(c)=\frac{\zeta(2(m+s)-1)}{\zeta(2(m+s))}. \quad (1.30)$$

We note that the eqs. (1.27), (1.28) and (1.29) can be related with the Selberg zeta-function for the modular group.

The Selberg zeta-function was introduced by Atle Selberg in the 1950s. It is analogous to the famous Riemann zeta function

$$\zeta(s)=\prod_{p\in P}\frac{1}{1-p^{-s}}, \quad (1.30b)$$

where P is the set of prime numbers. The Selberg zeta-function uses the lengths of simple closed geodesics instead of the prime numbers.
For the case where the surface is $\Gamma\backslash H^2$, where Γ is the modular group, the Selberg zeta-function is of special interest. For this special case the Selberg zeta-function is intimately connected to the Riemann zeta-function. In this case the scattering matrix is given by:

$$\varphi(s)=\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}. \quad (1.30c)$$

In particular, we see that if the Riemann zeta-function has a zero at s_0, then the scattering matrix has a pole at $s_0/2$, and hence the Selberg zeta-function has a zero at $s_0/2$.
Thence, **we have the following interesting mathematical connections**:

$$\kappa^m\frac{2^m\pi^{m+1/2}\Gamma(s+m-1/2)}{m!\Gamma(s+m)}\Rightarrow\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}, \quad (1.30e)$$

$$\cos\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{m-1}{2}+s\right)}{m!\Gamma(m+s)}2F_1\left(\frac{m-1}{2}+s,-\frac{m}{2}+s;\frac{1}{2};\kappa^2\right)+$$
$$+m\kappa\sin\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+s\right)}{m!\Gamma(m+s)}2F_1\left(\frac{1-m}{2},\frac{m}{2}+s;\frac{3}{2};\kappa^2\right)\Rightarrow$$
$$\Rightarrow\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}. \quad (1.30f)$$

$$\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\Rightarrow\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}. \quad (1.30g)$$

For $\mu=\pm1$, the Kloosterman sum becomes a special case of what is known as a Ramanujan's sum:

$$S(\hat{n},0;c)=\sum_{d\in(Z/cZ)^*}e^{2\pi i d/c}=\mu(c). \quad (1.30h)$$

Here $\mu(c)$ is the Mobius function, which is defined as follows: $\mu(c)=0$ if c is not square-free, while if c is the product of k distinct prime numbers, then $\mu(c)=(-1)^k$. The sum over c is given by

$$\sum_{c=1}^{\infty}c^{-2(s+m)}S(0,\pm1;c)=\sum_{c=1}^{\infty}c^{-2(s+m)}\mu(c)=\frac{1}{\zeta(2(s+m))}. \quad (1.30i)$$

The exact expression for the x independent part of our Poincaré series (1.18) is:

$$E_0(s,\kappa,\mu)=\sum_{m=0}^{\infty} w_m(s,\kappa,\mu)y^{1-m-s} . \quad (1.31)$$

If we take the $\mu=0$ case, we find that the coefficients $w_m(s,\kappa,\mu)$ are given by

$$w_m(1/2,\kappa,0)=\frac{2^m\pi^{m+1/2}\zeta(2m)}{m\Gamma(m+1/2)\zeta(2m+1)}\kappa^m . \quad (1.32)$$

To evaluate the $\mu=\pm 1$ terms, note that at $s=1/2$ the first two arguments of the relevant hypergeometric function appearing in (1.28) are integers. This means that the formula (1.28) simplifies considerably at $s=1/2$ - it is just a polynomial in κ. It is

$$I_{m,0}\left(\frac{1}{2},\kappa,\pm 1\right)=\frac{2\pi^{m+1/2}}{m\Gamma(m+1/2)}T_m(\kappa) \quad (1.33)$$

where $T_m(\kappa)$ denotes a Chebyshev polynomial of the first kind. So the coefficients appearing in (1.31) are

$$w_m(1/2,\kappa,\pm 1/2)=\frac{2\pi^{m+1/2}}{m\Gamma(m+1/2)\zeta(2m+1)}T_m(\kappa). \quad (1.34).$$

Also the eqs. (1.32), (1.33) and (1.34) can be related with the eq. (1.30c).
Now we will consider the $\hat{n}\neq 0$ terms. For $\mu=0$ and $\hat{n}\neq 0$, the integral (1.26) is a K-Bessel function

$$I_{m,\hat{n}}(s,\kappa,0)=\frac{2^{s+1}\pi^{2s+m}|\hat{n}|^{s+m-1/2}}{m!\Gamma(s+m)}y^{s+m-1/2}K_{s+m-1/2}(2\pi|\hat{n}|y). \quad (1.35)$$

The Kloosterman sum is now the general case of Ramanujan's sum

$$S(\hat{n},0;c)=\sum_{d\in(Z/cZ)^*} e^{2\pi i\hat{n}d/c}=\sum_{\delta|\hat{n}}\mu(\delta). \quad (1.36)$$

We will simply quote the answer for the sum over c

$$\sum_{c=1}^{\infty} c^{-2(s+m)}S(\hat{n},0;c)=\frac{1}{\zeta(2(s+m))}\sum_{\delta|\hat{n}}\delta^{1-2(s+m)} . \quad (1.37)$$

Taking $s=1/2$ gives a Fourier coefficient of the regularized partition function. Consider first the $m=0$ term. For $\hat{n}=0$, this was the dangerous term in the analytic continuation, but for $\hat{n}\neq 0$, it simply vanishes, because the $\zeta(2s)$ in (1.37) has a pole at $c=1/2$, causing the Kloosterman sum to vanish, and – unlike the $\hat{n}=0$ case – the integral (1.35) is finite at $s=1/2$. The other terms are non-zero, and give

$$E_{\hat{n}}(1/2,k,0)=\sum_{m=1}^{\infty}\frac{2^{3/2}\pi^{m+1}|\hat{n}|^{m}}{m!\Gamma(m+1/2)\zeta(2m+1)}\left(\sum_{\delta|\hat{n}}\delta^{-2m}\right)\sqrt{y}K_{m}(2\pi|\hat{n}|y). \quad (1.38)$$

Each term in the sum over m vanishes for large y as $e^{-2\pi|\hat{n}|y}$.
However, when $\mu\neq 0$, the sum over c

$$\sum_{c=1}^{\infty}c^{-2(m+s)}S(\hat{n},\mu,c), \quad (1.39)$$

though of considerable number-theoretic interest, cannot be expressed in terms of familiar number-theoretic functions such as the Riemann zeta function. The sum (1.39) defines a meromorphic function on the complex s plane. **This function is essentially the Selberg zeta function associated to the modular domain** $D=\mathrm{H}/SL(2,Z)$. When we take $s=1/2$, the function (1.39) remains regular for elementary reasons if $m>0$. Indeed, one can see directly that the sum converges for these values, by noting from the definition of the Kloosterman sum (1.23) that $|S(\hat{n},\mu,c)|\leq c$.
The sum over geometries, if we take account of only the classical action and not the one-loop correction, is

$$Z^{*}(\tau)=\sum_{c,d}\exp\{2\pi k\,\mathrm{Im}\,\gamma\tau\}. \quad (1.40)$$

The necessary s-dependent function was already introduced in eqn. (1.17); $Z^{*}(\tau)$ is formally given by the function $E(s,k,0)$ at $s=0$:

$$Z^{*}(\tau)=\lim_{s\to 0}E(s,k,0). \quad (1.41)$$

However, as an analytic function in s, $E(s,k,0)$ has a pole at $s=0$. To see this, consider the expansion (1.31) of the part of $E(s,k,0)$ which is constant in x. The $m=0$ term in this sum vanishes, because of the pole in $\Gamma(s)$ at $s=0$. The $m=1$ term gives

$$E(s,k,0)=\sqrt{\pi}\,\frac{\zeta(1+2s)\Gamma(s+1/2)}{\zeta(2+2s)\Gamma(1+s)}+O(y^{-1}) \quad (1.42)$$

which has a pole at $s=0$ coming from the harmonic series $\zeta(1)=\infty$.
We note that also the eq. (1.42) can be related with the eq. (1.30c), i.e. the Selberg zeta function for the modular group:

$$\sqrt{\pi}\,\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow\sqrt{\pi}\,\frac{\zeta(1+2s)\Gamma(s+1/2)}{\zeta(2+2s)\Gamma(1+s)}+O(y^{-1}). \quad (1.42b)$$

Now we recall some properties of the modular invariant $J(\tau)$. The J function has a q-expansion

$$J(\tau)=\sum_{m\geq -1}c(m)q^m=\frac{1}{q}+196884q+...$$

where the coefficients $c(m)$ are positive integers. As the $c(m)$ are positive, it follows that J obeys

$$J(\tau)=\bar{J}(-\bar{\tau})$$

and so is real along the imaginary τ axis.
Although the exact form of $J(q)$ is quite complicated, in many cases the “tree-level approximation”

$$J\approx q^{-1}+...$$

is very useful. To this end, we note that for any given value of $\tau=x+iy$, we have the bound

$$\left|J(\tau)-q^{-1}\right|=\left|\sum_{m\geq 1}c(m)q^m\right|\leq\sum_{m\geq 1}c(m)|q|^m=J(iy)-e^{2\pi y}.$$

The function $\sum_{m\geq 1}c(m)|q|^m$ depends only on y and is a monotonically decreasing function of y, so in the fundamental domain D, it is bounded above by its value at $y=\sqrt{3}/2$, which is the minimum of y in D. This value is

$$M=J\left(i\sqrt{3}/2\right)-e^{\pi\sqrt{3}}\approx 1335.$$

Note that $1335=3\cdot 5\cdot 89$, where 3, 5 and 89 are Fibonacci’s numbers. Furthermore, we have that

$$\Phi^{15}-\Phi^{7}=\left(\frac{\sqrt{5}+1}{2}\right)^{15}-\left(\frac{\sqrt{5}+1}{2}\right)^{7}\cong 1335.$$

Thence, there exist the connection both Fibonacci’s numbers and aureo ratio.

So in D, we have

$$\left|J(\tau)-q^{-1}\right|\leq M.$$

Applying the triangle inequality, it follows that, throughout D, we can bound the value of J by

$$|J(\tau)|\leq e^{2\pi y}+M.$$

We will now consider the action of the Hecke operator T_n on J, defined by

$$T_nJ=\sum_{\delta|n}\sum_{\beta=0}^{\delta-1}J\left((n\tau+\beta\delta)/\delta^2\right). \quad (1.43)$$

This is a new modular invariant function with an n-th order pole at $q=\infty$.

Let us first consider the $\delta=1$ term in (1.43), which is $J(n\sigma)$. For any σ in C, we can find an integer m such that $n\sigma+m$ lies in the fundamental domain. This allows us to apply the following equation

$$\left|J(\tau)-e^{-2\pi i\gamma\tau}\right|\le M \qquad (1.44)$$

with $\tau=n\sigma$ and $\gamma\tau=n\sigma+m$ to get

$$\left|J(n\sigma)-e^{-2\pi i n\sigma}\right|\le M\ . \quad (1.45)$$

Now, consider the $\delta=n$, $\beta=0$ term in (1.43), which is $J(\sigma/n)$. Since σ lies in C, we can find an integer m such that $-n/\sigma+m$ lies in the fundamental domain. So we may apply (1.44), with $\tau=\sigma/n$ and $\gamma=-n/\sigma+m$ to get

$$\left|J(\sigma/n)-e^{-2\pi i n/\sigma}\right|\le M\ . \quad (1.46)$$

Using these two equations, we may apply the triangle inequality to (1.43) to get

$$\left|T_nJ(\sigma)-e^{-2\pi i n\sigma}-e^{-2\pi i n/\sigma}\right|\le 2M+\sum{}'\left|J\left(\frac{n\sigma+\beta\delta}{\delta^2}\right)\right|. \quad (1.47)$$

We will now apply the following equation

$$|J(\tau)|\le \exp\left\{\max_{c,d}\frac{2\pi\,\mathrm{Im}\,\tau}{|c\tau+d|^2}\right\}+M \qquad (1.48)$$

to each term on the right hand side of (1.47). Consider first the term with $\delta=n,\beta=n-1,$ which is $J\left(\frac{\sigma-1}{n}\right)$. We would like to apply (1.48) with $\tau=\frac{\sigma-1}{n}$, so we must ask what possible values $|c\tau+d|$ can take for this value of τ. Now, since σ lies on C, it follows that $|\sigma-1|>1$ and hence $|\tau|=\left|\frac{\sigma-1}{n}\right|>1/n$. This implies that $|c\tau+d|\ge 1/n$ for all possible choices of c and d. Since $\mathrm{Im}\,\tau=y/n$, equation (1.48) gives

$$\left|J\left(\frac{\sigma+n+1}{n}\right)\right|\le e^{2\pi ny}+M\ . \quad (1.49)$$

Let us consider the case where $\delta=n$ and $0<\beta<n-1$, where we may apply (1.48) with $\tau=\frac{\sigma+\beta}{n}$. For this range of β, $|\sigma+\beta|>\sqrt{2}$, so $|\tau|>\sqrt{2}/n$. Hence $|c\tau+d|>\sqrt{2}/n$ and (1.44) gives

$$\left|J\left(\frac{\sigma+\beta}{n}\right)\right|\le e^{\pi ny}+M\ . \quad (1.50)$$

Finally, we consider the cases where $1<\delta<n$. In this case we apply (1.48) with $\tau=(n\sigma+\beta\delta)/\delta^2$. The fact that σ lies on C implies that $|c\tau+d|>\sqrt{3}n/\delta^2$. So we end up with the bound

$$\left|J\left(\frac{\sigma+\beta}{n}\right)\right|\leq e^{\frac{2\pi n}{3}y}+M\ . \quad (1.51)$$

Putting this all together, equation (1.47) becomes

$$\left|T_nJ(\sigma)-e^{-2\pi in\sigma}-e^{-2\pi in/\sigma}\right|\leq e^{2\pi ny}+ne^{\pi ny}+n^2e^{\frac{2\pi n}{3}y}+n^2M\ . \quad (1.52)$$

The factors of n and n^2 in (1.52) come from the simple fact that there are less than n terms with $\delta=n$, and less than n^2 terms with $1<\delta<n$. Multiplying both sides of (1.52) by $e^{-2\pi ny}$, and using the fact that $e^{-2\pi in/\sigma}=e^{2\pi in\bar{\sigma}}$ for points on the curve C, this becomes

$$\left|T_nJe^{-2\pi ny}-2\cos(2\pi nx)\right|\leq 1+ne^{-\pi ny}+n^2e^{-\frac{4\pi n}{3}y}+Mn^2e^{-2\pi ny}\ . \quad (1.53)$$

For the moment, we concentrate on the case $n\geq 2$. Since $y\geq\sqrt{3}/2$ for any point on C, we may evaluate this right hand side of (1.53) to get

$$\left|T_nJe^{-2\pi ny}-2\cos(2\pi nx)\right|\leq 1.12\ . \quad (1.54)$$

This formula is valid for any point on the arc C and $n\geq 2$.
We have only proven the bound (1.54) for $n\geq 2$. For $n=0$, we define $T_0J=1$. So the bound (1.54) is trivial. For $n=1$, we have $T_1J=J$. In this case we have the slightly weaker bound

$$\left|J(\tau)e^{-2\pi y}-2\cos(2\pi x)\right|\leq 1.22 \quad (1.55)$$

for points on C. We note that

$$1.22=2\cdot 0.61\cong 2\cdot\phi\cong 2\cdot 0.618033=2\cdot\left(\frac{\sqrt{5}-1}{2}\right)$$

hence the mathematical connection with the aurea section ϕ.
We now describe the general case of a modular invariant partition function at level k:

$$Z_k(\tau)=\sum_{\Delta=-k}^{\infty}F_\Delta q^\Delta=\sum_{\Delta=-k}^{0}F_\Delta T_{|\Delta|}J(\tau) \quad (1.56)$$

where the F_Δ are non-negative integers and $F_{-k}=1$. As with $J(\tau)$, the fact that $Z_k(\tau)$ has real coefficients and is modular-invariant means that it is real on the imaginary τ axis as well as on the boundary of the fundamental region ∂D.
We have proved the estimate (1.54) for T_nJ on C with $n\geq 2$. Along with the estimate (1.55) on C, this implies that for all $n\geq 0$

$$T_n J = e^{2\pi ny}\left(2\cos(2\pi nx) + E_n\right) \quad (1.57)$$

where E_n is an error term obeying

$$|E_n| < 1.22 . \quad (1.58)$$

We note that

$$\phi \cdot 2 = \left(\frac{\sqrt{5}-1}{2}\right) \cdot 2 = 0{,}618033 \cdot 2 = 1{,}236066 \cong 2 \cdot 0{,}61 = 1{,}22$$

hence the mathematical connection with ϕ, i.e. with the aurea section.

By simply adding up the inequalities (1.57) for $n = 0,\dots,k$, with coefficients F_{-n}, and using the fact that $F_{-k} = 1$, we get a similar estimate for Z_k:

$$Z_k e^{-2\pi ky} - 2\cos(2k\pi x) = E_k + \sum_{\Delta=-k+1}^{0} F_\Delta e^{-2\pi(k+\Delta)y}\left(2\cos(2\pi\Delta x) + E_{|\Delta|}\right). \quad (1.59)$$

To bound the location of the zeroes of Z_k, we must show that the right hand side is less than 2. Then the zeroes of Z_k will lie on C and become dense in the large k limit. For this to be the case, F_Δ must not grow too quickly with Δ. For example, assume that

$$F_\Delta < Ae^{2\pi\alpha(\Delta+k)} \quad \text{for} \quad -k \le \Delta \le 0 \quad (1.60)$$

where α and A are positive constants. In this case

$$\sum_{\Delta=-k+1}^{0} F_\Delta e^{-2\pi(k+\Delta)y} < A \sum_{\Delta=-k+1}^{0} e^{2\pi(k+\Delta)(\alpha-y)} < A\sum_{n=1}^{k} e^{-2\pi n\left(\alpha-\sqrt{3}/2\right)} < \frac{A}{e^{2\pi\left(\alpha-\sqrt{3}/2\right)}-1} . \quad (1.61)$$

In the second line, we have used the fact that $y > \sqrt{3}/2$ on C. Since

$$\left|2\cos(2\pi\Delta x) + E_{|\Delta|}\right| \le 2 + \left|E_{|\Delta|}\right| < 3.22 , \quad (1.62)$$

it follows from (1.58) that

$$\left|Z_k e^{-2\pi ky} - 2\cos(2k\pi x)\right| \le |E_k| + 3.22\frac{A}{e^{2\pi\left(\sqrt{3}/2-\alpha\right)}-1} , \quad (1.63)$$

which is less than 2 for certain values of A and α. In particular, using $|E_k| < 1.22$ and setting $A = 1$, we find that the right hand side is less than 2 provided that

$$\alpha \le 0.61 . \quad (1.64)$$

We note that

$$\Phi \cdot 2 = 2 \cdot \left(\frac{\sqrt{5}+1}{2}\right) = 3.2360678 \cong 2 \cdot 1{,}61 = 3.22 ,$$

hence the mathematical connection with Φ, i.e. with the aurea ratio.
Now we describe the supergravity partition functions.
In ordinary gravity, the thermal excitations of left-movers are obtained by acting on the ground state $|\Omega\rangle$ with Virasoro generators L_{-n}, $n \geq 2$. When the boundary theory has $N=1$ supersymmetry, we can also act on $|\Omega\rangle$ with superconformal generators $G_{-n+1/2}$, $n \geq 2$. Writing $-k^*/2$ for the ground state energy, the partition function of left-moving excitations is therefore

$$q^{-k^*/2}\prod_{n=2}^{\infty}\frac{1+q^{n-1/2}}{1-q^n}. \quad (1.65)$$

Including both left- and right-moving excitations, the contribution of $M_{0,1}$ to $F(q,\bar{q}) = Tr_{NS}\exp(-\beta H - i\theta J)$ is

$$F_{0,1} = \left|q^{-k^*/2}\prod_{n=2}^{\infty}\frac{1+q^{n-1/2}}{1-q^n}\right|^2. \quad (1.66)$$

This formula can be justified exactly as for the bosonic case.
The NS partition function $F(\tau) = Tr_{NS}\exp(-\beta H - i\theta J)$ corresponds to $\mu = \nu = 1/2$. This condition is invariant under the subgroup of $SL(2,Z)$ characterized by saying that $c+d$ and $a+b$ are both odd. If $c+d$ is odd, we can make $a+b$ odd by adding to (a,b) a multiple of (c,d). F, or at least the sum of known contributions to it, can therefore be computed by summing $F_{0,1}$ over modular images with $c+d$ odd:

$$F(\tau) = \sum_{c,d|c+d odd} F_{c,d}(\tau) \quad (1.67)$$

or equivalently

$$F(\tau) = \sum_{c,d|c+d odd} F_{0,1}((a\tau+b)/(c\tau+d)). \quad (1.68)$$

It is also of interest to compute partition functions with other spin structures. However, the three even spin structures on Σ are permuted by $SL(2,Z)$ and so the associated partition functions are not really independent functions. If we set $\mu = 0, \nu = 1/2$, we get $G(\tau) = Tr_{NS}(-1)^F\exp(-\beta H - i\theta J)$. The contribution of $M_{0,1}$ to this partition function is obtained by reversing the sign of all fermionic contributions in (1.66):

$$G_{0,1}(\tau) = \left|q^{-k^*/2}\prod_{n=2}^{\infty}\frac{1-q^{n-1/2}}{1-q^n}\right|^2. \quad (1.69)$$

The subgroup of $SL(2,Z)$ that preserves the conditions $\mu = 0, \nu = 1/2$ is characterized by requiring that b should be even, which implies that a and d are odd. Hence

$$G(\tau) = \sum_{c,d|d odd} G_{c,d} = \sum_{c,d|d odd} G_{0,1}((a\tau+b)/(c\tau+d)), \quad (1.70)$$

where for given c,d, we pick a,b so that $ad-bc=1$ and b is even. A modular transformation $T:\tau\to\tau+1$ exchanges $(\mu,\nu)=(0,1/2)$ with $(\mu,\nu)=(1/2,1/2)$, so in particular

$$F(\tau)=G(\tau+1)=F(\tau+2), \quad (1.71)$$

and the summand in (1.70) does not depend on the choice of a,b.
The Ramond partition function $K=Tr_R\exp(-\beta H-i\theta J)$ is computed from $\mu=1/2,\nu=0$, so

$$K(\tau)=G(-1/\tau). \quad (1.72)$$

We will now compute the partition functions by summing over smooth geometries. We will consider the partition function $G(\tau)=Tr_{NS}(-1)^F\exp(-\beta H-i\theta J)$, as it is technically the simplest; the two other non-zero partition functions are then given by (1.71) and (1.72).
From (1.69) and (1.70) we have

$$G(\tau)=\sum_{c,d|\ d\,odd}\left(|q|^{-k^*}\prod_{n\geq 2}\left|\frac{1-q^{n-1/2}}{1-q^n}\right|^2\right)\Bigg|_\gamma . \quad (1.73)$$

To understand the modular transformation properties of this sum, it is useful to rewrite the infinite product in terms of Dedekind eta functions. Using the identities

$$\prod_{n=2}^{\infty}(1-q^n)^{-1}=\frac{q^{1/24}(1-q)}{\eta(\tau)} \quad (1.74)$$

and

$$\prod_{n=2}^{\infty}\left(1-q^{n-1/2}\right)=\frac{q^{1/48}}{\left(1-q^{1/2}\right)}\frac{\eta(\tau/2)}{\eta(\tau)} \quad (1.75)$$

this may be written as

$$G(\tau)=\sum\left(|q|^{-k^*+3/24}\left|1+q^{1/2}\right|^2\frac{|\eta(\tau/2)|^2}{|\eta(\tau)|^4}\right)\Bigg|_\gamma . \quad (1.76)$$

We may now extract these Dedekind eta functions from the sum, using the fact that $\sqrt{\mathrm{Im}\,\tau}|\eta(\tau)|^2$ is modular invariant:

$$G(\tau)=\frac{|\eta(\tau/2)|^2}{y^{1/2}|\eta(\tau)|^4}\sum\left(y^{1/2}|q|^{-k^*+3/24}\left|1-q^{1/2}\right|^2\right)_\gamma =$$

$$=\frac{|\eta(\tau/2)|^2}{y^{1/2}|\eta(\tau)|^4}\left(\hat{E}(k^*-3/24,0)+\hat{E}(k^*+1-3/24,0)+\hat{E}(k^*+1/2-3/24,1/2)+\hat{E}(k^*+1/2-3/24,-1/2)\right) \quad (1.77)$$

In the second line we have defined

$$\hat{E}(\kappa,\mu)=\sum_{c,d|\ dodd}\frac{y^{1/2}}{|c\tau+d|}\exp\{2\pi\kappa\,\mathrm{Im}\,\gamma\tau+2\pi i\mu\,\mathrm{Re}\,\gamma\tau\}. \quad (1.78)$$

This is the supersymmetric version of the Poincaré series.
This sum is divergent. In particular, at large c and d the exponential approaches one and we are left with the linearly divergent sum $\sum_{c,d}|c\tau+d|^{-1}$. The sum may be regularized by considering the more general Poincaré series

$$\hat{E}(s,\kappa,\mu)=\sum_{c,d|\ dodd}\frac{y^{s}}{|c\tau+d|^{2s}}\exp\{2\pi\kappa\,\mathrm{Im}\,\gamma\tau+2\pi i\mu\,\mathrm{Re}\,\gamma\tau\} \quad (1.79)$$

as an analytic function of s and taking $s\to 1/2$.
Now we start by letting $d=d'+2nc$, where $d'=d \bmod 2c$. The sum in (1.79) can be written as a sum over c,d', and n:

$$E(s,\kappa,\mu)=y^{s}e^{2\pi(\kappa y+i\mu x)}+\sum_{c>0}\sum_{d'\in Z/2cZ}\sum_{n\in Z}f(c,d',n) \quad (1.80)$$

where

$$f(c,d',n)=\frac{y^{s}}{|c(\tau+2n)+d'|^{2s}}\exp\left\{\frac{2\pi\kappa y}{|c(\tau+2n)+d'|^{2}}+2\pi i\mu\left(\frac{a}{c}-\frac{cx+d}{c|c(\tau+2n)+d'|^{2}}\right)\right\}. \quad (1.81)$$

We will now apply the Poisson summation formula to the sum over n, as for the bosonic partition function. We must compute the Fourier transform

$$\hat{f}(c,d',\hat{n})=\int_{-\infty}^{\infty}dne^{2\pi in\hat{n}}f(c,d',n)$$
$$=\frac{1}{2}\exp\left(2\pi i\frac{(2\mu)a-\hat{n}d'}{2c}-\pi i\hat{n}x\right)\int_{-\infty}^{\infty}dte^{\pi i\hat{n}t}\left(\frac{y}{c^{2}(t^{2}+y^{2})}\right)^{s}\exp\left\{\frac{2\pi(\kappa y-i\mu t)}{c^{2}(t^{2}+y^{2})}\right\}. \quad (1.82)$$

The integral appearing in (1.82) is precisely that defined in (1.26). The Fourier coefficients of the Poincaré series (1.79) are therefore given by

$$\hat{E}(s,\kappa,\mu)=y^{s}e^{2\pi(\kappa y+i\mu x)}+\sum_{\hat{n}}e^{\pi i\hat{n}x}\hat{E}_{\hat{n}}(s,\kappa,\mu) \quad (1.83)$$

where

$$\hat{E}_{\hat{n}}(s,\kappa,\mu)=\frac{1}{2}\sum_{m=0}^{\infty}I_{m,\hat{n}/2}(s,\kappa,\mu)y^{1-m-s}\left(\sum_{c=1}^{\infty}c^{-2(m+s)}S(-\hat{n},2\mu;2c)\right) \quad (1.84)$$

is defined in terms of the integrals (1.26).
We will now restrict our attention to the $\hat{n}=0$ case. First consider $\mu=0$. In this case, the integrals were given in (1.27). To do the sum over c, note first that the Kloosterman sum $S(0,0,2c)$ is equal to Euler's totient function $\phi(2c)$. The sum over c is

$$\sum_{c>0} c^{-2(m+s)} S(0,0,2c) = \sum_{c>0} c^{-2(m+s)} \phi(2c) = \frac{2^{2(m+s)}}{2^{2(m+s)}-1} \frac{\zeta(2(m+s)-1)}{\zeta(2(m+s))}. \quad (1.85)$$

As in the bosonic case, we must be careful when taking $s \to 1/2$. For $m = 0$ (and $\hat{n} = 0$), the sum (1.85) vanishes at $s = 1/2$, whereas the integral $I_{0,0}$ has a pole at $m = 0, s = 1/2$, as we see in (1.27). The product of the two factors is finite; in fact, $\Gamma(s-1/2)/\zeta(2s) \to 2$ at $s \to 1/2$. The $m > 0$ terms are finite without any such subtleties. Evaluating the first three terms in the expansion of (1.79) gives

$$\hat{E}_0(1/2, \kappa, 0) = -y^{1/2} + \left(\frac{8\pi^3}{21\zeta(3)} \kappa \right) y^{-1/2} + \left(\frac{64\pi^6}{4185\zeta(5)} \kappa^2 \right) y^{-3/2} + O\left(y^{-5/2}\right). \quad (1.86)$$

Let us now consider the $\mu = \pm 1/2$ terms. For $m = 0$, we find that

$$I_{0,0}(s, \kappa, \pm 1/2) = \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)}. \quad (1.87)$$

This is the analogous of equation (1.29) used in the computation of the bosonic partition function. For $m > 0$ the integrals are complicated hypergeometric functions of the sort written down in (1.28). To do the sum over c, we use the fact that the Kloosterman sum $S(0, \pm 1, 2c)$ is equal to the Mobius function $\mu(2c)$. The sum over c is given by

$$\sum_{c>0} c^{-2(m+s)} S(0, \pm 1, 2c) = \sum_{c} c^{-2(m+s)} \mu(2c) = -\frac{2^{2(m+s)}}{\left(2^{2(m+s)}-1\right)\zeta(2(m+s))}. \quad (1.88)$$

At $s = 1/2$, we again must be careful to cancel the zero in (1.88) at $m = 0$ against a pole in (1.87), using the fact that $\Gamma(s-1/2)/\zeta(2s) \to 2$ as $s \to 1/2$. Including the next two terms in the series, we find

$$E_0(1/2, \kappa, \pm 1/2) = -2y^{1/2} - \left(\frac{16\pi}{7\zeta(3)} \kappa \right) y^{-1/2} - \left(\frac{16\pi^2}{93\zeta(5)} \left(8\kappa^2 - 1\right) \right) y^{-3/2} + O\left(y^{-5/2}\right). \quad (1.89)$$

Putting this all together gives the following expansion for the partition function

$$G(\tau) = G_{0,1}(\tau) + \frac{|\eta(\tau/2)|^2}{|\eta(\tau)|^4} \left(-6 + \frac{(6 + 16k^*)(\pi^3 - 6\pi)}{21\zeta(3)} y^{-1} \right.$$
$$\left. - \frac{4\pi^2\left(2880k^{*2} - 16k^*\left(2\pi^4 - 135\right) + 45 - 12\pi^4\right)}{4185\zeta(5)} y^{-2} + O\left(y^{-3}\right) \right). \quad (1.90)$$

This is the supergravity partition function.
Now, with regard the eqs. (1.89) and (1.90), we take the following pure numbers:

3, 5, 6, 7, 8, 12, 16, 21, 45, 93, 135, 2880, 4185.

We observe that four numbers are p(n), i.e. partitions of numbers: 3, 5, 7 and 135. Four numbers are of type $p(n)\pm 1$: 8 = 7 + 1, 12 = 11 + 1, 16 = 15 + 1, 21 = 22 – 1.
But we obtain also the number 31 (prime natural number with 5 and 7, i.e of type $6n \pm 1$ with n = 1 and 5 that are Fibonacci's numbers). Indeed:

$93 = 3\times 31$, $2880 = 45\times 64$, $4185 = 45\times 93$. We note that $93 = 31\times 3$ and $64 = 31\times 2 + 2 = 8\times 8$.

Furthermore, we have that $31 = 21 + 8 + 2$ that are all Fibonacci's numbers and the number 8 is related to the following Ramanujan modular equation that has 8 "modes" that correspond to the physical vibrations of a superstring.
Indeed, we have that:

$$8 = \frac{1}{3}\cdot\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.90a)$$

We have also that:

$2880 = (31\times 93) - 3$, $4185 = 45\times 93 = 31\times 135$, $93 = (77 + 101)/2 + 4 = 89 + 4$ with 89 that is Fibonacci's number. We have also that:

$6 = 2\times 3$, $12 = 2^2\times 3$, $16 = 2\times 8$, $21 = 3\times 7$, $45 = 3^2\times 5$, $135 = 3^3\times 5$,

with 2, 3, 5, 8 and 21 that are Fibonacci's numbers. It is important to observe that the number 31 is a factor of $248 = 31\times 8$, where 248 is the number related to the dimensions of the Lie group E_8, while 8 is a Fibonacci's number and is related to the physical vibrations of the superstrings.
Furthermore, we remember that (with regard the numbers of partitions):

$p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5, p(5) = 7, p(6) = 11, p(7) = 15, p(8) = 22, ..., p(14) = 135$

and we note that: 7 + 1 = 8, 11 + 1 = 12, 15 + 1 = 16 and 22 – 1 = 21, and that

21 + 16 + 8 = 45, with 21 and 8 that are Fibonacci's numbers. We note also that, with regard $p(3), p(4)$ and $p(5)$, we have: 3 + 4 + 5 = 12, while $p(14) = 135$ with 14 = 2 + 3 + 4 + 5 sum of n of $p(2), p(3), p(4)$ and $p(5)$.
Furthermore the sum of 3, 5, 6, 7, 8 and 16 is 45 and $135 = 45\times 3$. Thence, also the number 45 is very important. We have observed that 45 = 3 + 8 + 13 + 21, that are all Fibonacci's numbers, and 8 is related to the physical vibrations of a superstring.
Furthermore, we have also that $2880 = 12\times 240$ and $2880 - 45 = 2835 = 21\times 135$. Also here, we note that 21 is a Fibonacci's number, while the number 24 ($12 = 24/2$, $240 = 24\times 10$), is related to the physical vibrations of a bosonic string, thence to the following Ramanujan's modular equation:

$$24 = \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]} . \qquad (1.90b)$$

In conclusion, we have also that:

$$p(2) + p(5) + p(8) = 2 + 7 + 22 = 2 + 8 + 21 = 31,$$
$$p(1) + p(5) + p(7) + p(8) = 1 + 7 + 15 + 22 = 3 + 8 + 13 + 21 = 45,$$

with 2, 3, 8, 13 and 21 that are Fibonacci's numbers.

1.1 *Gravity and Chern-Simons Theory with negative cosmological constant*

As noted by Achucarro and Townsend and subsequently extensively developed by Witten, vacuum Einstein gravity in three spacetime dimensions is equivalent to a Chern-Simons gauge theory. We will be interested in the case of a negative cosmological constant $\Lambda = -1/\ell^2$, hence $\ell = \sqrt{-1/\Lambda}$. Then the coframe $e^a = e^a_\mu dx^\mu$ and the spin connection $\omega^a = \frac{1}{2}\varepsilon^{abc}\omega_{\mu bc} dx^\mu$ can be combined into two SL(2,R) connections one-forms

$$A^{(\pm)a} = \omega^a \pm \frac{1}{\ell} e^a . \qquad (1.91)$$

It is straightforward to show that up to possible boundary terms, the first-order form of the usual Einstein-Hilbert action can be written as

$$I = \frac{1}{8\pi G} \int_M \left\{ e^a \wedge \left(d\omega_a + \frac{1}{2}\varepsilon_{abc}\omega^b \wedge \omega^c \right) + \frac{\Lambda}{6}\varepsilon_{abc} e^a \wedge e^b \wedge e^c \right\} = I_{CS}\left[A^{(+)}\right] - I_{CS}\left[A^{(-)}\right] \qquad (1.92)$$

where $A^{(\pm)} = A^{(\pm)a} T_a$ are SL(2,R)-valued gauge potentials, and the Chern-Simons action I_{CS} is

$$I_{CS} = \frac{\ell}{4G} \frac{1}{4\pi} \int_M Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) . \qquad (1.93)$$

Similarly, the Chern-Simons field equations

$$F^{(\pm)} = dA^{(\pm)} + A^{(\pm)} \wedge A^{(\pm)} = 0 \qquad (1.94)$$

are easily seen to be equivalent to the requirement that the connection be torsion-free and that the metric have constant negative curvature, as required by the vacuum Einstein field equations.

Banados, Teitelboim and Zanelli showed that vacuum (2+1)-dimensional gravity with $\Lambda < 0$ admitted a black hole solution. The BTZ black hole in "Schwarzschild" coordinates is given by the metric

$$ds^2 = \left(N^{\perp}\right)^2 dt^2 - f^{-2}dr^2 - r^2\left(d\phi + N^{\phi}dt\right)^2 \quad (1.95)$$

with lapse and shift functions and radial metric

$$N^{\perp} = f = \left(-8GM + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2}\right)^{1/2}, \quad N^{\phi} = -\frac{4GJ}{r^2} \qquad \left(|J| \le M\ell\right). \quad (1.96)$$

The metric (1.95) is stationary and axially symmetric, with Killing vectors ∂_t and ∂_ϕ, and generically has no other symmetries. Although it describes a spacetime of constant negative curvature, it is a true black hole: it has a genuine event horizon at r_+ and, when $J \neq 0$, an inner Cauchy horizon at r_-, where

$$r_{\pm}^2 = 4GM\ell^2\left\{1 \pm \left[1 - \left(\frac{J}{M\ell}\right)^2\right]^{1/2}\right\}, \quad (1.97)$$

i.e.,

$$M = \frac{r_+^2 + r_-^2}{8G\ell^2}, \qquad J = \frac{r_+ r_-}{4G\ell}. \quad (1.98)$$

Another useful coordinate system is based on proper radial distance ρ and two light-cone-like coordinates $u, v = t/\ell \pm \phi$; the metric then takes the form

$$ds^2 = 4G\ell\left(L^+du^2 + L^-dv^2\right) - \ell^2 d\rho^2 + \left(\ell^2 e^{2\rho} + 16G^2L^+L^-e^{-2\rho}\right)dudv \quad (1.99)$$

with

$$L^{\pm} = \frac{(r_+ \pm r_-)^2}{16G\ell}. \quad (1.100)$$

In these coordinates, the Chern-Simons connections (1.91) take the simple form

$$A^{(+)} = \begin{pmatrix} \frac{1}{2}d\rho & -\frac{4G}{\ell}L^+e^{-\rho}du \\ -e^{\rho}du & -\frac{1}{2}d\rho \end{pmatrix}, \qquad A^{(-)} = \begin{pmatrix} -\frac{1}{2}d\rho & -e^{\rho}dv \\ -\frac{4G}{\ell}L^-e^{-\rho}dv & \frac{1}{2}d\rho \end{pmatrix}. \quad (1.101)$$

It is easy to check that these connections satisfy the equations of motion (1.94). This solution may be generalized: the Einstein field equations are still satisfied if one allows L^+ to be an arbitrary function of u and L^- to be an arbitrary function of v.

As a constant curvature spacetime, the BTZ black hole is locally isometric to anti-de Sitter space. In fact, it is globally a quotient space of AdS_3 by a discrete group. We can identify AdS_3 with the universal covering space of the group SL(2,R); the BTZ black hole is then obtained by the identification

$$g \approx \rho^{-} g \rho^{+}, \qquad \rho^{\pm} = \begin{pmatrix} e^{\pi(r_{+} \pm r_{-})/\ell} & 0 \\ 0 & e^{-\pi(r_{+} \pm r_{-})/\ell} \end{pmatrix}. \quad (1.102)$$

Up to a gauge transformation, the group elements $\rho^{\pm}$ can be identified with the holonomies of the SL(2,R) connections (1.101).
The most important feature of the BTZ black hole is that it has thermodynamic properties closely analogous to those of realistic (3+1)-dimensional black holes: it radiates at a Hawking temperature of

$$T = \frac{\hbar \kappa}{2\pi} = \frac{\hbar\left(r_{+}^{2} - r_{-}^{2}\right)}{2\pi \ell^{2} r_{+}}, \quad (1.103)$$

where κ is the surface gravity, and has an entropy

$$S = \frac{2\pi r_{+}}{4\hbar G} \quad (1.104)$$

equal to a quarter of its area.

2. On some equations concerning the Selberg Supertrace formula for super Riemann surfaces, analytic properties of Selberg super zeta-functions and multiloop contributions for the fermionic string. [3] [4]

We have formulated the Selberg Supertrace formula on super Riemannian surfaces for operator valued functions of the Laplace-Dirac operator $\square_m$. Let h be a test-function with the properties:

i) $h\left(\frac{1}{2} + ip\right) \in C^{\infty}(R)$,

ii) $h\left(\frac{1}{2} + ip\right)$ need not be an even function in p,

iii) $h\left(\frac{1}{2} + ip\right) \propto O\left(\frac{1}{p^{2}}\right)(p \to \pm\infty)$,

iv) $h\left(\frac{1}{2} + ip\right)$ is holomorphic in the strip $\left|\operatorname{Im}(p)\right| \le 1 + \frac{m}{2} + \varepsilon$, $\varepsilon > 0$ to guarantee absolute convergence in the sums of eq. (2.5) below.

Its Fourier transform g is given by:

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{-iup} h\left(ip + \frac{m+1}{2}\right). \quad (2.1)$$

The term $A_{0}^{(m)}$ corresponding to the identity transformation reads

$$A_0^{(m)} = i(g-1)\int_{-\infty}^{\infty} h\left(ip + \frac{m+1}{2}\right)\tanh \pi p\,dp + (1-g)\sum_{k=1}^{m/2}\left[h\left(\frac{m}{2}+k\right) - h\left(\frac{m}{2}-k-1\right)\right], \quad (m \text{ even}) \qquad (2.2)$$

$$A_0^{(m)} = i(g-1)\int_{-\infty}^{\infty} h\left(ip + \frac{m+1}{2}\right)\coth \pi p\,dp + (1-g)\sum_{k=1}^{(m-1)/2}\left[h\left(\frac{m+1}{2}+k\right) - h\left(\frac{m-1}{2}-k\right)\right], \quad (m \text{ odd}) \qquad (2.3)$$

The last two equations can be combined and stated in a compact form yielding

$$A_0^{(m)} = (1-g)\int_0^{\infty} \frac{g(u)-g(-u)}{\sinh\frac{u}{2}} T_m\left(\cosh\frac{u}{2}\right)du, \quad (m \in Z), \quad (2.4)$$

where $T_m\left(\cosh\frac{u}{2}\right) = \cosh\frac{m}{2}u$ denotes the m^{th} Chebyshev-polynomial in $\cosh\frac{u}{2}$. Thus for the supertrace formula we get (l_γ primitive geodesic, $\lambda_n^{B(F)} = \frac{1}{2} + p_n^{B(F)}$ $(n \in N)$ are denoting the Bose and Fermi Eigenvalues of $\square$, respectively):

$$\sum_{n=0}^{\infty}\left[h_m\left(p_n^B\right) - h_m\left(p_n^F\right)\right] =$$

$$= (1-g)\int_0^{\infty} \frac{g(u)-g(-u)}{\sinh\frac{u}{2}} \cosh\frac{m}{2}du + \sum_{\{\gamma\}_p}\sum_{k=1}^{\infty} \frac{l_\gamma \chi_\gamma^{km}}{e^{kl_\gamma/2} - e^{-kl_\gamma}}\left[g(kl_\gamma) + g(-kl_\gamma) - \chi_\gamma^k\left(g(kl_\gamma)e^{-kl_\gamma/2} + g(-kl_\gamma)e^{kl_\gamma/2}\right)\right] \qquad (2.5)$$

The Selberg super zeta-functions are defined by

$$Z_q(s) := \prod_{\{\gamma\}_p}\prod_{k=0}^{\infty}\left[1 - \chi_\gamma^q e^{-(s+k)l_\gamma}\right], \quad (\mathrm{Re}(s) > 1), \quad (2.6)$$

where q can take on the values $q = 0{,}1$, respectively. χ_γ describes the spin structure and l_γ is the length of a primitive geodesic. The γ product is taken over all primitive conjugacy classes $\gamma \in \Gamma$. The Selberg super R-functions are defined by

$$R_q(s) := \frac{Z_q(s)}{Z_q(s+1)} = \prod_{\{\gamma\}_p}\left[1 - \chi_\gamma^q e^{-sl_\gamma}\right], \quad (\mathrm{Re}(s) > 1). \quad (2.7)$$

To study the analytic properties of Z_0 and Z_1 let us consider the Selberg supertrace formula for $m = 0$, i.e.:

$$\sum_{n=0}^{\infty}\left[h\left(p_n^B\right)-h\left(p_n^F\right)\right]=$$

$$=i(g-1)\int_{-\infty}^{\infty}h\left(ip+\frac{1}{2}\right)\tanh\pi p dp+\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma}{e^{kl_\gamma/2}-e^{-kl_\gamma/2}}\left[g\left(kl_\gamma\right)+g\left(-kl_\gamma\right)-\chi_\gamma^k\left(g\left(kl_\gamma\right)e^{-kl_\gamma/2}+g\left(-kl_\gamma\right)e^{kl_\gamma/2}\right)\right] \quad (2.8)$$

To get information for Z_1 or R_1, respectively, one has to choose a test function $h(p)$ so that the first two terms in the square bracket in the supertrace formula cancel, i.e. $g(u)=-g(-u)$. We choose the function $(\mathrm{Re}(s)>1,\mathrm{Re}(\sigma)>1)$:

$$h_s(p)=2\left[\frac{\lambda-\frac{1}{2}}{s^2-\left(\lambda-\frac{1}{2}\right)^2}-\frac{\lambda-\frac{1}{2}}{\sigma^2-\left(\lambda-\frac{1}{2}\right)^2}\right]\Bigg|_{\lambda=1/2+ip}=2ip\left(\frac{1}{s^2+p^2}-\frac{1}{\sigma^2+p^2}\right). \quad (2.9)$$

The second term plays the role of a regulator so that all the involved terms in the supertrace formula are convergent. Thus for $g(u)$:

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iup}h_s(p)dp=\frac{2}{\pi}\int_0^{\infty}p\sin up\left(\frac{1}{s^2+p^2}-\frac{1}{\sigma^2+p^2}\right)dp, \quad (2.10)$$

and we see that $g(u)$ is an odd function as required. Using

$$\int_0^{\infty}\frac{x\sin ax}{\beta^2+x^2}dx=\frac{\pi}{2}e^{-\alpha\beta}, \quad (2.11)$$

we get $(u>0)$ $g(u)=\left(e^{-su}-e^{-\sigma u}\right)$ and for $u\in R$

$$g(u)=sign(u)\left(e^{-s|u|}-e^{-\sigma|u|}\right), \quad (2.12)$$

thus finally for $G(u,\chi)$

$$G(u,\chi_\gamma)=2\chi_\gamma\left(e^{-s|u|}-e^{-\sigma|u|}\right)\sinh\frac{u}{2}. \quad (2.13)$$

Therefore only the χ_γ-term remains in the supertrace formula which allows to study the properties of Z_1 alone. Inserting $G(u,\chi)$ into the length term yields

$$\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma}{2\sinh\frac{kl_\gamma}{2}}G\left(kl_\gamma,\chi_\gamma\right)=\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}l_\gamma\chi_\gamma^k\left(e^{-skl_\gamma}-e^{-\sigma kl_\gamma}\right)=\sum_{\{\gamma\}_p}\left[\frac{l_\gamma\chi_\gamma e^{-sl_\gamma}}{1-\chi_\gamma e^{-sl_\gamma}}-\frac{l_\gamma\chi_\gamma e^{-\sigma l_\gamma}}{1-\chi_\gamma e^{-\sigma l_\gamma}}\right]=\frac{R_1'(s)}{R_1(s)}-\frac{R_1'(\sigma)}{R_1(\sigma)}. \quad (2.14)$$

In the last step the property of the logarithmic derivative of the Selberg super R-functions has been used, i.e. for $\mathrm{Re}(s) > 1$:

$$\frac{d}{ds}\ln R_q(s) = \frac{d}{ds}\ln\prod_{\{\gamma\}_p}\left[1-\chi_\gamma^q e^{-sl_\gamma}\right] = \sum_{\{\gamma\}_p}\frac{l_\gamma \chi_\gamma^q e^{-sl_\gamma}}{1-\chi_\gamma^q e^{-sl_\gamma}} . \quad (2.15)$$

The A_0 term gives

$$A_0 = i(g-1)\int_{-\infty}^{\infty} h_s(p)\tanh \pi p dp = \frac{4}{\pi}(1-g)\int_0^\infty \frac{du}{\sinh\frac{u}{2}}\left[\int_0^\infty \frac{p\sin up}{p^2+s^2}dp - \int_0^\infty \frac{p\sin up}{p^2+\sigma^2}dp\right] =$$

$$= 4(1-g)\int_0^\infty e^{-((s+\sigma)/2)u}\frac{\sinh\frac{\sigma-s}{2}}{\sinh\frac{u}{2}}du , \quad (2.16)$$

where the integrals (2.11) and

$$\int_0^\infty \frac{\sin(ax)}{\sinh(bx)}dx = \frac{\pi}{2b}\tanh\frac{a\pi}{2b} , \quad (2.16b)$$

have been used. Using now

$$\int_0^\infty e^{-\mu x}\frac{\sinh\beta x}{\sinh bx}dx = \frac{1}{2b}\left[\Psi\left(\frac{1}{2}+\frac{\mu+\beta}{2b}\right) - \Psi\left(\frac{1}{2}+\frac{\mu-\beta}{2b}\right)\right], \quad (2.17)$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z)$, $z \in C$, we obtain finally for A_0

$$A_0 = 4(g-1)\left[\Psi\left(s+\frac{1}{2}\right) - \Psi\left(\sigma+\frac{1}{2}\right)\right]. \quad (2.18)$$

Let us denote by $\Delta n_0^{(0)} = n_0^B - n_0^F$ the difference between the number of even and odd zero modes of the Dirac operator □. Thus we get the supertrace formula for the function h_s

$$\sum_{n=1}^\infty\left[h_s(p_n^B) - h_s(p_n^F)\right] - \Delta n_0^{(0)}\left[\frac{1}{\left(s-\frac{1}{2}\right)\left(s+\frac{1}{2}\right)} - \frac{1}{\left(\sigma-\frac{1}{2}\right)\left(\sigma+\frac{1}{2}\right)}\right] =$$

$$= 4(g-1)\left[\Psi\left(s+\frac{1}{2}\right) - \Psi\left(\sigma+\frac{1}{2}\right)\right] + \frac{R_1'(s)}{R_1(s)} - \frac{R_1'(\sigma)}{R_1(\sigma)}. \quad (2.19)$$

Let us consider the eq. (2.19) in the limit $\sigma \to \frac{1}{2}$ and get

$$\lim_{\sigma\to 1/2}\left[\frac{R_1^{'}(\sigma)}{R_1(\sigma)}+\frac{\Delta n_0^{(0)}}{\sigma^2-\frac{1}{4}}-4(g-1)\Psi\left(\sigma+\frac{1}{2}\right)\right]=A_1+4(g-1)\gamma_E\,,\quad (2.20)$$

where $\Psi(1)=-\gamma_E$ is the Euler's constant $\gamma_E=0.57721...$ and A_1 is given by

$$A_1=\frac{R_1^{'}\left(\frac{1}{2}\right)}{R_1\left(\frac{1}{2}\right)},\quad \left(\Delta n_0^{(0)}=0\right),\qquad A_1=\frac{R_1^{\left(1-\Delta n_0^{(0)}\right)}\left(\frac{1}{2}\right)}{\left(1-\Delta n_0^{(0)}\right)R_1^{\left(-\Delta n_0^{(0)}\right)}\left(\frac{1}{2}\right)},\quad \left(\Delta n_0^{(0)}<0\right),$$

$$A_1=\frac{\oint\left(\sigma-\frac{1}{2}\right)^{\Delta n_0^{(0)}-2}R_1(\sigma)d\sigma}{\oint\left(\sigma-\frac{1}{2}\right)^{\Delta n_0^{(0)}-1}R_1(\sigma)d\sigma},\quad \left(\Delta n_0^{(0)}>0\right).\quad (2.21)$$

Therefore

$$\sum_{n=1}^{\infty}\left[\frac{2ip_n^B}{s^2+\left(p_n^B\right)^2}-\frac{2ip_n^B}{\frac{1}{4}+\left(p_n^B\right)^2}-\frac{2ip_n^F}{s^2+\left(p_n^F\right)^2}+\frac{2ip_n^F}{\frac{1}{2}+\left(p_n^F\right)^2}\right]-4(g-1)\gamma_E-A_1=$$

$$=\frac{\Delta n_0^{(0)}}{\left(s-\frac{1}{2}\right)\left(s+\frac{1}{2}\right)}+4(g-1)\Psi\left(s+\frac{1}{2}\right)+\frac{R_1^{'}(s)}{R_1(s)}.\quad (2.22)$$

h_s has the symmetry $h_s=h_{-s}$. Writing down eq. (2.22) for $s\to -s$ and subtracting it from eq. (2.22) gives with $\Psi\left(\frac{1}{2}+s\right)=\Psi\left(\frac{1}{2}-s\right)+\pi\tan\pi s$ the functional equation in differential form for the R_1-function,

$$\frac{d}{ds}\ln R_1(s)R_1(-s)=-4(g-1)\pi\tan\pi s\,.\quad (2.23)$$

This equation can be integrated yielding

$$R_1(s)R_1(-s)=\tilde{A}_1(\cos\pi s)^{4(g-1)},\quad (2.24)$$

where $\tilde{A}_1$ is a constant given e.g. by $\tilde{A}_1=R_1(s_0)R_1(-s_0)(\cos\pi s_0)^{4(1-g)}$ with some $s_0\in C$, which is however, independent of s_0. We have, e.g. for $s_0=0$: $\tilde{A}_1=R_1^2(0)$.

Now, we derive the analytic properties of the Selberg super zeta-function Z_0 and present a functional equation connecting the two Selberg super zeta-functions Z_0 and Z_1. Let us consider the test function $\left(\operatorname{Re}(s)>\frac{3}{2}\right)$:

$$h_s(p)=\frac{1}{\lambda(1-\lambda)-s(1-s)}\Big|_{\lambda=ip+(1/2)}=\frac{1}{p^2+\left(s-\frac{1}{2}\right)^2}. \quad (2.25)$$

This gives at once $A_0=0$ because h_s is an even function in p. Furthermore for $g(u)$:

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iup}}{\left(s-\frac{1}{2}\right)^2+p^2}dp=\frac{1}{2s-1}e^{-(s-(1/2))|u|}. \quad (2.26)$$

Thus for $G(u,\chi)$:

$$G(kl_\gamma,\chi_\gamma)=\frac{e^{-(s-(1/2))kl_\gamma}}{s-\frac{1}{2}}\left(1-\chi_\gamma^k\cosh\frac{kl_\gamma}{2}\right). \quad (2.27)$$

Therefore we get for the right-hand side of the supertrace formula

$$\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma}{2\sinh\frac{kl_\gamma}{2}}\frac{e^{-(s-(1/2))kl_\gamma}}{s-\frac{1}{2}}\left(1-\chi_\gamma^k\cosh\frac{kl_\gamma}{2}\right)=$$

$$=\frac{1}{2s-1}\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma}{1-e^{-kl_\gamma}}\left[2e^{-skl_\gamma}-\chi_\gamma^k e^{-(s-(1/2))kl_\gamma}-\chi_\gamma^k e^{-(s+(1/2))kl_\gamma}\right]=\frac{1}{2s-1}\frac{d}{ds}\ln\left[\frac{Z_0^2(s)}{Z_1\left(s-\frac{1}{2}\right)Z_1\left(s+\frac{1}{2}\right)}\right]. \quad (2.28)$$

Here use has been made of the properties of the logarithmic derivative of the super zeta-functions:

$$\frac{d}{ds}\ln Z_q(s)=\frac{d}{ds}\ln\sum_{\{\gamma\}_p}\prod_{k=0}^{\infty}\left[1-\chi_\gamma^q e^{-(s+k)l_\gamma}\right]=\sum_{\{\gamma\}_p}\sum_{n=1}^{\infty}\frac{l_\gamma\chi_\gamma^{qn}e^{-snl_\gamma}}{1-e^{-nl_\gamma}}. \quad (2.29)$$

Thus we find the supertrace formula for the test function h_s

$$\sum_{n=1}^{\infty}\left[h_s(p_n^B)-h_s(p_n^F)\right]-\frac{\Delta n_0^{(0)}}{s(1-s)}=\frac{1}{2s-1}\frac{d}{ds}\ln\left[\frac{Z_0^2(s)}{Z_1\left(s-\frac{1}{2}\right)Z_1\left(s+\frac{1}{2}\right)}\right]. \quad (2.30)$$

The test function h_s is invariant under the change $s\to 1-s$. Performing this substitution in eq. (2.30) and subtracting it from (2.30) yields the functional equation

$$\frac{d}{ds}\ln\left[\frac{Z_0^2(s)}{Z_1\left(s-\frac{1}{2}\right)Z_1\left(s+\frac{1}{2}\right)}\right]=\frac{d}{ds}\ln\left[\frac{Z_0^2(1-s)}{Z_1\left(\frac{1}{2}-s\right)Z_1\left(\frac{3}{2}-s\right)}\right]. \quad (2.31)$$

Let us consider the functional equation (2.23) for the R_1-function and perform the substitution $s\to\frac{1}{2}-s$. By expressing the R_1-function by the quotient of the Z_1-functions, this yields

$$\frac{d}{ds}\ln\left[\frac{Z_1\left(\frac{1}{2}-s\right)Z_1\left(s-\frac{1}{2}\right)}{Z_1\left(\frac{3}{2}-s\right)Z_1\left(s+\frac{1}{2}\right)}\right]=4\pi(g-1)\cot\pi s. \quad (2.32)$$

Thus we find by combining eqs. (2.31) and (2.32) the functional equation in differential form connecting Z_0 and Z_1:

$$\frac{d}{ds}\ln\left[\frac{Z_1\left(\frac{1}{2}-s\right)Z_0(s)}{Z_1\left(\frac{1}{2}+s\right)Z_0(1-s)}\right]=2\pi(g-1)\cot\pi s. \quad (2.33)$$

The functional equation can be integrated yielding:

$$\frac{Z_1\left(\frac{1}{2}-s\right)Z_0(s)}{Z_1\left(\frac{1}{2}+s\right)Z_0(1-s)}=C_0(\sin\pi s)^{2(g-1)}, \quad (2.34)$$

where C_0 is, e.g. given by

$$C_0=\frac{Z_1\left(\frac{1}{2}-s_0\right)Z_0(s_0)}{\left[Z_1\left(\frac{1}{2}+s_0\right)Z_0(1-s_0)(\sin\pi s_0)^{2(1-g)}\right]} \quad (2.34b)$$

with some $s_0\in C$ which is, however, independent of s_0, e.g. for $s_0=\frac{1}{2}$: $C_0=Z_1(0)/Z_1(1)=R_1(0)=\sqrt{\tilde{A}_1}$.

To get around the difficulties of the combination of the Z_0 and Z_1 functions for general test functions h in the Selberg supertrace formula let us define the super zeta-function Z_s:

$$Z_s(s) := \prod_{\{\gamma\}_p}\prod_{n=0}^{\infty} s\det\left[1 - diag\left(1, e^{-l_\gamma}, \chi_\gamma e^{-(l_\gamma/2)}, \chi_\gamma e^{-(l_\gamma/2)} e^{-(s+n)l_\gamma}\right)\right] =$$

$$= \prod_{\{\gamma\}_p}\prod_{n=0}^{\infty} \frac{\left[1 - e^{-(s+n)l_\gamma}\right]\left[1 - e^{-(s+n+1)l_\gamma}\right]}{\left[1 - \chi_\gamma e^{-(s+n+1/2)l_\gamma}\right]^2} = \frac{Z_0(s)Z(s+1)}{Z_1^2\left(s+\frac{1}{2}\right)}. \quad (2.35)$$

Let us consider the resolvent of $\square_0^2 : R_s(\square_0^2) = (s^2 - \square_0^2)^{-1}$ $(\mathrm{Re}(s) > 1)$. Therefore

$$h(p) = \frac{1}{s^2 - \lambda^2}\Big|_{\lambda=(1/2)+ip} = \frac{1}{\left(s^2 - \frac{1}{4}\right) - ip + p^2}. \quad (2.36)$$

We first calculate the Fourier transform of $h(p)$:

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(p) e^{-iup} dp = g_1(u) + g_2(u), \quad (2.37)$$

where

$$g_1(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos up}{\left(s^2 - \frac{1}{2}\right) - ip + p^2} dp = g_1(-u); \quad g_2(u) = \frac{-i}{2\pi}\int_{-\infty}^{\infty} \frac{\sin up}{\left(s^2 - \frac{1}{2}\right) - ip + p^2} dp = -g_2(-u).$$

(2.38)

Using the integrals:

$$\int_{-\infty}^{\infty} \frac{(b+cx)\sin ax}{p + qx + x^2} dx = \left[\frac{cq - b}{\sqrt{p - q^2}}\sin aq + c\cos aq\right]\pi e^{-a\sqrt{p-q^2}},$$

$$\int_{-\infty}^{\infty} \frac{(b+cx)\cos ax}{p + qx + x^2} dx = \left[\frac{b - cq}{\sqrt{p - q^2}}\cos aq + c\sin aq\right]\pi e^{-a\sqrt{p-q^2}}. \quad (2.39)$$

We get for $u > 0$:

$$g_1(u) = \frac{1}{2s}\cosh\frac{u}{2}e^{-us}, \quad g_2(u) = \frac{1}{2s}\sinh\frac{u}{2}e^{-us}. \quad (2.40)$$

Therefore $(u \in R)$:

$$g(u) = \frac{1}{2s}e^{(u/2) - s|u|}, \quad (2.41)$$

which gives for $G(u, \chi)$

$$G(u, \chi_\gamma) = \frac{1}{s}e^{-us}\left(\cosh\frac{u}{2} - \chi_\gamma\right), \quad (2.42)$$

and the right-hand side of the supertrace formula reads:

$$\frac{1}{2s}\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma}{e^{kl_\gamma/2}-e^{-kl_\gamma/2}}e^{-skl_\gamma}\left(e^{kl_\gamma/2}+e^{-kl_\gamma/2}-2\chi_\gamma^k\right)=\frac{1}{2s}\frac{Z_s^{'}(s)}{Z_s(s)}. \quad (2.43)$$

For the A_0-term:

$$A_0=i(g-1)\int_{-\infty}^{\infty}h(p)\tanh\pi p\,dp=i\frac{g-1}{\pi}\int_0^{\infty}\frac{du}{\sinh\frac{u}{2}}\int_{-\infty}^{\infty}\frac{\sin up}{\left(s^2-\frac{1}{4}\right)-ip+p^2}dp=\frac{1-g}{s^2}. \quad (2.44)$$

Therefore we have for the resolvent kernel the supertrace formula

$$\sum_{n=1}^{\infty}\left[\frac{1}{s^2-\left(\lambda_n^B\right)^2}-\frac{1}{s^2-\left(\lambda_n^F\right)^2}\right]+\frac{\Delta n_n^{(0)}+g-1}{s^2}=\frac{1}{2s}\frac{Z_s^{'}(s)}{Z_s(s)}. \quad (2.45)$$

Equation (2.45) and Z_s can be extended meromorphically to all $s\in\Lambda_\infty$.
A very simple functional relation can be deduced from eq. (2.45), reading

$$\frac{d\ln Z_s(s)}{ds}=\frac{d\ln Z_s(-s)}{ds}. \quad (2.46)$$

In terms of Z_0 and Z_1 eq. (2.46) gives

$$\frac{d}{ds}\ln\frac{Z_0(s)Z_0(s+1)}{Z_1^2\left(s+\frac{1}{2}\right)}=\frac{d}{ds}\ln\frac{Z_0(-s)Z_0(1-s)}{Z_1^2\left(\frac{1}{2}-s\right)}. \quad (2.47)$$

Equation (2.46) or (2.47), respectively, integrated gives $Z_s(s)=Z_s(-s)$, thus $Z_s(s)$ is an even function in s. Combining eqs. (2.23), (2.33) and (2.47) we deduce the functional equation for the R_0 function, which reads:

$$\frac{d}{ds}\ln R_0(s)R_0(-s)=4\pi(g-1)\cot\pi s. \quad (2.48)$$

Equation (2.48) can be integrated to give

$$R_0(s)R_0(-s)=B_0(\sin\pi s)^{4(g-1)}, \quad (2.49)$$

where the constant B_0 is e.g. given by $B_0=R_0(s_0)R_0(-s_0)(\sin\pi s_0)^{4(1-g)}$ with some $s_0\in C$, where B_0 is independent of s_0. We have, e.g. for $s=\pm\frac{1}{2}$ $B_0=Z_0\left(-\frac{1}{2}\right)/Z_0\left(\frac{3}{2}\right)=R_0\left(-\frac{1}{2}\right)$.
A similar relation holds also for the ordinary Selberg zeta-function:

$$R(s)R(-s)=\frac{Z(s)Z(-s)}{Z(1+s)Z(1-s)} \quad (2.50)$$

however, in this case the integration constant is given by $B = 2^{4(g-1)}$. From eqs. (2.24), (2.34) and (2.49) many relations linking Z_0 and Z_1 for particular arguments can be deduced, e.g.

$$B_0 = \frac{Z_0\left(-\frac{1}{2}\right)}{Z_0\left(\frac{3}{2}\right)} = \frac{Z_1(-1)Z_1(1)}{Z_1(2)Z_1(0)} = \frac{Z_1^2(0)}{Z_1^2(1)} = C_0^2 = \tilde{A}_1 . \quad (2.51)$$

Let be $m \in N_0$. Let us calculate the superdeterminants by the ζ-function regularization. We get:

$$s\det\left(c^2 - \square_m^2\right) = \exp\left[-\frac{\partial}{\partial s}\zeta_m(s;c)\Big|_{s=0}\right]$$

$$\zeta_m(s;c) = str\left[\left(c^2 - \square_m^2\right)^{-s}\right] = \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1} str\left\{\exp\left[\left(-t^2 - \square_m^2\right)\right]\right\}, \quad (2.52)$$

where use has been made of the following integral:

$$\int_0^\infty x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu}\Gamma(\nu). \quad (2.53)$$

The function h corresponding to the heat-kernel of $\left(c^2 - \square_m^2\right)$ reads

$$h_{hk}(s) = e^{t\left[(s+(m/2))^2 - c^2\right]}. \quad (2.54)$$

Therefore for $g(u)$

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iup} h_{hk}\left(ip + \frac{1}{2}\right)dp = \frac{1}{\sqrt{4\pi t}}\exp\left[-\frac{u^2}{4t} - c^2 t + (m+1)\frac{u}{2}\right]. \quad (2.55)$$

This gives

$$G(u,\chi) = \frac{1}{\sqrt{\pi t}} e^{-u^2/4t - c^2 t}\left[\cosh(m+1)\frac{u}{2} - \chi\cosh m\frac{u}{2}\right],$$

$$g(u) - g(-u) = \frac{1}{\sqrt{\pi t}} e^{-u^2/4t - c^2 t}\sinh(m+1)\frac{u}{2}. \quad (2.56)$$

Splitting the calculation of $\zeta_m(s;c)$ into two terms corresponding to the identity transformation and the length term, respectively, gives:

$$\zeta_m(s;c) = \zeta_m^I(s;c) + \zeta_m^\Gamma(s;c). \quad (2.57)$$

Let us first calculate ζ_m^I:

$$\zeta_m^I(s;c)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}A_0^{(m)}(t)dt\,,$$

$$A_0^{(m)}(t)=\frac{1-g}{\sqrt{\pi t}}e^{-c^2t}\int_0^\infty e^{-\left(u^2/4t\right)}\frac{\sinh(m+1)\frac{u}{2}}{\sinh\frac{u}{2}}\cosh m\frac{u}{2}du=(1-g)e^{-c^2t}\sum_{k=0}^{m}e^{k^2t}\,, \quad (2.58)$$

hence:

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\frac{1-g}{\sqrt{\pi t}}e^{-c^2t}\int_0^\infty e^{-\left(u^2/4t\right)}\frac{\sinh(m+1)\frac{u}{2}}{\sinh\frac{u}{2}}\cosh m\frac{u}{2}dudt=(1-g)e^{-c^2t}\sum_{k=0}^{m}e^{k^2t}\,. \quad (2.58b)$$

Similarly:

$$A_0^{(-1)}=0 \qquad A_0^{(-m)}(t)=(g-1)e^{-c^2t}\sum_{k=0}^{m-2}e^{k^2t} \qquad (m=2,3,\ldots). \quad (2.59)$$

This gives for ζ_m^I:

$$\zeta_m^I(s;c)=\frac{1-g}{\Gamma(s)}\sum_{k=0}^{m}\int_0^\infty t^{s-1}e^{-\left(c^2-k^2\right)t}dt=(1-g)\sum_{k=0}^{m}\left(c^2-k^2\right)^{-s}\,. \quad (2.60)$$

Furthermore, we can easily calculate

$$\frac{\partial}{\partial s}\zeta_m^I(s;c)\Big|_{s=0}=(g-1)\sum_{k=0}^{m}\ln\left(c^2-k^2\right). \quad (2.61)$$

Let us calculate ζ_m^Γ in two alternative ways. The first is appropriate to the analysis of the spectrum, the second to the calculation of the superdeterminants.

1) The supertrace formula for the heat-kernel now reads:

$$\sum_{n=1}^{\infty}\left\{e^{t\left[\left(\lambda_{n,m}^B\right)^2-c^2\right]}-e^{\left[t\left(\lambda_{n,m}^F\right)^2-c^2\right]}\right\}=(1-g)e^{-c^2t}\sum_{k=1}^{m}e^{k^2t}+$$
$$+\frac{e^{-c^2t}}{\sqrt{4\pi t}}\sum_{\gamma\in\Gamma}\sum_{k=1}^{\infty}\frac{l_\gamma\chi_\gamma^{km}}{\sinh\frac{kl_\gamma}{2}}e^{-\left(k^2l_\gamma^2/4t\right)}\left[\cosh(m+1)\frac{kl_\gamma}{2}-\chi_\gamma^k\cosh m\frac{kl_\gamma}{2}\right], \quad (2.62)$$

and the A_0 term appropriately replaced for negative integers. With the help of eqs. (2.53), (2.58) and the integral:

$$\int_0^\infty x^{\nu-1}e^{-(\beta/x)-\gamma x}dx=2\left(\frac{\beta}{\gamma}\right)^{\nu/2}K_\nu\left(2\sqrt{\beta\gamma}\right), \quad (2.63)$$

we get for the supertrace formula of the generalized resolvent kernel:

$$\sum_{n=0}^{\infty}\left\{\frac{1}{\left[c^2-\left(\lambda_{n,m}^B\right)^2\right]^s}-\frac{1}{\left[c^2-\left(\lambda_{n,m}^F\right)^2\right]^s}\right\}=(1-g)\sum_{k=0}^{m}\frac{1}{\left(c^2-k^2\right)^s}+\frac{1}{\Gamma(s)}\sum_{\gamma\in\Gamma}\sum_{k=1}^{\infty}\frac{l_\gamma\chi_\gamma^{km}}{e^{kl_\gamma/2}-e^{-(kl_\gamma/2)}}\left(\frac{kl_\gamma}{2c}\right)^{s-(1/2)}$$
$$K_{s-(1/2)}(ckl_\gamma)\left[\cosh\left(\frac{m+1}{2}kl_\gamma\right)-\chi_\gamma^k\cosh\left(\frac{m}{2}kl_\gamma\right)\right].\quad(2.64)$$

This gives explicitly for $s=1$ (m even):

$$\sum_{n=0}^{\infty}\left[\frac{1}{c^2-\left(\lambda_{n,m}^B\right)^2}-\frac{1}{c^2-\left(\lambda_{n,m}^F\right)^2}\right]=(1-g)\sum_{k=0}^{m}\frac{1}{c^2-k^2}+\frac{1}{2c}\frac{d}{dc}\ln\left[\frac{Z_0\left(\frac{m}{2}+c+1\right)Z_0\left(c-\frac{m}{2}\right)}{Z_1\left(c+\frac{m+1}{2}\right)Z_1\left(c+\frac{1-m}{2}\right)}\right],\quad(2.65)$$

where the logarithmic derivative of the super zeta-functions has been used. For $s=1$ and m odd:

$$\sum_{n=0}^{\infty}\left[\frac{1}{\left[c^2-\left(\lambda_n^B\right)^2\right]}-\frac{1}{\left[c^2-\left(\lambda_n^F\right)^2\right]}\right]=(1-g)\sum_{k=0}^{m}\frac{1}{c^2-k^2}+\frac{1}{2c}\frac{d}{dc}\ln\left[\frac{Z_1\left(\frac{m}{2}+c+1\right)Z_1\left(c-\frac{m}{2}\right)}{Z_0\left(\frac{m+1}{2}+c\right)Z_0\left(c+\frac{1-m}{2}\right)}\right].\quad(2.66)$$

2) Let first m be an even number. Let us consider the representation $(\mathrm{Re}\,s(s)<1)$:

$$t^{s-1}=\frac{2}{\Gamma(1-s)}\int_0^{\infty}\frac{\lambda+c}{[\lambda(\lambda+2c)]^s}e^{-\lambda(\lambda+2c)t}d\lambda.\quad(2.67)$$

Therefore we get for $\zeta_m\Gamma(c;s)$ with the help of eq. (2.62) and the representation $K_{\pm1/2}(z)=\sqrt{\pi/2z}e^{-z}$:

$$\zeta_m^{\Gamma}(s;c)=\frac{\sin\pi s}{\pi}\int_0^{\infty}\frac{d\lambda}{[\lambda(\lambda+2c)]^s}\cdot2\sum_{\gamma\in\Gamma}\sum_{k=1}^{\infty}\frac{l_\gamma}{1-e^{-kl_\gamma}}e^{-kl_\gamma(\lambda+c+1/2)}\left[\cosh\left(\frac{m+1}{2}kl_\gamma\right)-\chi_\gamma^k\cosh\left(\frac{m}{2}kl_\gamma\right)\right]=$$
$$=\frac{\sin\pi s}{\pi}\int_0^{\infty}\frac{d\lambda}{[\lambda(\lambda+2c)]^s}\frac{d}{d\lambda}\ln\left[\frac{Z_0\left(\frac{m}{2}+\lambda+c+1\right)Z_0\left(\lambda+c-\frac{m}{2}\right)}{Z_1\left(\frac{m+1}{2}+\lambda+c\right)Z_1\left(\frac{m-1}{2}+\lambda+c\right)}\right].\quad(2.68)$$

Let be $f(s)=\sin(\pi s)[\lambda(\lambda+2c)]^{-s}$. Then $f'(s)\big|_{s=0}=\pi$ and we get for $\zeta'(0;c)(\mathrm{Re}(s)>m)$:

$$\zeta'(0;c)=(g-1)\sum_{k=0}^{m}\ln\left(c^2-k^2\right)+\int_0^{\infty}d\lambda\frac{d}{d\lambda}\ln\left[\frac{Z_0\left(\lambda+\frac{m}{2}+c+1\right)Z_0\left(\lambda-\frac{m}{2}+c\right)}{Z_1\left(\lambda+\frac{m+1}{2}+c\right)Z_1\left(\lambda+\frac{1-m}{2}+c\right)}\right]=$$

$$= (g-1)\sum_{k=0}^{m}\ln(c^2-k^2) - \ln\left[\frac{Z_0\left(c+1+\frac{m}{2}\right)Z_0\left(c-\frac{m}{2}\right)}{Z_1\left(c+\frac{m+1}{2}\right)Z_1\left(c+\frac{1-m}{2}\right)}\right]. \quad (2.69)$$

Here it was used that $\lim_{s\to\infty} Z_q(s) = 1$, which follows at once from the Euler product representation of the Selberg super zeta-functions. Therefore $(m = 0,2,...)$:

$$s\det(c^2 - \square_m^2) = \frac{Z_0\left(c+\frac{m}{2}+1\right)Z_0\left(c-\frac{m}{2}\right)}{Z_1\left(c+\frac{m+1}{2}\right)Z_1\left(c+\frac{1-m}{2}\right)c^{2(g-1)}}\prod_{k=1}^{m}(c^2-k^2)^{1-g}. \quad (2.70)$$

Similarly $(m = 2,4,...)$:

$$s\det(c^2 - \square_{-m}^2) = \frac{Z_0\left(c-\frac{m}{2}+1\right)Z_0\left(c+\frac{m}{2}\right)}{Z_1\left(c+\frac{m+1}{2}\right)Z_1\left(c+\frac{1-m}{2}\right)}\prod_{k=0}^{m-2}(c^2-k^2)^{g-1}. \quad (2.71)$$

For m an odd number the roles of Z_0 and Z_1 are just reversed and it follows immediately $(m = 1,3,...)$:

$$s\det(c^2 - \square_m^2) = \frac{Z_1\left(c+1+\frac{m}{2}\right)Z_1\left(c-\frac{m}{2}\right)}{Z_0\left(c+\frac{m+1}{2}\right)Z_0\left(c+\frac{1-m}{2}\right)c^{2g-2}}\prod_{k=0}^{m-2}(c^2-k^2)^{1-g}. \quad (2.72)$$

Similarly $(m = 1,3,...)$:

$$s\det(c^2 - \square_{-m}^2) = \frac{Z_1\left(c+1-\frac{m}{2}\right)Z_1\left(c+\frac{m}{2}\right)}{Z_0\left(c+\frac{m+1}{2}\right)Z_0\left(c+\frac{1-m}{2}\right)}\prod_{k=1}^{m-2}(c^2-k^2)^{g-1}. \quad (2.73)$$

Equations (2.70) – (2.73) are the starting points for the calculation of determinants. Because the super zeta-functions are meromorphic functions in Λ_∞, the same holds for the superdeterminants. Let us denote by $\hat{\Theta}(t) := str\left[\exp(t\square_0^2)\right]$. Then we have:

$$s\det(c^2 - \square_0^2) = \exp\left\{-\frac{\partial}{\partial s}\left[\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}e^{-tc^2}\hat{\Theta}(t)\Big|_{s=0}\right]\right\}. \quad (2.74)$$

We can make some statement about $\hat{\Theta}$ and can derive an equation expressing $\hat{\Theta}$ by the zeta-function Z_s. Let us consider the supertrace formula for the resolvent kernel:

$$str\left(c^2-\Box_0^2\right)^{-1}=\int_0^\infty e^{-tc^2}\hat{\Theta}(t)dt=\frac{1}{2c}\frac{Z_s'(c)}{Z_s(c)}-\frac{g-1+\Delta n_0^{(0)}}{c^2}. \quad (2.75)$$

This equation can be inverted by the theory of Laplace transformations yielding:

$$\hat{\Theta}(t)=\frac{1}{2\pi}\int_{b-i\infty}^{b+i\infty}\left[\frac{1}{2c}\frac{Z_s'(c)}{Z_s(c)}-\frac{g-1+\Delta n_0^{(0)}}{c^2}\right]dc^2=-\frac{1}{\sqrt{4\pi t}}\int_0^\infty ue^{-u^2/4t}\left(\mathcal{L}^{-1}\ln Z_s\right)(u)du-\left(g-1+\Delta n_0^{(0)}\right), \quad (2.76)$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transformation. In particular this gives

$$\hat{\Theta}(0)=-\left(g-1+\Delta n_0^{(0)}\right); \quad (2.77)$$

this result is consistent with eq. (2.65). Equation (2.65) gives also that for $t\to\infty$ the supertrace for the heat-kernel for $\Box_m^2$ diverges according to $(m\in N)$

$$\hat{\Theta}_m(t)=str\left[\exp\left(t\Box_m^2\right)\right]\cong\left(1-g-\Delta n_0^{(0)}\right)e^{m^2t}, \quad (t\to\infty). \quad (2.78)$$

The starting points for the calculation of determinants of the operator $\Box_m^2$ are eqs. (2.70) – (2.73) which all can be analytically continued to $c=0$. Let us first consider eq. (2.70) for $m=0$. Performing the limit $c\to\varepsilon$ for $|\varepsilon|<<1$ one gets

$$s\det\left(-\Box_0^2\right)=\frac{1}{(2g-2)!}\frac{Z_0(1)Z_0^{(2g-2)}(0)}{\left[\tilde{Z}_1\left(\frac{1}{2}\right)\right]^2}\varepsilon^{2\Delta n_0^{(0)}}. \quad (2.79)$$

Here we have denoted by $\tilde{Z}_1\left(\frac{1}{2}\right)$ the appropriate derivative of residuum of Z_1 at $s=\frac{1}{2}$, depending whether $\Delta n_0^{(0)}\le 0$ or $\Delta n_0^{(0)}>0$ respectively. To make this quantity well-defined we subtract from $s\det\left(-\Box_0^2\right)$ the zero-mode which is denoted by priming the sdet. Using further the functional relation for Z_0 and Z_1 we get finally:

$$s\det'\left(-\Box_0^2\right)=(-1)^{\Delta n_0^{(0)}}\left[\pi^{g-1}\frac{Z_0(1)}{\tilde{Z}_1\left(\frac{1}{2}\right)}\right]^2\frac{Z_1(0)}{Z_1(1)}. \quad (2.80)$$

For calculating the superdeterminant for m even and $m\ge 2$ a subtraction of zero- or trivial-modes is not necessary. Proceeding similarly as for $m=0$ we get for $m=2,4,\ldots$:

$$s\det\left(-\Box_m^2\right)=\left[\left(\frac{\pi}{m!}\right)^{g-1}\frac{Z_0\left(1+\frac{m}{2}\right)}{Z_1\left(\frac{m+1}{2}\right)}\right]^2\frac{Z_1(0)}{Z_1(1)}. \quad (2.81)$$

Similarly $(m=2,4,\ldots)$:

$$s\det\left(-\Box_{-m}^2\right)=\left[\left(\frac{(m-2)!}{\pi}\right)^{g-1}\frac{Z_0\left(\frac{m}{2}\right)}{Z_1\left(\frac{m+1}{2}\right)}\right]^2\frac{Z_1(1)}{Z_1(0)}. \quad (2.82)$$

For $m=1,3,\ldots$:

$$s\det\left(-\Box_m^2\right)=\left[\left(\frac{\pi}{im!}\right)^{g-1}\frac{Z_1\left(1+\frac{m}{2}\right)}{Z_0\left(\frac{m+1}{2}\right)}\right]^2\frac{Z_1(0)}{Z_1(1)}; \quad (2.83)$$

and $m=3,5,\ldots$:

$$s\det\left(-\Box_{-m}^2\right)=\left[\left(\frac{(m-2)!i}{\pi}\right)^{g-1}\frac{Z_1\left(\frac{m}{2}\right)}{Z_0\left(\frac{m+1}{2}\right)}\right]^2\frac{Z_1(1)}{Z_1(0)}. \quad (2.84)$$

The case of $\Box_{-1}^2$ must be treated separately because of the appearance of zero-modes which must be subtracted. Therefore denoting the omission of zero-modes by priming the super determinant we get

$$s\det'\left(-\Box_{-1}^2\right)=(-1)^{\Delta n_0^{(0)}}\left[\pi^{1-g}\frac{\tilde{Z}_1\left(\frac{1}{2}\right)}{Z_0(1)}\right]^2\frac{Z_1(1)}{Z_1(0)}. \quad (2.85)$$

The relevant string integrand is given by $s\det'\left(-\Box_0^2\right)$ and $s\det\left(-\Box_2^2\right)$. Equations (2.80) and (2.81) yield:

$$\left[s\det'\left(-\Box_0^2\right)\right]^{-5/2}\left[s\det\left(-\Box_2^2\right)\right]^{1/2}=(-1)^{5/2\Delta n_0^{(0)}}\left(\pi^{g-1}\frac{Z_0(1)}{\tilde{Z}_1\left(\frac{1}{2}\right)}\right)^{-5}\left(\frac{\pi}{2}\right)^{g-1}\frac{Z_0(2)}{Z_1\left(\frac{3}{2}\right)}\left(\frac{Z_1(1)}{Z_1(0)}\right)^2=$$

$$=(-1)^{5/2\Delta n_0^{(0)}}\left(\pi^{g-1}\frac{Z_0(1)}{\tilde{Z}_1\left(\frac{1}{2}\right)}\right)^{-5}\left(\frac{\pi}{2}\right)^{g-1}\frac{Z_0(2)}{Z_1\left(\frac{3}{2}\right)}\frac{Z_0\left(\frac{3}{2}\right)}{Z_0\left(-\frac{1}{2}\right)}, \quad (2.86)$$

and we conclude that this expression is well defined. Furthermore for Z_g of the following string equation

$$Z_g = \int_{S\mathcal{M}_g} d\mu_{SWP} \left[s\det'\left(-\square_0^2\right)\right]^{-5/2} \left[s\det\left(-\square_2^2\right)\right]^{1/2}, \quad (2.86b)$$

where $S\mathcal{M}_g$ is the super moduli space, $d\mu_{SWP}$ the super Weil-Peterson measure and the factor $\left[s\det\left(-\square_2^2\right)\right]^{1/2}$ is the contribution from the Faddev-Poppov ghost determinant, we have that:

$$Z_g = \int_{sM_g} d\mu_{SWP} \left[s\det'\left(-\square_0^2\right)\right]^{-5/2} \left[s\det\left(-\square_2^2\right)\right]^{1/2} = \left(\frac{1}{2\pi^4}\right)^{g-1} \int_{sM_g} d\mu_{SWP} (-1)^{5/2\Delta n_0^{(0)}} \left(\frac{Z_0(1)}{\tilde{Z}_1\left(\frac{1}{2}\right)}\right)^{-5} \frac{Z_0(2)}{Z_1\left(\frac{3}{2}\right)} \left(\frac{Z_1(1)}{Z_1(0)}\right)^2$$
(2.87).

This is the **fermionic string integrand**. We note the appearance of the various ratios of the Selberg super zeta-functions.

In order to continue the discussion on the Selberg super-zeta functions, let us first introduce the classical Selberg zeta function $Z(s)$ defined by

$$Z(s) = \prod_{\gamma\in\Gamma}\prod_{k=0}^{\infty}\left[1 - e^{-(s+k)l_\gamma}\right], \quad \mathcal{R}(s) > 1. \quad (2.88)$$

As long as only the elliptic terms are maintained, the principal analytic structure is not very much altered. However, the parabolic terms give rise to additional poles and do in fact matter a lot. A functional equation can be derived which has the form

$$Z(1-s) = \Delta(s)\exp\left\{-A\dim V \int_0^{s-\frac{1}{2}} t\tan\pi t dt\right\} \left[\frac{\Gamma\left(\frac{3}{2}-s\right)}{\Gamma\left(s+\frac{1}{2}\right)}\right]^{\kappa_0} \Psi_Z(s) Z(s), \quad (2.89)$$

where $\Psi_Z(s)$ is defined as

$$\Psi_Z(s) = \exp\left\{ \sum_{\{R\}}\sum_{k=1}^{\nu-1} \frac{tr_V\left[U^k(R)\right]}{\nu\sin(2k\pi/\nu)} \sum_{l=1}^{\infty}\left(2l\frac{k\pi}{\nu}\right)\ln\left|\frac{(s+l)(s+l-1)}{(s-l)(s-l-1)}\right| + (1-2s) \right.$$
$$\left.\left[\kappa_0 \ln 2 + \sum_{\alpha=1}^{\nu}\sum_{l=1+k_\alpha}^{\dim V} \ln\left|1 - e^{2\pi i\theta_{l\alpha}}\right|\right] - i\arg\Delta\left(\frac{1}{2}\right)\right\}. \quad (2.90)$$

Hence, we can write

$$Z(1-s)=\Delta(s)\exp\left\{-A\dim V\int_0^{s-\frac{1}{2}} t\tan\pi t dt\right\}\left[\frac{\Gamma\left(\frac{3}{2}-s\right)}{\Gamma\left(s+\frac{1}{2}\right)}\right]^{\kappa_0} Z(s)\times$$

$$\times\exp\left\{\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{tr_V\left[U^k(R)\right]}{\nu\sin(2k\pi/\nu)}\sum_{l=1}^{\infty}\left(2l\frac{k\pi}{\nu}\right)\ln\left|\frac{(s+l)(s+l-1)}{(s-l)(s-l-1)}\right|+(1-2s)\right.$$

$$\left.\left[\kappa_0\ln 2+\sum_{\alpha=1}^{\nu}\sum_{l=1+k_\alpha}^{\dim V}\ln\left|1-e^{2\pi i\theta_{l\alpha}}\right|\right]-i\arg\Delta\left(\frac{1}{2}\right)\right\}. \quad (2.90b)$$

Note the relation

$$R(s)R(-s):=\frac{Z(s)Z(-s)}{Z(1+s)Z(1-s)}=(2\sin\pi s)^{A\dim V/\pi}\left(\frac{1}{4}-s^2\right)^{-\kappa_0}\frac{\Delta(1+s)}{\Delta(s)}\frac{1}{\Psi(s)\Psi(-s)}. \quad (2.91)$$

Let us consider the two Selberg super-zeta functions Z_0 and Z_1, respectively, defined by

$$Z_0(s)=\prod_{\{\gamma\}}\prod_{k=0}^{\infty}s\det\left[1_V-U(\gamma)e^{-(s+k)l_\gamma}\right],\quad \mathscr{R}(s)>1, \quad (2.92)$$

$$Z_1(s)=\prod_{\{\gamma\}}\prod_{k=0}^{\infty}s\det\left[1_V-U(\gamma)\chi_\gamma e^{-(s+k)l_\gamma}\right],\quad \mathscr{R}(s)>1. \quad (2.93)$$

For convenience we will consider the functions

$$R_0(s)=\frac{Z_0(s)}{Z_0(s+1)},\quad R_1(s)=\frac{Z_1(s)}{Z_1(s+1)},\quad \mathscr{R}(s)>1, \quad (2.94)$$

and the analytic properties of the $Z_{0,1}$-functions can be easily derived from the $R_{0,1}$-functions. As we shall see, only functional relations for the $R_{0,1}$-functions can be derived, but not for the $Z_{0,1}$-functions.
We first discuss the function $Z_1(s)$. In order to do this we choose the test function $(\mathscr{R}(s,a)>1)$

$$h_1\left(\frac{1}{2}+ip,s,a\right)=2ip\left(\frac{1}{s^2+p^2}-\frac{1}{a^2+p^2}\right), \quad (2.95)$$

with the Fourier transformed function $g_1(u)$ given by

$$g_1(u,s,a)=sign(u)\left(e^{-s|u|}-e^{-a|u|}\right). \quad (2.96)$$

The hyperbolic- and zero-length term has been evaluated with result

$$i\dim V\frac{A}{4\pi}\int_{-\infty}^{\infty}h_1\left(ip+\frac{1}{2},s,a\right)\tanh\pi p dp=\dim V\frac{A}{\pi}\left[\Psi\left(s+\frac{1}{2}\right)-\Psi\left(a+\frac{1}{2}\right)\right], \quad (2.97)$$

$$\sum_{\{\gamma\}}\sum_{k=1}^{\infty}\frac{l_\gamma}{2\sinh\frac{kl_\gamma}{2}}\left[g_1(kl_\gamma,s,a)+g_1(-kl_\gamma,s,a)-\chi_\gamma^k\left(g_1(kl_\gamma,s,a)e^{-kl_\gamma/2}+g_1(-kl_\gamma,s,a)e^{kl_\gamma}\right)\right]=$$

$$=\frac{R_1'(s)}{R_1(s)}-\frac{R_1'(a)}{R_1(a)}. \quad (2.98)$$

Next we consider the elliptic terms. By the use of

$$\int_0^{\infty}\frac{e^{-\mu x}}{\cosh x-\cos t}dx=\frac{2}{\sin t}\sum_{l=1}^{\infty}\frac{\sin lt}{\mu+l} \quad (2.99)$$

we obtain

$$-\int_0^{\infty}\frac{g_1(u,s,a)e^{-u/2}+g_1(-u,s,a)e^{u/2}}{\cosh^2\frac{u}{2}-\cos^2\phi}du=\frac{4}{\cos\phi}\sum_{l=0}^{\infty}\cos[(2l+1)\phi]\left(\frac{1}{s+l+\frac{1}{2}}-\frac{1}{a+l+\frac{1}{2}}\right). \quad (2.100)$$

Let us turn to the parabolic terms. Quite easily we have $g_1(0)=0$, and $h_1\left(\frac{1}{2},s,a\right)=0$. Furthermore

$$\int_{-\infty}^{\infty}h_1\left(ip+\frac{1}{2},s,a\right)[\Psi(1+ip)+\Psi(1-ip)]=0,\quad \int_0^{\infty}g_1(\pm u,s,a)du=\pm\left(\frac{1}{s}-\frac{1}{a}\right). \quad (2.101)$$

Let us consider the contour integral

$$\frac{1}{2\pi i}\oint_C\frac{\Delta'(z)}{\Delta(z)}\left[\frac{2\left(z-\frac{1}{2}\right)}{s^2-\left(z-\frac{1}{2}\right)^2}-\frac{2\left(z-\frac{1}{2}\right)}{a^2-\left(z-\frac{1}{2}\right)^2}\right]dz=$$

$$=-\frac{1}{2\pi i}\oint_C\frac{\Delta'(z)}{\Delta(z)}\left[\frac{1}{z-\left(s+\frac{1}{2}\right)}+\frac{1}{z+\left(s-\frac{1}{2}\right)}-\frac{1}{z-\left(a+\frac{1}{2}\right)}+\frac{1}{z+\left(a-\frac{1}{2}\right)}\right], \quad (2.102)$$

where the contour is running from $z=\frac{1}{2}-iR$ to $z=\frac{1}{2}+iR$ on the line $\mathcal{R}(z)=\frac{1}{2}$, and closed by the semi-circle going through the points $z=\frac{1}{2}-iR$, $z=\frac{1}{2}-R$, $z=\frac{1}{2}+iR$ and the points are given in the direction they are tranversed by the contour. Considering $R\to\infty$, the integral over the semi-circle vanishes due to the properties of the logarithmic derivative of $\Delta(z)$ and the choice of the test function. The integral over the line yields by the Cuuchy residue theorem

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h_1\left(ip+\frac{1}{2},s,a\right)\frac{\Delta'\left(\frac{1}{2}+ip\right)}{\Delta\left(\frac{1}{2}+ip\right)}dp = -\frac{\Delta'\left(s+\frac{1}{2}\right)}{\Delta\left(s+\frac{1}{2}\right)}+\frac{\Delta'\left(a+\frac{1}{2}\right)}{\Delta\left(a+\frac{1}{2}\right)}+$$

$$-\sum_{j=1}^{\mathcal{M}}\left[\frac{1}{s-\left(\sigma_j-\frac{1}{2}\right)}-\frac{1}{s+\left(\sigma_j-\frac{1}{2}\right)}-\frac{1}{a-\left(\sigma_j-\frac{1}{2}\right)}+\frac{1}{a+\left(\sigma_j-\frac{1}{2}\right)}\right]+$$

$$-\sum_{\rho,\beta<\frac{1}{2}}\left[\frac{1}{s-\left(\rho-\frac{1}{2}\right)}-\frac{1}{s+\left(\rho-\frac{1}{2}\right)}-\frac{1}{a-\left(\rho-\frac{1}{2}\right)}+\frac{1}{a+\left(\rho-\frac{1}{2}\right)}\right]. \quad (2.103)$$

Therefore we obtain the Selberg super-trace formula for the test function $h_1\left(ip+\frac{1}{2},s,a\right)$ as follows:

$$\frac{R_1'(s)}{R_1(s)}-\frac{R_1'(a)}{R_1(a)} = -\Delta n_0^{(0)}\left(\frac{1}{s^2-\frac{1}{4}}-\frac{1}{a^2-\frac{1}{4}}\right)+$$

$$+2\sum_{n=1}^{\infty}\left[\frac{\lambda_n^B-\frac{1}{2}}{s^2-\left(\lambda_n^B-\frac{1}{2}\right)^2}-\frac{\lambda_n^B-\frac{1}{2}}{a^2-\left(\lambda_n^B-\frac{1}{2}\right)^2}-\frac{\lambda_n^F-\frac{1}{2}}{s^2-\left(\lambda_n^F-\frac{1}{2}\right)^2}+\frac{\lambda_n^F-\frac{1}{2}}{a^2-\left(\lambda_n^F-\frac{1}{2}\right)^2}\right]+$$

$$-2\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{str\left[U^k(R)\right]}{\nu}\chi_R^k\sum_{l=0}^{\infty}\cos\left[(2l+1)\frac{k\pi}{\nu}\right]\left(\frac{1}{s+l+\frac{1}{2}}-\frac{1}{a+l+\frac{1}{2}}\right)-\frac{A\dim V}{\pi}\left[\Psi\left(s+\frac{1}{2}\right)-\Psi\left(a+\frac{1}{2}\right)\right]+\kappa_-\left(\frac{1}{s}-\frac{1}{a}\right)$$

$$-\kappa_0\left(\frac{1}{s}-\frac{1}{a}\right)+\kappa_-\frac{\Delta'\left(s+\frac{1}{2}\right)}{\Delta\left(s+\frac{1}{2}\right)}-\kappa_-\frac{\Delta'\left(a+\frac{1}{2}\right)}{\Delta\left(a+\frac{1}{2}\right)}+\kappa_-\sum_{j=1}^{\mathcal{M}}\left[\frac{1}{s-\left(\sigma_j-\frac{1}{2}\right)}-\frac{1}{s+\left(\sigma_j-\frac{1}{2}\right)}-\frac{1}{a-\left(\sigma_j-\frac{1}{2}\right)}+\frac{1}{a+\left(\sigma_j-\frac{1}{2}\right)}\right]$$

$$+\kappa_-\sum_{\rho,\beta<\frac{1}{2}}\left[\frac{1}{s-\left(\rho-\frac{1}{2}\right)}-\frac{1}{s+\left(\rho-\frac{1}{2}\right)}-\frac{1}{a-\left(\rho-\frac{1}{2}\right)}+\frac{1}{a+\left(\rho-\frac{1}{2}\right)}\right]. \quad (2.104)$$

Here $\Delta n_0^{(0)} = n_0^B - n_0^F$ denotes the difference between the number of even- and odd zero-modes of the Dirac-Laplace operator $\Box$.

The Selberg super-zeta function $R_1(s)$ *is a meromorphic function on* Λ_∞. Of course, eq. (2.104) can be extended meromorphically to all $s\in\Lambda_\infty$. The test functions $h_1\left(ip+\frac{1}{2},s,a\right)$ is symmetric by the interchange $s \to -s$. Therefore subtracting the trace formula for $h_1\left(ip+\frac{1}{2},s,a\right)$ and $h_1\left(ip+\frac{1}{2},-s,a\right)$ yields the functional equation for R_1 in differential form

$$\frac{d}{ds}\ln\left[R_1(s)-R_1(-s)\right]=-A\dim V\tan\pi s+\frac{2(\kappa_- -\kappa_0)}{s}-2\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{str\left[U^k(R)\right]}{\nu}\chi_R^k\sum_{l=0}^{\infty}\cos\left[(2l+1)\frac{k\pi}{\nu}\right]\left(\frac{1}{s+l+\frac{1}{2}}+\frac{1}{s-\left(l+\frac{1}{2}\right)}\right). \quad (2.105)$$

The integrated functional equation therefore has the form

$$R_1(s)R_1(-s)=\text{const.}\,(\cos\pi s)^{A\dim V/\pi}s^{2(\kappa_- -\kappa_0)}\Psi_1(s), \quad (2.106)$$

with the function $\Psi_1(s)$ given by

$$\Psi_1(s)=\exp\left\{-2\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{str\left[U^k(R)\right]}{\nu}\chi_R^k\times\sum_{l=0}^{\infty}\cos\left[(2l+1)\frac{k\pi}{\nu}\right]\ln\left|s^2-\left(l+\frac{1}{2}\right)^2\right|\right\}. \quad (2.107)$$

Hence we can write:

$$R_1(s)R_1(-s)=\text{const.}\,(\cos\pi s)^{A\dim V/\pi}s^{2(\kappa_- -\kappa_0)}$$
$$\exp\left\{-2\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{str\left[U^k(R)\right]}{\nu}\chi_R^k\times\sum_{l=0}^{\infty}\cos\left[(2l+1)\frac{k\pi}{\nu}\right]\ln\left|s^2-\left(l+\frac{1}{2}\right)^2\right|\right\}. \quad (2.107b)$$

We check easily the consistence of the functional equation with respect to the analytical properties of the Selberg super-zeta function R_1.

Let us turn to the discussion of the Selberg super-zeta function R_0. We consider the test function $(\mathcal{R}(s,a)>1)$

$$h_0\left(ip+\frac{1}{2},s,a\right)=2\left(\frac{1}{2}+ip\right)\left(\frac{1}{s^2-\left(\frac{1}{2}+ip\right)^2}-\frac{1}{a^2-\left(\frac{1}{2}+ip\right)^2}\right), \quad (2.108)$$

with the Fourier transform $g_0(u,s,a)$ given by

$$g_0(u,s,a)=sign(u)e^{u/2}\left(e^{-s|u|}-e^{-a|u|}\right). \quad (2.109)$$

Again a regularization term is needed to match the requirements of a valid test function for the trace formula. Similarly as for R_1 we obtain the Selberg super trace formula for the test function $h_0\left(ip+\frac{1}{2},s,a\right)$ as follows:

$$\frac{R_0'(s)}{R_0(s)}-\frac{R_0'(a)}{R_0(a)}=2\sum_{n=1}^{\infty}\left[\frac{\lambda_n^B}{s^2-(\lambda_n^B)^2}-\frac{\lambda_n^B}{a^2-(\lambda_n^B)^2}-\frac{\lambda_n^F}{s^2-(\lambda_n^F)^2}+\frac{\lambda_n^F}{a^2-(\lambda_n^F)^2}\right]-\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{str\left[U^k(R)\right]}{\nu\sin(2k\pi/\nu)}\times$$
$$\times\sum_{l=1}^{\infty}\sin\left(\frac{2lk\pi}{\nu}\right)\left[\frac{1}{s+l-1}-\frac{1}{s+l+1}-\frac{1}{a+l-1}+\frac{1}{a+l+1}\right]-\frac{A\dim V}{2\pi}\left[\Psi(s)+\Psi(s+1)-\Psi(a)-\Psi(a+1)\right]+$$

$$-\frac{\kappa_0}{2}\left(\frac{1}{s-\frac{1}{2}}+\frac{1}{s+\frac{1}{2}}-\frac{1}{a-\frac{1}{2}}-\frac{1}{a+\frac{1}{2}}\right)+\frac{1}{2}tr\left[\mathcal{J}\left(\frac{1}{2}\right)\right]\left(\frac{1}{s^2-\frac{1}{4}}-\frac{1}{a^2-\frac{1}{4}}\right)+\kappa_-\frac{\Delta'(s+1)}{\Delta(s+1)}-\kappa_-\frac{\Delta'(a+1)}{\Delta(a+1)}+$$
$$+\kappa_-\sum_{\rho,\beta<\frac{1}{2}}\left[\frac{1}{s-\rho}-\frac{1}{s+\rho}-\frac{1}{a-\rho}+\frac{1}{a+\rho}\right]-\kappa_-\sum_{j=1}^{\mathcal{M}}\left[\frac{1}{s+(\sigma_j-1)}-\frac{1}{s-(\sigma_j-1)}-\frac{1}{a+(\sigma_j-1)}+\frac{1}{a-(\sigma_j-1)}\right]$$
(2.110)

The Selberg super-zeta function $R_0(s)$ *is a meromorphic function on* Λ_∞. Of course, eq. (2.110) can be extended meromorphically to all $s\in\Lambda_\infty$. The test function $h_0\left(ip+\frac{1}{2},s,a\right)$ is symmetric with respect to $s\to -s$. Therefore subtracting the trace formulae of $h_0\left(ip+\frac{1}{2},s,a\right)$ and $h_0\left(ip+\frac{1}{2},-s,a\right)$ from each other yields the functional equation for the R_0-function in differential form

$$\frac{d}{ds}\ln\left[R_0(s)R_0(-s)\right]=\frac{A\dim V}{\pi}\frac{d}{ds}\ln(\sin\pi s)+\left[\frac{\Delta'(s)}{\Delta(s)}-\frac{\Delta'(1+s)}{\Delta(1+s)}\right]^{\kappa_-}+\kappa_0\left(\frac{1}{s-\frac{1}{2}}+\frac{1}{s+\frac{1}{2}}\right)+$$
$$-\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{str\left[U^k(R)\right]}{\nu\sin(2k\pi/\nu)}\times\sum_{l=1}^{\infty}\sin\left(\frac{2lk\pi}{\nu}\right)\left[\frac{1}{s+l-1}+\frac{1}{s-(l-1)}-\frac{1}{s+l+1}-\frac{1}{s-(l+1)}\right].\quad(2.111)$$

In integrated form, this gives the functional equation

$$R_0(s)R_0(-s)=\text{const.}(\sin\pi s)^{A\dim V/\pi}\left(\frac{\Delta(s+1)}{\Delta(s)}\right)^{\kappa_-}\ln\left(s^2-\frac{1}{4}\right)^{-\kappa_0}\Psi_0(s),\quad(2.112)$$

with the function $\Psi_0(s)$ given by

$$\Psi_0(s)=\exp\left\{-\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{str\left[U^k(R)\right]}{\nu\sin(2k\pi/\nu)}\sum_{l=1}^{\infty}\sin\left(\frac{2lk\pi}{\nu}\right)\ln\left|\frac{\left(s^2-(l-1)^2\right)}{\left(s^2-(l+1)^2\right)}\right|\right\}.\quad(2.113)$$

Hence, we can write

$$R_0(s)R_0(-s) = \text{const.}\,(\sin \pi s)^{A\dim V/\pi}\left(\frac{\Delta(s+1)}{\Delta(s)}\right)^{\kappa_-}\ln\left(s^2-\frac{1}{4}\right)^{-\kappa_0}\times$$
$$\times\exp\left\{-\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{str[U^k(R)]}{\nu\sin(2k\pi/\nu)}\sum_{l=1}^{\infty}\sin\left(\frac{2lk\pi}{\nu}\right)\ln\left|\frac{\left(s^2-(l-1)^2\right)}{\left(s^2-(l+1)^2\right)}\right|\right\}. \quad (2.113b)$$

We check easily the consistence of the functional equation with respect to the analytical properties of the Selberg super-zeta function R_0. Note the similarity of the corresponding relation (2.91) for the classical Selberg zeta function.

We can also introduce the Selberg super-zeta function $Z_S(s)$ defined by

$$Z_S(s) = \frac{Z_0(s)Z_0(s+1)}{Z_1^2\left(s+\frac{1}{2}\right)}. \quad (2.114)$$

The appropriate test function is $(\mathscr{R}(s)>1)$

$$h_S(p,s) = \frac{1}{s^2-\lambda^2}\bigg|_{\lambda=\frac{1}{2}+ip} = \frac{1}{\left(s^2-\frac{1}{4}\right)-ip+p^2}. \quad (2.115)$$

The corresponding Fourier transform g_S is given by

$$g_S(u,s) = \frac{1}{2s}e^{u/2-s|u|}. \quad (2.116)$$

The evaluation of the various terms in the Selberg super-trace formula is straightforward and we obtain similarly to the previous two cases

$$\frac{1}{2s}\frac{Z_S'(s)}{Z_S(s)} = \sum_{n=1}^{\infty}\left[\frac{1}{s^2-\left(\lambda_n^B\right)^2}-\frac{1}{s^2-\left(\lambda_n^F\right)^2}\right]+\left(\Delta_0^{(0)}+\frac{A\dim V}{4\pi}\right)\frac{1}{s^2}-\frac{1}{2s}\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{str[U^k(R)]}{\nu\sin(2k\pi/\nu)}\sum_{l=1}^{\infty}\sin\left(\frac{2lk\pi}{\nu}\right)\times$$
$$\times\left[\frac{4\left(1-\chi_R^k\cos\left(\frac{k\pi}{\nu}\right)\right)}{s+l}+\frac{1}{s+l-1}+\frac{1}{s+l+1}-\frac{2}{s+l}\right]+\frac{1}{s}\left[\tilde{\kappa}_0+\kappa_-\ln\left|s\det(1-U(S))\right|\right]+\frac{\kappa_-}{2}\frac{tr\left[\mathscr{J}\left(\frac{1}{2}\right)\right]}{s^2-\frac{1}{4}}+$$
$$-\frac{\kappa_-}{2s}\frac{1}{s+\frac{1}{2}}+\frac{\kappa_0}{4s}\left(\frac{1}{s+\frac{1}{2}}-\frac{1}{s-\frac{1}{2}}\right)+\frac{\tilde{\kappa}_0}{2s}\left[\Psi\left(s+\frac{1}{2}\right)+\Psi\left(s+\frac{3}{2}\right)\right]-\frac{\kappa_-}{2s}\frac{\Delta'(1+s)}{\Delta(1+s)}+$$
$$-\kappa_-\sum_{j=1}^{\mathscr{M}}\frac{1}{s^2-(\sigma_j-1)^2}+\kappa_-\sum_{\rho,\beta<\frac{1}{2}}\frac{1}{s^2-\rho^2}. \quad (2.117)$$

The Selberg super-zeta function Z_S *is a meromorphic function on* Λ_∞. Of course, eq. (2.117) can be extended meromorphically to all $s \in \Lambda_\infty$.The test function $h_S\left(ip+\frac{1}{2}, s\right)$ is symmetric with respect to $s \to -s$ and therefore we can deduce the functional relation

$$\frac{Z_S(s)}{Z_S(-s)} = \text{const.}\, e^{4[\kappa_0+\kappa_- \ln|s \det(1U(S))|]} \times \left(\frac{1}{\Delta(s)\Delta(s+1)}\right)^{\kappa_-} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\kappa_- - \kappa_0} \left(\frac{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(\frac{3}{2}-s\right)}\right)^{\tilde{\kappa}_0} \Psi_S(s), \tag{2.118}$$

with the function $\Psi_S(s)$ given by

$$\Psi_S(s) = \exp\left\{-2\sum_{\{R\}}\sum_{k=1}^{\nu-1} \frac{str\left[U^k(R)\right]}{\nu \sin(2k\pi/\nu)} \sum_{l=1}^{\infty} \sin\left(\frac{2lk\pi}{\nu}\right) \times \right.$$
$$\left. \times \left[2\left(1-2\chi_R^k \cos\frac{k\pi}{\nu}\right)\ln\left|\frac{s+l}{s-l}\right| + \ln\left|\frac{(s+l-1)(s+l+1)}{(s-l+1)(s-l-1)}\right|\right]\right\}. \quad (2.119)$$

Hence, we can write

$$\frac{Z_S(s)}{Z_S(-s)} = \text{const.}\, e^{4[\kappa_0+\kappa_- \ln|s \det(1U(S))|]} \times \left(\frac{1}{\Delta(s)\Delta(s+1)}\right)^{\kappa_-} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\kappa_- - \kappa_0} \left(\frac{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(\frac{3}{2}-s\right)}\right)^{\tilde{\kappa}_0}$$
$$\exp\left\{-2\sum_{\{R\}}\sum_{k=1}^{\nu-1} \frac{str\left[U^k(R)\right]}{\nu \sin(2k\pi/\nu)} \sum_{l=1}^{\infty} \sin\left(\frac{2lk\pi}{\nu}\right) \times \right.$$
$$\left. \times \left[2\left(1-2\chi_R^k \cos\frac{k\pi}{\nu}\right)\ln\left|\frac{s+l}{s-l}\right| + \ln\left|\frac{(s+l-1)(s+l+1)}{(s-l+1)(s-l-1)}\right|\right]\right\}. \quad (2.119b)$$

We check easily the consistence of the functional equation with respect to the analytical properties of the Selberg super-zeta function Z_S. In the case, where only hyperbolic conjugacy classes are present in the super Fuchsian group, eq. (2.118) reduces to the simple functional equation

$$Z_S(s) = Z_S(-s). \quad (2.120)$$

Let us note that the relation

$$\frac{d}{ds}\ln\left[\frac{Z_0(s)Z_0(s+1)}{Z_1\left(s+\frac{1}{2}\right)}\right] - \frac{d}{ds}\ln\left[\frac{Z_0(s+1)Z_0(s+2)}{Z_1\left(s+\frac{3}{2}\right)}\right] = \frac{R_0'(s)}{R_0(s)} + \frac{R_0'(s+1)}{R_0(s+1)} - 2\frac{R_1'\left(s+\frac{1}{2}\right)}{R_1\left(s+\frac{1}{2}\right)}. \quad (2.121)$$

provides a consistency check for the zeta functions R_0, R_1 and Z_S, respectively. This concludes the discussion. With regard the mathematical connections, in the **Section 5**, we'll show as some equations of this Section are related with various equations of **Section 1** and **Section 4**.

3. On some equations concerning the ten-dimensional anomaly cancellations and the vanishing of cosmological constant.

3.1 The ten-dimensional anomaly cancellations [5]

Now we describe a simplified demonstration of the ten-dimensional anomaly cancellation. Gauge and gravitational anomalies arising from loop diagrams (with chiral particles going around the loop) in D dimensions can be succinctly characterized by gauge-invariant D + 2 forms, which are derived form D +2 dimensional index theorems. These forms are constructed out of Yang-Mills field strengths and gravitational curvatures. In the language of forms, the Yang-Mills field strength is given by

$$F = \frac{1}{2} F_{\mu\nu} dx^{\mu} \wedge dx^{\nu} = dA + A^2 , \quad (3.1)$$

where

$$A = A_{\mu}^{a} \lambda^{a} dx^{\mu} . \quad (3.2)$$

The matrix λ^a is anti-hermitian, and in this subsection it is taken to be in the adjoint representation of the Yang-Mills algebra. The Lorentz-curvature two form is given by

$$R = \frac{1}{2} R_{\mu\nu} dx^{\mu} \wedge dx^{\nu} = d\omega + \omega^2 , \quad (3.3)$$

where $\omega = \omega_{\mu} dx^{\mu}$ is a 10×10 antisymmetric matrix (the spin connection), corresponding to the fundamental representation of the Lorentz algebra SO(9,1). We now consider N = 1 supergravity coupled to N = 1 super Yang-Mills theory in ten dimensions, with a gauge group G_{10}.
All the Yang-Mills, gravitational, and mixed anomalies due to these loops are characterized by a 12-form proportional to

$$I_{12} = -\frac{1}{15} TrF^6 + \frac{1}{24} TrF^4 trR^2 - \frac{1}{960} TrF^2 \left[4trR^4 + 5\left(trR^2\right)^2 \right] + \left(\frac{1}{32} + \frac{n-496}{13824} \right) \left(trR^2\right)^3 + \\ + \left(\frac{1}{8} + \frac{n-496}{5760} \right) trR^2 trR^4 + \left(\frac{n-496}{7560} \right) trR^6 . \quad (3.4)$$

We now demonstrate the existence of a local counterterm that cancels the anomalies whenever eq. (3.4) can be factorized into an expression of the form

$$I_{12} = \left(trR^2 + kTrF^2 \right) X_8 , \quad (3.5)$$

where X_8 is gauge-invariant eight-form made out of the $F's$ and $R's$. A crucial role is played in the anomaly-cancellation mechanism by a second-rank potential

$$B = B_{\mu\nu} dx^{\mu} \wedge dx^{\nu} , \quad (3.6)$$

which is part of the N =1 D = 10 supergravity multiplet. A three-form field strength

$$H = dB + \omega_{3L} + k\omega_{3Y} \quad (3.7)$$

is formed from this potential, where the Chern-Simons forms ω_{3Y} and ω_{3L} satisfy

$$d\omega_{3Y} = trF^2, \quad (3.8a) \qquad d\omega_{3L} = trR^2. \quad (3.8b)$$

There also exist two-forms ω^1_{2Y} and ω^1_{2L} such that under an infinitesimal Yang-Mills gauge transformation

$$\delta\omega_{3Y} = -d\omega^1_{2Y} \quad (3.9a)$$

and under an infinitesimal local-Lorentz transformation

$$\delta\omega_{3L} = -d\omega^1_{2L}. \quad (3.9b)$$

Substituting in eq. (3.7) it is clear that H is a gauge invariant provided that

$$\delta B = \omega^1_{2L} + k\omega^1_{2Y}. \quad (3.10)$$

Now let us return to the expression I_{12} in eq. (3.5) and replace X_8 by X_{2N} so that the analysis applies for other dimensions as well. In analogy with ω_3 and ω^1_2, we can introduce forms X_{2N-1} and X^1_{2N-2} satisfying

$$X_{2N} = dX_{2N-1}, \quad (3.11) \qquad \delta X_{2N-1} = -dX^1_{2N-2}. \quad (3.12)$$

There are solutions of these equations for arbitrary invariant forms X_{2N} such that the anomaly associated with I_{2N+4} is proportional to

$$G = \int \left[2\left(\omega^1_{2L} + k\omega^1_{2Y}\right)X_{2N} + N\left(trR^2 + kTrF^2\right)X^1_{2N-2}\right]. \quad (3.13)$$

The problem then is to find a local interaction S_c such that

$$\delta S_c + G = 0. \quad (3.14)$$

The solution to this equation is easily seen to be given by

$$S_c = \int \left[N\left(\omega_{3L} + k\omega_{3Y}\right)X_{2N-1} - (N+2)BX_{2N}\right]. \quad (3.15)$$

This result is unique up to terms that are gauge invariant. Now let us investigate when eq. (3.4) reduces to the form (3.5). Clearly, two necessary conditions are that $n = \dim G_{10} = 496$ and that TrF^6 not be an independent sixth-order Casimir invariant of G_{10}. Both these properties are satisfied by $E_8 \times E_8$ and $SO(32) = D_{16}$. In the case of a single E_8

$$TrF^4 = \frac{1}{100}\left(TrF^2\right)^2, \quad (3.16) \qquad TrF^6 = \frac{1}{7200}\left(TrF^2\right)^3. \quad (3.17)$$

Neither of these is valid, of course, for $E_8 \times E_8$. However, in that case the weaker condition

$$TrF^6 = \frac{1}{48} TrF^2 TrF^4 - \frac{1}{14400}\left(TrF^2\right)^3 \quad (3.18)$$

is satisfied. Remarkably, eq. (3.18) is also valid for D_{16}. Substituting this relation and $n = 496$ into eq. (3.4) gives a factorized expression of the form in eq. (3.5) with

$$k = -\frac{1}{30} \quad (3.19)$$

and

$$X_8 = \frac{1}{24} TrF^4 - \frac{1}{7200}\left(TrF^2\right)^2 - \frac{1}{240} TrF^2 trR^2 + \frac{1}{8} trR^4 + \frac{1}{32}\left(trR^2\right)^2. \quad (3.20)$$

This proves the cancellation of all anomalies for both D_{16} and $E_8 \times E_8$. It is easy to show that $n = 496$ and eq. (3.18), with precisely the coefficients given, are both necessary and sufficient for the factorization of eq. (3.4).

We found that the three-form field strength is given by

$$H = dB - \frac{1}{30}\omega_{3Y} + \omega_{3L}. \quad (3.21)$$

The requirement that H be globally well-defined gives a topological condition on possible spatial compactifications. Specifically, since

$$dH = -\frac{1}{30} TrF^2 + trR^2, \quad (3.22)$$

it is necessary that background fields R_0, F_0 satisfy

$$\int_{M_4} \left(trR_0^2 - \frac{1}{30} TrF_0^2 \right) = 0 \quad (3.23)$$

for any closed four-dimensional submanifold M_4 of the ten-dimensional space-time.

In the background specified by R_0 and F_0, the effective six-dimensional theory has a reduced gauge symmetry. Specifically, if the nonzero fields F_0 span a subgroup $H \subset G_{10}$, then there is a unique maximal subalgebra G of G_{10}, all of whose generators commute with those of H, such that

$$G_{10} \supset G \times H. \quad (3.24)$$

The adjoint representation of G_{10} can be decomposed into a sum of representations

$$\text{adjoint of } G_{10} = \sum_i (L_i, C_i), \quad (3.25)$$

where L_i and C_i are irreducible representations of G and H, respectively. In particular,

$$\sum_i \dim L_i \dim C_i = \dim G_{10} = 496 . \quad (3.26)$$

If X is a matrix in the adjoint representation of G_{10} that corresponds to a generator of the subgroup G, it can be decomposed as follows

$$X = \oplus_i (X_i \otimes 1_i). \quad (3.27a)$$

Similarly, if Y corresponds to a generator of H

$$Y = \oplus_i (1_i \otimes Y_i). \quad (3.27b)$$

From these formulas it is evident that

$$XY = YX = \oplus_i (X_i \otimes Y_i) \quad (3.28)$$

and

$$Tr(XY) = \sum_i trX_i trY_i = 0 , \quad (3.29)$$

since G_{10} is semisimple. Therefore we have six-dimensional two-forms F with generators restricted to G (i.e., of X type) and background fields F_0 associated with H (i.e., of Y type). The total anomaly in six dimensions is characterized by a formal eight-form

$$I = n_{3/2} I^0_{3/2} + n_{1/2} I^0_{1/2} + \sum_i n^i_{1/2} I^i_{1/2} . \quad (3.30)$$

$I^0_{3/2}$ characterizes the anomaly due to a single left-handed spin $3/2$ field, which in the present case is a singlet of the gauge group G. It is given by

$$I^0_{3/2} = \frac{1}{(4\pi)^4}\left[-\frac{43}{288}\left(trR^2\right)^2 + \frac{49}{72} trR^4\right]. \quad (3.31)$$

The two-form R is a 6×6 matrix in the fundamental representation of the algebra SO(5,1).

$$n_{3/2} = n^L_{3/2} - n^R_{3/2} \quad (3.32)$$

is the net number of left-handed gravitinos in six dimensions. This number is given by an index theorem:

$$n_{3/2} = \frac{1}{8\pi^2}\frac{1}{48}\int_{M_4} trR_0^2 . \quad (3.33)$$

This formula is a consequence of a spin $1/2$ index theorem, which is relevant since spin $3/2$ in six dimensions requires the internal part of the ten-dimensional gravitino field to be spin $1/2$. It gives the number of six-dimensional gravitinos for left-handed gravitino in the D = 10 theory, but there is only one of them. Similarly, $I^0_{1/2}$ is the anomaly due to a left-handed singlet spin $1/2$ field in six dimensions

$$I^0_{1/2} = \frac{1}{(4\pi)^4}\left[\frac{1}{288}\left(trR^2\right)^2 + \frac{1}{360}trR^4\right]. \quad (3.34)$$

The net number of these fields in six dimensions arising from one left-handed gravitino and one right-handed spinor in the D = 10 supergravity multiplet is given by a combination of the spin $1/2$ and spin $3/2$ index theorems in the internal space. The result is

$$n_{1/2} = n^L_{1/2} - n^R_{1/2} = -\frac{1}{8\pi^2}\frac{7}{16}\int_{M_4} trR_0^2 \,. \quad (3.35)$$

The anomaly contribution of a multiplet of left-handed spinors in the representation L_i of G is

$$I^i_{1/2} = \frac{1}{(4\pi)^4}\left[\frac{2}{3}tr_{L_i}F^4 - \frac{1}{6}tr_{L_i}F^2 trR^2 + \frac{\dim L_i}{288}\left(trR^2\right)^2 + \frac{\dim L_i}{360}trR^4\right] \quad (3.36)$$

and the number of such multiplets is given by a spin $1/2$ index theorem

$$n_i = n^L_i - n^R_i = \frac{1}{8\pi^2}\int_{M_4}\left[-\frac{1}{2}tr_{C_i}F_0^2 + \frac{\dim C_i}{48}trR_0^2\right]. \quad (3.37)$$

These results can now be assembled to give the complete expression I of eq. (3.30). Combining eqs. (3.30-3.37) gives

$$I = \frac{1}{2(4\pi)^6}\int_{M_4}\left[-\frac{4}{3}TrF^4F_0^2 + \frac{1}{3}TrF^2F_0^2trR^2 - \frac{1}{144}\left(trR^2\right)^2 TrF_0^2 - \frac{11}{72}\left(trR^2\right)^2 trR_0^2 + \right.$$
$$\left. -\frac{1}{180}trR^4TrF_0^2 + \frac{1}{6}trR^4trR_0^2 + \frac{1}{18}TrF^4trR_0^2 - \frac{1}{72}TrF^2trR^2trR_0^2\right]. \quad (3.38)$$

To simplify this expression we note first that eq. (3.18) applied to an arbitrary linear combination of F and F_0, and eq. (3.29) imply that

$$TrF^4F_0^2 = \frac{1}{120}TrF^2TrF^2F_0^2 + \frac{1}{720}TrF_0^2TrF^4 - \frac{1}{72000}TrF_0^2\left(TrF^2\right)^2. \quad (3.39)$$

Using this and eq. (3.23) gives

$$I=\frac{1}{2(4\pi)^6}\int_{M_4}\left[-\frac{1}{90}TrF^2TrF^2F_0^2+\frac{1}{1800}\left(TrF^2\right)^2trR_0^2+\frac{1}{3}trR^2TrF^2F_0^2-\frac{1}{72}TrF^2trR^2trR_0^2-\frac{1}{12}\left(trR^2\right)^2trR_0^2\right]=$$
$$=\frac{1}{2(4\pi)^6}\left(trR^2-\frac{1}{30}TrF^2\right)\int_{M_4}\left[\frac{1}{3}TrF^2F_0^2-\frac{1}{60}trR_0^2TrF^2-\frac{1}{12}trR_0^2trR^2\right]. \quad (3.40)$$

Since this is a factorized expression of the same type as in eq. (3.5), a local counterterm of the form in eq. (3.15) (now using N = 2) can be constructed, thereby showing that all anomalies cancel. Using eqs. (3.23), (3.26), (3.33), and (3.37) one can show that the net number of left-handed spin $1/2$ fermions arising from the D = 10 Yang-Mills supermultiplet is

$$\sum_i n_i \dim L_i = -224 n_{3/2}\,. \quad (3.41)$$

3.2 The vanishing of cosmological constant. [6] [7] [8] [9]

In the **Section 1**, we have described some equations regarding the three-dimensional pure quantum gravity with *negative cosmological constant*. Now we consider some equations concerning the Coleman's approach that describes that wormholes have the effect of making *the cosmological constant vanish* and some equations regarding the cosmological constant problem in Kaluza-Klein theories, describing a mechanism to solve this problem by allowing extra time-like variables in D = 11 supergravity. Furthermore, we describe also an intersecting brane configuration in six-dimensional space with one extra space-like and one extra time-like dimensions. With a certain additional symmetry imposed on the extra space-time we obtain that *effective four-dimensional cosmological constant vanishes* automatically.

We start by considering pure gravity and will follow Coleman's argument in application to the d-dimensional Euclidean space-time. The Euclidean path integral for quantum gravity is

$$\int e^{-I(g,\lambda)}Dg \quad (3.42)$$

where g_{MN} is the metric, λ are coupling constants and $I(g,\lambda)$ is the action functional. The integration over Dg includes summation over all compact topologies of the d-dimensional space-time M^d. The integral (3.42) will be dominated by the following expression

$$\int d\alpha\rho(\alpha)e^{-S}\,.$$

Here $\alpha=(\alpha_1)$ are wormholes parameters, $\rho(\alpha)$ is a probability function and the effective action S has the form

$$S=\frac{1}{G}\int d^d\sqrt{g}\,(2\Lambda-R). \quad (3.43)$$

This action is calculated at the stationary point which is a solution of the Einstein equation

$$R_{MN}-\frac{1}{2}g_{MN}R+\Lambda g_{MN}=0\,. \quad (3.43a)$$

Here the parameters G and Λ depend on α and the dimensionality d . The probability function ρ is

$$\rho(\alpha) \approx Z(\alpha) e^{Z(\alpha)}$$

where

$$Z(\alpha) \approx \int D g e^{-S(g, \lambda+\alpha)} \approx e^{-S} . \quad (3.43b)$$

In the last integral the integration is performed only over smooth large universes without wormholes. Therefore

$$\rho(\alpha) \approx e^{-S} \exp\left(e^{-S}\right).$$

Now transform eq. (3.43) to a more simple form. From eq. (3.43a) one gets

$$R = \frac{2\Lambda d}{d-2} . \quad (3.43c)$$

Substituting this into eq. (3.43) we find

$$S = -\frac{4\Lambda}{(d-2)G} \int \sqrt{g} d^d x . \quad (3.44)$$

Note that from eqs. (3.43a) and (3.43c) one gets

$$R_{MN} = \frac{2\Lambda}{d-2} g_{MN} . \quad (3.44b)$$

Let us find the value of the action (3.44) for the d-sphere. For the d-sphere of radius r on has

$$R_{MN} = \frac{d-1}{r^2} g_{MN} . \quad (3.44c)$$

Therefore eqs. (3.44b) and (3.44c) yield

$$r^2 = \frac{(d-1)(d-2)}{2\Lambda}$$

and the action (3.44) for the sphere is equal to

$$S = -\frac{4\Lambda}{(d-2)G} r^d \Omega_d = -\frac{(d-1)^{\frac{d}{2}} (d-2)^{\frac{d}{2}-1} \Omega_d}{2^{\frac{d}{2}-2} G \Lambda^{\frac{d}{2}-1}} . \quad (3.45)$$

Here Ω_d is the volume of the unite d-sphere

$$\Omega_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} .$$

Hence, for eq. (3.45), the eq. (3.44) can be write also

$$S = -\frac{4\Lambda}{(d-2)G}\int\sqrt{g}\,d^d x = -\frac{(d-1)^{\frac{d}{2}}(d-2)^{\frac{d}{2}-1}\Omega_d}{2^{\frac{d}{2}-2}G\Lambda^{\frac{d}{2}-1}}. \quad (3.45b)$$

Equation (3.45) ***implies that*** $\rho(\alpha)$ ***is infinitely peaked on the values of*** α_1 ***for which the cosmological constant*** Λ ***vanishes***. This is Coleman's solution to the cosmological constant problem in d-dimensional space-time. If there exist several solutions of the Einstein eq. (3.43a), then instead eq. (3.43b) one would obtain

$$Z(\alpha) \approx \int Dg e^{-I(g,\lambda+\alpha)} \approx \sum_n e^{-S_n} \quad (3.45c)$$

where S_n is the action functional on the corresponding solution number n. Now the probability function is

$$\rho(\alpha) \approx \left(\sum_n e^{-S_n}\right)\exp\left(\sum_n e^{-S_n}\right). \quad (3.45d)$$

We consider solutions which correspond to the spontaneous compactifications of the form

$$M^d = M^m \times B^r.$$

It is clear from eq. (3.45d) that the more probable solution corresponds to the smaller value of the Euclidean action.
Wormholes are topology-changing configurations in Euclidean quantum gravity. Coleman showed that if wormholes exist, they have the effect of making the cosmological constant vanish.
Now, we describe some equations concerning this subject and their mathematical connections with the Hartle-Hawking wave function.
We begin with some (possibly disconnected) manifold, M, with long-wavelength background fields. We also specify the initial number of baby universes of each type, n_i, and the final number, $n_i^{'}$. We then integrate over the fluctuations and sum over all possible locations of the wormholes. The result of this process is

$$\sum_{\substack{fluctuations\\ and wormholes\\ n_i, n_i^{'} fixed}} e^{-S} = \left\langle n_1^{'}, n_2^{'} \ldots \right| e^{-S_{eff}} \left| n_1, n_2 \ldots \right\rangle. \quad (3.46)$$

Here,

$$S_{eff} = \int_M d^4x\sqrt{g}\,\mathcal{L}_{eff}, \quad (3.47)$$

where

$$\mathcal{L}_{eff} = \mathcal{L}_0 + \sum_i \left(a_i^{|} + a_{i*}\right)\mathcal{L}_i, \quad (3.48)$$

$\mathcal{L}_0$ is the result of integration over the fluctuations, and

$$\mathcal{L}_i = e^{-S_i} K_i . \quad (3.49)$$

Hence, we can write eq. (3.47) also

$$S_{eff} = \int_M d^4 x \sqrt{g} \mathcal{L}_0 + \sum_i \left(a_i^{|} + a_{i*} \right) e^{-S_i} K_i . \quad (3.49b)$$

It will be useful to have an explicit formula relating A eigenstates to baby-universe-number eigenstates:

$$\left\langle \alpha_1, \alpha_2 ... \middle| n_1, n_2 ... \right\rangle = \prod_i \psi_{n_i} \left(\alpha_i / \sqrt{2} \right), \quad (3.50)$$

where $\psi_n(q)$ is the n[th] energy eigenfunction of the harmonic oscillator, $H = \frac{1}{2}\left(p^2 + q^2\right)$. If we rewrite eq. (3.46) in terms of A-eigenstates, we find

$$\left\langle \alpha_1^{'} ... \middle| e^{-S_{eff}} \middle| \alpha_1 ... \right\rangle = e^{-S_{eff}(\alpha)} \prod_i \delta\left(\alpha_i^{'} - \alpha_i\right), \quad (3.51)$$

where

$$S_{eff}(\alpha) = \int_M d^4 x \sqrt{g} \left(\mathcal{L}_0 + \sum_i \mathcal{L}_i \alpha_i \right). \quad (3.52)$$

Let us define an amplitude

$$A(B_1, B_2, t) \equiv \sum e^{-S} , \quad (3.53)$$

where S is the (gauge-invariant) Euclidean action and the sum is over all motions that go from B_2 to B_1 in Euclidean time t. This amplitude obeys a composition law

$$\int A(B_1, B_2, t) d\mu(B_2) A(B_2, B_3, t') = A(B_1, B_3, t + t'), \quad (3.54)$$

for appropriate measure $\mu(B)$. Further, given any function $\Phi(B)$,

$$\Psi(B,t) = \int A(B, B', t) \Phi(B') d\mu(B'), \quad (3.55)$$

is a possible wave function of the universe for $t > 0$.
Hartle and Hawking showed that

$$\Psi(B) = \int A(B, B') \Phi(B') d\mu(B'), \quad (3.56)$$

is a possible wave function of the universe, for any Φ and any measure μ. Hartle and Hawking paid special attention to the wave function defined by the simplest boundary condition of all, no boundaries. In our notation, $\Phi(B)$ is proportional to δ_{B0}, and

$$\Psi^{HH}(B) = A(B,0) = A(0,B)^* . \quad (3.57)$$

They suggested that this was in a sense the ground-state wave function of the universe. A naïve generalization of the composition law, eq. (3.54) would lead us to believe that

$$A(B_1,B_3)=\int A(B_1,B_2)d\mu(B_2)A(B_2,B_3), \quad (3.58)$$

for appropriate measure μ. If we define an inner product between allowable wave functions by

$$(\Psi_1,\Psi_2)=\int \Psi_1^*(B)\Psi_2(B)d\mu(B), \quad (3.59)$$

then it follows from eqs. (3.56) and (3.58) that

$$(\Psi_1,\Psi_2)=\int \Phi_1(B_1)^* A(B_1,B_2)\Phi_2(B_2)d\mu(B_1)d\mu(B_2). \quad (3.60)$$

Now, we shall attempt to compute the Hartle-Hawking wave function of the universe in the presence of wormholes. Let us consider a theory in which the action in $S_{eff}(\alpha)$, for some fixed value of the α's, and in which we integrate only over configurations that are slowly varying on the wormhole scale. In this case, the Hartle-Hawking wave function is

$$\Psi_\alpha^{HH}(B)=\sum e^{-S_{eff}(\alpha)}, \quad (3.61)$$

where the sum is over all manifolds that go from no boundary to B (B is slowly varying on the wormhole scale). A general manifold will have several components. Some of these will be connected to B (only one, if B is itself connected). However, there may be other components which are closed, that is to say, which have no boundary at all. The action is a sum over the various components. Thus the sum over four-manifolds factorizes,

$$\Psi_\alpha^{HH}(B)=\psi_\alpha^{HH}(B)Z(\alpha), \quad (3.62)$$

where ψ_α^{HH} is given by the sum over manifolds connected to B, and

$$Z(\alpha)=\sum_{CM} e^{-S_{eff}(\alpha)}, \quad (3.63)$$

where CM denotes closed manifolds. Hence, the eq. (3.62) can be rewritten also:

$$\Psi_\alpha^{HH}(B)=\psi_\alpha^{HH}(B)\sum_{CM} e^{-S_{eff}(\alpha)}. \quad (3.63b)$$

Let us now compute the expectation value of some scalar field, ϕ. By following equation

$$A(x_1,x_2)=\sum_{paths} e^{-S}, \quad (3.64)$$

we have that

$$\langle\phi\rangle_\alpha^{HH} = \frac{\sum_{CM} e^{-S_{eff}(\alpha)}\phi(x)}{\sum_{CM} e^{-S_{eff}(\alpha)}}. \quad (3.65)$$

The denominator is just $Z(\alpha)$ again. Hence,

$$\sum_{CM} e^{-S_{eff}(\alpha)}\phi(x) = \langle\phi\rangle_\alpha^{HH} Z(\alpha). \quad (3.66)$$

Furthermore, we have that

$$\langle\phi\rangle_\alpha^{HH} = \frac{\sum_{CCM} e^{-S_{eff}(\alpha)}\phi(x)}{\sum_{CCM} e^{-S_{eff}(\alpha)}}, \quad (3.67)$$

where CCM denotes closed connected manifolds. This equation tell us that, if the α's are constants, we make no error if we simply ignore disconnected closed components in the path integral. Now let us turn to the real thing, the theory with wormholes. The argument of Ψ is now not just B, but also the number of baby universes. Equation (3.50) tell us how to write the no-baby-universe state in terms of the α's:

$$\langle\alpha|0\rangle = e^{-\alpha^2/4}, \quad (3.68)$$

times an irrelevant normalization constant. Thus we can directly aplly the wormhole summation formula, eq. (3.51), to find

$$\Psi^{HH}(B,\alpha) = e^{-\alpha^2/4}\psi_\alpha^{HH}(B)Z(\alpha). \quad (3.69)$$

This equation strongly suggests that $Z(\alpha)$ governs the probability of finding given values of α in the Hartle-Hawking state. To get a more precise idea of what is going on, let us compute $\langle\phi\rangle^{HH}$. For this computation, we must sum over closed manifolds. Thus both the initial and final state contain no baby universes. It then follows from the wormhole summation formula and eq. (3.66) that

$$\langle\phi\rangle^{HH} = \frac{\int d\alpha e^{-\alpha^2/2}\langle\phi\rangle_\alpha^{HH} Z(\alpha)}{\int d\alpha e^{-\alpha^2/2} Z(\alpha)}. \quad (3.70)$$

We see that the probability distribution in α is

$$dP = e^{-\alpha^2/2}Z(\alpha)d\alpha, \quad (3.71)$$

up to a normalization.
We shall now show that $Z(\alpha)$ displays the announced peak. We have that

$$Z(\alpha) = \exp\left[\sum_{CCM} e^{-S_{eff}(\alpha)}\right]. \quad (3.72)$$

The sum over closed connected manifolds can be expressed in terms of a background-gravitational-field effective action, Γ. The path integral of e^{-S} is then equal to $e^{-\Gamma}$, evaluated at the stationary point of Γ. The sum in eq. (3.72) runs over manifolds of all possible topologies.

Thus we will define an independent effective action for each topology, and write

$$\sum_{CCM} e^{-S_{eff}(\alpha)} = \sum_{topol.} e^{-\Gamma_\alpha(g)}, \quad (3.73)$$

where g denotes the background metric on each topology, and each term on the right is to be evaluated at its stationary point. The leading term in Γ for large volume is given by

$$\Gamma = \lambda \int d^4x \sqrt{g} + ..., \quad (3.74)$$

where λ is the cosmological constant. The first correction to eq. (3.73) is also known,

$$\Gamma = \int d^4x \sqrt{g} \left[\lambda - \frac{1}{16\pi G} R \right] + ..., \quad (3.75)$$

where G is Newton's constant, again including all renormalization effects of all interactions. The stationary points of eq. (3.75) are Einstein spaces,

$$R_{\mu\nu} = 8\pi G \lambda g_{\mu\nu}. \quad (3.76)$$

For these,

$$\Gamma = -\lambda \int d^4x \sqrt{g}. \quad (3.77)$$

Thus for positive λ we want the Einstein space of maximum volume, for negative λ that of minimum volume. For positive λ, the space of maximum volume is known; it is the four-sphere of radius $\sqrt{3/8\pi G\lambda}$, for which

$$\Gamma = -\frac{3}{8G^2\lambda}. \quad (3.78)$$

For sufficiently small λ, the neglected terms in eq. (3.75) are negligible compared to this. For negative λ, the minimum volume space is not known. Nevertheless, whatever it is, it makes a positive contribution to Γ proportional to $1/\lambda$. Thus,

$$\ln Z \begin{cases} \propto e^{3/(8G^2\lambda)}, \lambda \to 0^+, \\ \to 0, \lambda \to 0^-. \end{cases} \quad (3.79)$$

If an infrared cutoff is introduced, say by restricting the path integral to manifolds with diameters less than some maximum value, D, Γ approaches a finite limit, proportional to D^2/G, as λ goes to zero. ***If, in the presence of such a cutoff, we normalize the probability distribution in α, eq.*** (3.71)***, and then let D go to infinity, the probability distribution becomes concentrated on that submanifold of α space on which λ vanishes***.
We note that the eqs. (3.45b), (3.45c), (3.69) and (3.71), can be mathematically connected. Indeed, we have:

$$S = -\frac{4\Lambda}{(d-2)G} \int \sqrt{g}\, d^d x = -\frac{(d-1)^{\frac{d}{2}} (d-2)^{\frac{d}{2}-1} \Omega_d}{2^{\frac{d}{2}-2} G \Lambda^{\frac{d}{2}-1}} \Rightarrow$$

$$\Rightarrow Z(\alpha) \approx \int Dg e^{-I(g,\lambda+\alpha)} \approx \sum_n e^{-S_n} \Rightarrow \Psi^{HH}(B,\alpha) = e^{-\alpha^2/4}\psi_\alpha^{HH}(B)Z(\alpha) \Rightarrow$$

$$\Rightarrow dP = e^{-\alpha^2/2}Z(\alpha)d\alpha. \quad (3.79b)$$

Now it will be shown that extra time-like coordinates can solve the cosmological constant problem in D = 11 supergravity. We will obtain it by a double use of the Freund-Rubin ansatz in the internal seven-dimensional space. Contributions to the Einstein equations from the antisymmetric field strength tensors depend on the signature and a compensating mechanism may take place. It is known that compactified time-like dimensions, as a rule, cause the appearance of ghosts and tachyons in the effective four-dimensional theory. The massive ghosts and tachyons can be ignored, because their masses are of the Planck scale. In this Kaluza-Klein approach we deal only with the massless sector. Therefore it is enough to avoid the appearance of massless ghosts only.

The vacuum solutions of D = 11 supergravity with zero cosmological constant presented now, satisfy the above mentioned criteria.

The equations of motion for the bosonic part of D = 11 supergravity are:

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{3}\left[F_{MPQR}F_N^{PQR} - \frac{1}{8}g_{MN}F_{PQRS}F^{PQRS}\right], \quad (3.80)$$

$$\nabla_M F^{MNPQ} = -\left(\frac{1}{576}\sqrt{|g|}\right)\varepsilon^{M_1\ldots M_8 NPQ}F_{M_1\ldots M_4}F_{M_5\ldots M_8}, \quad (3.81)$$

with the Bianchi identity

$$\partial_{[M}F_{NPQR]} = 0, \quad (3.82)$$

where $M, N, \ldots = 0,1,\ldots,10$.

Suppose that the eleven-dimensional manifold has the direct product form

$$M_4^{11} = M_1 \times M_0^3 \times M_0^1 \times M_3^3, \quad (3.83)$$

where upper indices denote dimensionality of manifolds and lower indices denote the number of time-like coordinates. This topology yields the block diagonal form of the metric tensor

$$g_{MN} = \begin{bmatrix} g_{\mu\nu}(x) & & 0 & \\ & g_{ab}(y) & & \\ 0 & & g_{77}(z) & \\ & & & g_{mn}(w) \end{bmatrix}, \quad (3.84)$$

where $\mu,\nu = 0,1,2,3; a,b = 4,5,6; m,n = 8,9,10$. Non-zero components of the rank-four field strengths are

$$F_{\hat a\hat b\hat c\hat d} = \lambda_1\left[|g(y,z)|\right]^{1/2}\varepsilon_{\hat a\hat b\hat c\hat d}, \quad (3.85a) \qquad F_{\hat m\hat n\hat p\hat q} = \lambda_2\left[|g(w,z)|\right]^{1/2}\varepsilon_{\hat m\hat n\hat p\hat q}, \quad (3.85b)$$

where

$$g(y,z) = g_{77}(z)\det g_{ab}(y), \quad g(w,z) = g_{77}(z)\det g_{mn}(w), \quad \hat a,\hat b,\hat c,\hat d = 4,5,6,7; \hat m,\hat n,\hat p,\hat q = 7,8,9,10.$$

The right hand-side of eq. (3.81) vanishes due to overlapping the seventh coordinate in the ansatze (3.85a) and (3.85b). The left hand-side of eq. (3.81) is also zero, so eq. (3.81) is really satisfied. The Bianchi identity is true by virtue of the supposed topology (3.84). Eqs. (3.80) take the form

$$R_{\mu\nu}(x) = -\frac{2}{3}\left(\lambda_1^2 - \lambda_2^2\right)g_{\mu\nu}(x), \quad (3.86a) \qquad R_{ab}(y) = \frac{2}{3}\left(2\lambda_1^2 + \lambda_2^2\right)g_{ab}(y), \quad (3.86b)$$

$$R_{77}(z) = \frac{4}{3}\left(\lambda_1^2 - \lambda_2^2\right)g_{77}(z), \quad (3.86c) \qquad R_{mn}(w) = -\frac{2}{3}\left(\lambda_1^2 + 2\lambda_2^2\right)g_{mn}(w). \quad (3.86d)$$

Since the left hand-side of eq. (3.86c) is zero, it follows that

$$\lambda_1^2 = \lambda_2^2 = \lambda^2. \quad (3.87)$$

Therefore eqs. (3.86) get the form

$$R_{\mu\nu}(x) = 0, \quad (3.88a) \qquad R_{ab}(y) = 2\lambda^2 g_{ab}(y), \quad (3.88b) \qquad R_{mn}(w) = -2\lambda^2 g_{mn}(w). \quad (3.88c)$$

To avoid the appearance of ghosts in the massless four-dimensional theory, the internal space with time-like compactified coordinates should satisfy the following conditions:

(i) the internal space has no Killing vectors, and

(ii) all Betti numbers b_{2k+1} of the internal space vanish for $2k+1 \le l$ if antisymmetric tensor fields of rank l are present.

If we take M_3^3 are the quotient space S^3/Γ with a discrete group of isometry Γ acting on S^3 non-freely, then the conditions (i) and (ii) are satisfied. In this case M_3^3 is a finite-volume manifold with singular points. So, we have the following compactification:

$$M_4^{11} = M_1^4 \times S^3 \times S^1 \times S^3/\Gamma,$$

where M_1^4 is a Minkowski space-time. According to the standard Kaluza-Klein ansatz such a compactification leads to the appearance of the gauge fields with the $SO(4)\times U(1)$ group.

The result obtained can be generalized to the compactification of the form

$$M_T^{11} = M_1^4 \times M_{t_1}^3 \times S^1 \times M_{t_2}^3, \quad (3.89)$$

where the lower indices denote the numbers of the time-like dimensions in the corresponding manifolds $(T = 1 + t_1 + t_2)$. In this case eq. (3.80) takes the form

$$R_{\mu\nu}(x) = -\frac{2}{3}\left[(-1)^{t_1}\lambda_1^2 + (-1)^{t_2}\lambda_2^2\right]g_{\mu\nu}(x), \qquad R_{ab}(y) = \frac{2}{3}\left[(-1)^{t_1}2\lambda_1^2 - (-1)^{t_2}\lambda_2^2\right]g_{ab}(y),$$

$$R_{77}(z) = \frac{4}{3}\left[(-1)^{t_1}\lambda_1^2 + (-1)^{t_2}\lambda_2^2\right]g_{77}(z), \qquad R_{mn}(w) = -\frac{2}{3}\left[(-1)^{t_1}\lambda_1^2 - (-1)^{t_2}2\lambda_2^2\right]g_{mn}(w). \quad (3.90)$$

We have a Ricci flat four-dimensional space-time if $\lambda_1^2 = \lambda_2^2 = \lambda^2$ and $(-1)^{t_1} + (-1)^{t_2} = 0$.

Among many vacuum solutions, which follow from eqs. (3.90), we wish to point out one (3.83), because it is the most favourable. This solution is analogous to the Minkowski four-dimensional

one in non-chiral N = 2, D = 10 supergravity, obtained taking twice (without overlapping) F_{MNP} in the internal six-dimensional space.
We note that the eq. (3.81) can be mathematically connected with the eqs. (3.31), (3.34), (3.36) and (3.38). Indeed, we have:

$$\nabla_M F^{MNPQ} = -\left(\frac{1}{576}\sqrt{|g|}\right)\varepsilon^{M_1...M_8NPQ}F_{M_1...M_4}F_{M_5...M_8} \Rightarrow$$
$$\Rightarrow \frac{1}{(4\pi)^4}\left[-\frac{43}{288}\left(trR^2\right)^2+\frac{49}{72}trR^4\right] \Rightarrow \frac{1}{(4\pi)^4}\left[\frac{1}{288}\left(trR^2\right)^2+\frac{1}{360}trR^4\right] \Rightarrow$$
$$\Rightarrow \frac{1}{(4\pi)^4}\left[\frac{2}{3}tr_{L_i}F^4-\frac{1}{6}tr_{L_i}F^2trR^2+\frac{\dim L_i}{288}\left(trR^2\right)^2+\frac{\dim L_i}{360}trR^4\right] \Rightarrow$$
$$\Rightarrow \frac{1}{2(4\pi)^6}\int_{M_4}\left[-\frac{4}{3}TrF^4F_0^2+\frac{1}{3}TrF^2F_0^2trR^2-\frac{1}{144}\left(trR^2\right)^2TrF_0^2-\frac{11}{72}\left(trR^2\right)^2trR_0^2+\right.$$
$$\left.-\frac{1}{180}trR^4TrF_0^2+\frac{1}{6}trR^4trR_0^2+\frac{1}{18}TrF^4trR_0^2-\frac{1}{72}TrF^2trR^2trR_0^2\right]. \quad (3.90b)$$

We have also that:

$$\nabla_M F^{MNPQ} = -\left(\frac{1}{576}\sqrt{|g|}\right)\varepsilon^{M_1...M_8NPQ}F_{M_1...M_4}F_{M_5...M_8} \Rightarrow \frac{1}{2(4\pi)^4}\left[-\frac{43}{576}\left(trR^2\right)^2+\frac{49}{144}trR^4\right]. \quad (3.90c)$$

We note that $576 = 24 \times 24$ and that the number 24 correspond to the Ramanujan function that has 24 "modes" that correspond to the physical vibrations of a bosonic string.
Indeed, we have that:

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Furthermore, if the (3.81) is an equation of motion for the bosonic part of D = 11 supergravity, then it is possible also the mathematical connections with the fundamental equation of Palumbo-Nardelli model:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right],$$

Thence, we have the following connections:

$$\nabla_M F^{MNPQ} = -\left(\frac{1}{576}\sqrt{|g|}\right)\varepsilon^{M_1 \ldots M_8 NPQ} F_{M_1 \ldots M_4} F_{M_5 \ldots M_8} \Rightarrow \frac{1}{2(4\pi)^4}\left[-\frac{43}{576}\left(trR^2\right)^2 + \frac{49}{144}trR^4\right] \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]. \quad (3.90d)$$

This equation can be related easily also with eq. (3.90b).

Now, we consider 6-dimensional space-time $M^{(4,2)}$ with one extra time-like dimension τ and one extra dimension y, i.e. space-time with a signature (4,2). Suppose that there are two branes with a world-volume signature (4,1) (“time brane”) and (3,2) (“space brane”) embedded in $M^{(4,2)}$ with tensions T_τ and T_y, respectively. The intersection of these branes, which we take to be at $(y=0,\tau=0)$ point for definiteness, is a 4-dimensional subspace (3-brane) of $M^{(4,2)}$ with signature (3,1) which can be identified with a visible world. The relevant action describing such a set-up is:

$$S = \int d^6x\sqrt{\det g}\left(\frac{1}{\kappa_6^2}R - \Lambda_b\right) - \int d^6x\sqrt{\det g}\left[T_\tau\delta(\tau) + T_y\delta(y)\right] = \int_{M^{(4,2)}} d^4x dy d\tau\sqrt{\det g}\left(\frac{1}{\kappa_6^2}R - \Lambda_b\right) +$$
$$- \int_{M^{(4,1)}} d^4x dy\sqrt{-\det g^{\tau=0}}T_\tau - \int_{M^{(3,2)}} d^4x d\tau\sqrt{\det g^{y=0}}T_y. \quad (3.91)$$

Here $\kappa_6^2 = 16\pi M_6^{-4}$, where M_6 is the six-dimensional fundamental scale of the theory and Λ_b is a bulk cosmological constant. We can rewrite the eq. (3.91) also

$$S = \int_{M^{(4,2)}} d^4x dy d\tau\sqrt{\det g}\left(\frac{1}{16\pi M_6^{-4}}R - \Lambda_b\right) - \int_{M^{(4,1)}} d^4x dy\sqrt{-\det g^{\tau=0}}T_\tau - \int_{M^{(3,2)}} d^4x d\tau\sqrt{\det g^{y=0}}T_y.$$

(3.91b)

The induced metrics on the branes, $g_{ab}^{\tau=0}(a,b=\mu,y)$ and $g_{\alpha\beta}^{y=0}(\alpha,\beta=\mu,\tau)$, are defined as:

$$g_{ab}^{\tau=0} = g_{ab}\left(x^\mu, y, \tau=0\right), \qquad g_{\alpha\beta}^{y=0} = g_{\alpha\beta}\left(x^\mu, y=0, \tau\right), \quad (3.92)$$

where $g_{MN}, M, N = \mu(0,1,2,3), y, \tau,$ is a six-dimensional metric. We use metric with mostly positive signature $(-++++-)$. The field equations followed from the above action (3.91) are:

$$R_N^M - \frac{1}{2}\delta_N^M R = \frac{\kappa_6^2}{2}T_N^M, \quad (3.93)$$

where the energy momentum tensor T_N^M is expressed through the bulk cosmological constant Λ_b and brane tensions T_τ and T_y as:

$$T_N^M = -\Lambda_b \delta_N^M - \sqrt{\frac{-\det g^{\tau=0}}{\det g}} T_\tau \delta(\tau) \delta_a^M \delta_N^a - \sqrt{\frac{\det g^{y=0}}{\det g}} T_y \delta(y) \delta_\alpha^M \delta_N^\alpha . \quad (3.94)$$

Hence, the eq. (3.93) can be rewritten also

$$R_N^M - \frac{1}{2}\delta_N^M R = \frac{\kappa_6^2}{2} \left[-\Lambda_b \delta_N^M - \sqrt{\frac{-\det g^{\tau=0}}{\det g}} T_\tau \delta(\tau) \delta_a^M \delta_N^a - \sqrt{\frac{\det g^{y=0}}{\det g}} T_y \delta(y) \delta_\alpha^M \delta_N^\alpha \right]. \quad (3.94b)$$

We are looking for a static solution of the above equations (3.93) that respects 4-dimensional Poincare invariance in the x^μ direction.
A 6-dimensional line element satisfying this ansatz can be written as:

$$ds^2 = A^2(y,\tau)\eta_{\mu\nu}dx^\mu dx^\nu + B^2(y,\tau)dy^2 - C^2(y,\tau)d\tau^2 , \quad (3.95)$$

where $\eta_{\mu\nu}$ is a 4-dimensional flat Minkowski metric. It is more convenient, however, to perform the actual calculations within a conformally flat metric ansatz

$$ds^2 = A^2(z,\theta)\eta_{MN}dx^M dx^N , \quad (3.96)$$

which can be obtained from (3.95) by the following coordinate transformations:

$$dz = \frac{B}{A}dy , \qquad d\theta = \frac{C}{A}d\tau . \quad (3.97)$$

Now using the well-known conformal transformation formulae for the Einstein tensor

$$G_N^M = R_N^M - \frac{1}{2}\delta_N^M R$$

$$\tilde{G}_{MN} = G_{MN} + 4(\nabla_M \ln A \nabla_N \ln A - \nabla_M \nabla_N \ln A) + 4\eta_{MN}\left(\nabla^2 \ln A + \frac{3}{2}(\nabla \ln A)^2\right), \quad (3.98)$$

we easily obtain:

$$G_\nu^\mu = \frac{2}{A^2}\left[\left(\frac{A'}{A}\right)^2 - \left(\frac{\dot{A}}{A}\right)^2 + 2\left(\frac{A''}{A} - \frac{\ddot{A}}{A}\right)\right]\delta_\nu^\mu , \quad (3.99) \qquad G_z^z = \frac{2}{A^2}\left[5\left(\frac{A'}{A}\right)^2 - \left(\frac{\dot{A}}{A}\right)^2 - 2\frac{\ddot{A}}{A}\right], \quad (3.100)$$

$$G_\theta^\theta = \frac{2}{A^2}\left[-5\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{A'}{A}\right)^2 + 2\frac{A''}{A}\right], \quad (3.101) \qquad G_\theta^z = -G_z^\theta = \frac{4}{A^2}\left[2\frac{\dot{A}A'}{A^2} - \frac{\dot{A}'}{A}\right], \quad (3.102)$$

where primes and overdots denote the derivatives with respect to space-like z and time-like θ coordinates, respectively. Taking the conformal factor (warp factor) in (3.96) as

$$A = \frac{1}{k_y|z| + k_\tau|\theta| + 1}, \quad (3.103)$$

one can easily check that non-diagonal elements (3.102) of the Einstein tensor vanish, $G^z_\theta = -G^\theta_z = 0$, and thus $(z\theta)$ Einstein's equations are satisfied identically, while the remaining equations will be satisfied if the following relations are fulfilled:

$$k_y^2 - k_\tau^2 = -\frac{\kappa_6^2\Lambda_b}{10}, \quad (3.104) \qquad k_y = \frac{\kappa_6^2 T_y}{4}, \quad (3.105) \qquad k_\tau = -\frac{\kappa_6^2 T_\tau}{4}. \quad (3.106)$$

Here we will assume that the space brane has a positive tension $T_y > 0$, while the time brane the negative one, $T_\tau < 0$, so that both k_y and k_τ are positive. ***If we demand that the Einstein equations*** (3.93) ***are invariant under the*** $\theta \leftrightarrow z$ ***exchange than among the solutions*** (3.103 – 3.106) ***the one with*** $\Lambda_b = 0$ ***survives***. The fine tuning problem now is resolved since ***the above invariance demands*** $T_\tau = -T_y$ ***and ensures automatic cancellation of the 4-dimensional cosmological constant***. Now, we start from the more general ansatz by taking

$$\tilde{g}_{\mu\nu} = \left(1 - \frac{1}{4}H^2\eta_{\mu\nu}x^\mu x^\nu\right)^{-2}\eta_{\mu\nu} \quad (3.107)$$

instead of the flat 4-dimensional metric $\eta_{\mu\nu}$ in (3.96). Here H is a “Hubble constant” on the intersection. Now the ansatz (3.96) with (3.107) instead of $\eta_{\mu\nu}$ describes maximally symmetric 4-dimensional space-times of the intersection of branes, i.e. de Sitter $(H^2 > 0)$ or anti-de Sitter $(H^2 < 0)$. Then the components of the Einstein tensor (3.99) and (3.100), (3.101) will be changed by the additional term $+\frac{3H^2}{A^2}\delta^\mu_\nu$ and $+\frac{6H^2}{A^2}$, respectively, while (3.102) will remain unchanged. ***It is easy to see that the corresponding Einstein equations will remain invariant under the discrete symmetry*** $\theta \leftrightarrow z$ ***if and only if*** $H = 0$ ***and*** $\Lambda_b = 0$. This can be easily understood from the fact that the origin for the non-zero Hubble constant is a non-zero 4-dimensional cosmological constant on the intersection of branes which in turn is indeed forbidden if one demands that the theory is invariant under the discrete symmetry imposed above. ***The above invariance can be viewed as a constraint imposed on the system described by the action*** (3.91) ***which holds for the special class of metrics*** g_{MN} ***including the background one given by*** (3.96, 3.103 – 3.106) ***with*** $\Lambda_b = 0$. Notice that the vanishing of the bulk cosmological constant, $\Lambda_b = 0$, and the relation $T_\tau = -T_y$ emerge merely from the discrete symmetry imposed and are not consequence of any fine-tuning.
In conclusion, with regard the action (3.91), we note that it is possible the mathematical connection with the eqs. (3.81), (3.90b) and (3.90d). Thence, we obtain:

$$S = \int d^6x\sqrt{\det g}\left(\frac{1}{\kappa_6^2}R - \Lambda_b\right) - \int d^6x\sqrt{\det g}\left[T_\tau\delta(\tau) + T_y\delta(y)\right] = \int_{M^{(4,2)}} d^4x dy d\tau\sqrt{\det g}\left(\frac{1}{\kappa_6^2}R - \Lambda_b\right) +$$

$$-\int_{M^{(4,1)}} d^4xdy\sqrt{-\det g^{\tau=0}}T_\tau - \int_{M^{(3,2)}} d^4xd\tau\sqrt{\det g^{y=0}}T_y \Rightarrow$$

$$\Rightarrow \nabla_M F^{MNPQ} = -\left(\frac{1}{576}\sqrt{|g|}\right)\varepsilon^{M_1...M_8NPQ} F_{M_1...M_4} F_{M_5...M_8} \Rightarrow$$

$$\Rightarrow \frac{1}{(4\pi)^4}\left[-\frac{43}{288}\left(trR^2\right)^2 + \frac{49}{72}trR^4\right] \Rightarrow \frac{1}{(4\pi)^4}\left[\frac{1}{288}\left(trR^2\right)^2 + \frac{1}{360}trR^4\right] \Rightarrow$$

$$\Rightarrow \frac{1}{(4\pi)^4}\left[\frac{2}{3}tr_{L_i}F^4 - \frac{1}{6}tr_{L_i}F^2trR^2 + \frac{\dim L_i}{288}\left(trR^2\right)^2 + \frac{\dim L_i}{360}trR^4\right] \Rightarrow$$

$$\Rightarrow \frac{1}{2(4\pi)^6}\int_{M_4}\left[-\frac{4}{3}TrF^4F_0^2 + \frac{1}{3}TrF^2F_0^2trR^2 - \frac{1}{144}\left(trR^2\right)^2TrF_0^2 - \frac{11}{72}\left(trR^2\right)^2trR_0^2 + \right.$$

$$\left. -\frac{1}{180}trR^4TrF_0^2 + \frac{1}{6}trR^4trR_0^2 + \frac{1}{18}TrF^4trR_0^2 - \frac{1}{72}TrF^2trR^2trR_0^2\right] \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\dfrac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\dfrac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (3.108)$$

Note that also the relation (3.79b) can be mathematically connected with this last relation.

4. On some equations concerning p-adic strings, p-adic and adelic zeta functions, zeta strings and zeta nonlocal scalar fields. [10] [11] [12] [13] [14]

Like in the ordinary string theory, the starting point of p-adic strings is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following forms:

$$A_\infty(a,b) = g^2\int_R |x|_\infty^{a-1}|1-x|_\infty^{b-1}dx = g^2\left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} + \frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)}\right] = g^2\frac{\zeta(1-a)}{\zeta(a)}\frac{\zeta(1-b)}{\zeta(b)}\frac{\zeta(1-c)}{\zeta(c)} =$$

$$= g^2\int DX\exp\left(-\frac{i}{2\pi}\int d^2\sigma\partial^\alpha X_\mu\partial_\alpha X^\mu\right)\prod_{j=1}^4\int d^2\sigma_j\exp\left(ik_\mu^{(j)}X^\mu\right), \quad (4.1-4.4)$$

where $\hbar = 1$, $T = 1/\pi$, and $a = -\alpha(s) = -1 - \frac{s}{2}$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $s+t+u = -8$, i.e. $a+b+c=1$.

The p-adic generalization of the above expression

$$A_\infty(a,b)=g^2\int_R |x|_\infty^{a-1}|1-x|_\infty^{b-1}dx\,,$$

is:

$$A_p(a,b)=g_p^2\int_{Q_p}|x|_p^{a-1}|1-x|_p^{b-1}dx\,, \quad (4.5)$$

where $|\ldots|_p$ denotes p-adic absolute value. In this case only string world-sheet parameter x is treated as p-adic variable, and all other quantities have their usual (real) valuation.
Now, we remember that the Gauss integrals satisfy adelic product formula

$$\int_R \chi_\infty\left(ax^2+bx\right)d_\infty x\prod_{p\in P}\int_{Q_p}\chi_p\left(ax^2+bx\right)d_p x=1\,, \quad a\in Q^\times\,, \quad b\in Q\,, \quad (4.6)$$

what follows from

$$\int_{Q_v}\chi_v\left(ax^2+bx\right)d_v x=\lambda_v(a)|2a|_v^{-\frac{1}{2}}\chi_v\left(-\frac{b^2}{4a}\right), \quad v=\infty,2,\ldots,p\ldots. \quad (4.7)$$

These Gauss integrals apply in evaluation of the Feynman path integrals

$$K_v(x'',t'';x',t')=\int_{x',t'}^{x'',t''}\chi_v\left(-\frac{1}{h}\int_{t'}^{t''}L(\dot{q},q,t)dt\right)D_v q\,, \quad (4.8)$$

for kernels $K_v(x'',t'';x',t')$ of the evolution operator in adelic quantum mechanics for quadratic Lagrangians. In the case of Lagrangian

$$L(\dot{q},q)=\frac{1}{2}\left(-\frac{\dot{q}^2}{4}-\lambda q+1\right),$$

for the de Sitter cosmological model one obtains

$$K_\infty(x'',T;x',0)\prod_{p\in P}K_p(x'',T;x',0)=1\,, \quad x'',x',\lambda\in Q\,,\ T\in Q^\times\,, \quad (4.9)$$

where

$$K_v(x'',T;x',0)=\lambda_v(-8T)|4T|_v^{-\frac{1}{2}}\chi_v\left(-\frac{\lambda^2T^3}{24}+[\lambda(x''+x')-2]\frac{T}{4}+\frac{(x''-x')^2}{8T}\right). \quad (4.10)$$

Also here we have the number 24 that correspond to the Ramanujan function that has 24 “modes”, i.e., the physical vibrations of a bosonic string. Hence, we obtain the following mathematical connection:

$$K_v(x'',T;x',0)=\lambda_v(-8T)|4T|_v^{-\frac{1}{2}}\chi_v\left(-\frac{\lambda^2T^3}{24}+[\lambda(x''+x')-2]\frac{T}{4}+\frac{(x''-x')^2}{8T}\right)\Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (4.10b)$$

The adelic wave function for the simplest ground state has the form

$$\psi_A(x)=\psi_\infty(x)\prod_{p\in P}\Omega\left(|x|_p\right)=\begin{cases}\psi_\infty(x), x\in Z\\ 0, x\in Q\setminus Z\end{cases}, \quad (4.11)$$

where $\Omega\left(|x|_p\right)=1$ if $|x|_p\le 1$ and $\Omega\left(|x|_p\right)=0$ if $|x|_p>1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to p-adic effects in adelic approach. The Gel'fand-Graev-Tate gamma and beta functions are:

$$\Gamma_\infty(a)=\int_R |x|_\infty^{a-1}\chi_\infty(x)d_\infty x=\frac{\zeta(1-a)}{\zeta(a)}, \quad \Gamma_p(a)=\int_{Q_p}|x|_p^{a-1}\chi_p(x)d_p x=\frac{1-p^{a-1}}{1-p^{-a}}, \quad (4.12)$$

$$B_\infty(a,b)=\int_R |x|_\infty^{a-1}|1-x|_\infty^{b-1}d_\infty x=\Gamma_\infty(a)\Gamma_\infty(b)\Gamma_\infty(c), \quad (4.13)$$

$$B_p(a,b)=\int_{Q_p}|x|_p^{a-1}|1-x|_p^{b-1}d_p x=\Gamma_p(a)\Gamma_p(b)\Gamma_p(c), \quad (4.14)$$

where $a,b,c\in C$ with condition $a+b+c=1$ and $\zeta(a)$ is the Riemann zeta function. With a regularization of the product of p-adic gamma functions one has adelic products:

$$\Gamma_\infty(u)\prod_{p\in P}\Gamma_p(u)=1, \quad B_\infty(a,b)\prod_{p\in P}B_p(a,b)=1, \quad u\ne 0,1, \quad u=a,b,c, \quad (4.15)$$

where $a+b+c=1$. We note that $B_\infty(a,b)$ and $B_p(a,b)$ are the crossing symmetric standard and p-adic Veneziano amplitudes for scattering of two open tachyon strings. Introducing real, p-adic and adelic zeta functions as

$$\zeta_\infty(a)=\int_R \exp\left(-\pi x^2\right)|x|_\infty^{a-1}d_\infty x=\pi^{-\frac{a}{2}}\Gamma\left(\frac{a}{2}\right), \quad (4.16)$$

$$\zeta_p(a)=\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|_p^{a-1}d_p x=\frac{1}{1-p^{-a}}, \quad \mathrm{Re}\,a>1, \quad (4.17)$$

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a), \quad (4.18)$$

one obtains

$$\zeta_A(1-a)=\zeta_A(a), \quad (4.19)$$

where $\zeta_A(a)$ can be called adelic zeta function. We have also that

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R \exp\left(-\pi x^2\right)|x|_\infty^{a-1}d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|_p^{a-1}d_p x. \quad (4.19b)$$

Let us note that $\exp\left(-\pi x^2\right)$ and $\Omega\left(|x|_p\right)$ are analogous functions in real and p-adic cases. Adelic harmonic oscillator has connection with the Riemann zeta function. The simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

$$\psi_A(x)=2^{\frac{1}{4}}e^{-\pi x_\infty^2}\prod_{p\in P}\Omega\left(|x_p|_p\right), \quad (4.20)$$

whose the Fourier transform

$$\psi_A(k)=\int\chi_A(kx)\psi_A(x)=2^{\frac{1}{4}}e^{-\pi k_\infty^2}\prod_{p\in P}\Omega\left(|k_p|_p\right) \quad (4.21)$$

has the same form as $\psi_A(x)$. The Mellin transform of $\psi_A(x)$ is

$$\Phi_A(a)=\int\psi_A(x)|x|^a d_A^\times x=\int_R\psi_\infty(x)|x|^{a-1}d_\infty x\prod_{p\in P}\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|^{a-1}d_p x=\sqrt{2}\Gamma\left(\frac{a}{2}\right)\pi^{-\frac{a}{2}}\zeta(a) \quad (4.22)$$

and the same for $\psi_A(k)$. Then according to the Tate formula one obtains (4.19).
The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p=\frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right], \quad (4.23)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L=\sum_{n\geq 1}C_n\mathcal{L}_n=\sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n=\frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1}n^{-\frac{\Box}{2}}\phi+\sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (4.24)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}}, \quad s=\sigma+i\tau, \quad \sigma>1. \quad (4.25)$$

Employing usual expansion for the logarithmic function and definition (4.25) we can rewrite (4.24) in the form

$$L=-\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi+\phi+\ln(1-\phi)\right], \quad (4.26)$$

where $|\phi|<1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x)=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk\,,\quad -k^2=k_0^2-\vec{k}^2>2+\varepsilon\,,\quad (4.27)$$

where $\tilde{\phi}(k)=\int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}\,,\quad (4.28)$$

which has an evident solution $\phi=0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}\,.\quad (4.29)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\geq1}\theta^{\frac{n(n-1)}{2}}\phi^n\,,\quad (4.30)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\geq1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right],\quad (4.31)$$

and one can easily see trivial solution $\phi=\theta=0$.

The exact tree-level Lagrangian of effective scalar field φ, which describes open p-adic string tachyon, is:

$$\mathcal{L}_p=\frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2m_p^2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right],\quad (4.32)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$, as above. Now, we want to introduce a model which incorporates all the above string Lagrangians (4.32) with p replaced by $n\in N$. Thence, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = \sum_{n=1}^{+\infty} C_n \frac{m_n^D}{g_n^2} \frac{n^2}{n-1} \left[-\frac{1}{2} \phi n^{-\frac{\Box}{2m_n^2}} \phi + \frac{1}{n+1} \phi^{n+1} \right], \quad (4.33)$$

whose explicit realization depends on particular choice of coefficients C_n, masses m_n and coupling constants g_n.

Now, we consider the following case

$$C_n = \frac{n-1}{n^{2+h}}, \quad (4.34)$$

where h is a real number. The corresponding Lagrangian reads

$$L_h = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2} - h} \phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right] \quad (4.35)$$

and it depends on parameter h. According to the Euler product formula one can write

$$\sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2} - h} = \prod_p \frac{1}{1 - p^{-\frac{\Box}{2m^2} - h}}. \quad (4.36)$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1, \quad (4.37)$$

which has analytic continuation to the entire complex s plane, excluding the point $s = 1$, where it has a simple pole with residue 1. Employing definition (4.37) we can rewrite (4.35) in the form

$$L_h = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \zeta\left(\frac{\Box}{2m^2} + h \right) \phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right]. \quad (4.38)$$

Here $\zeta\left(\frac{\Box}{2m^2} + h \right)$ acts as a pseudodifferential operator

$$\zeta\left(\frac{\Box}{2m^2} + h \right) \phi(x) = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{2m^2} + h \right) \tilde{\phi}(k) dk, \quad (4.39)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)} \phi(x) dx$ is the Fourier transform of $\phi(x)$.

We consider Lagrangian (4.38) with analytic continuations of the zeta function and the power series $\sum \frac{n^{-h}}{n+1} \phi^{n+1}$, i.e.

$$L_h = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \zeta\left(\frac{\Box}{2m^2} + h \right) \phi + AC \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right], \quad (4.40)$$

where AC denotes analytic continuation.
Potential of the above zeta scalar field (4.40) is equal to $-L_h$ at $\Box = 0$, i.e.

$$V_h(\phi) = \frac{m^D}{g^2}\left(\frac{\phi^2}{2}\zeta(h) - AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right), \quad (4.41)$$

where $h \neq 1$ since $\zeta(1) = \infty$. The term with ζ -function vanishes at $h = -2, -4, -6, \ldots$. The equation of motion in differential and integral form is

$$\zeta\left(\frac{\Box}{2m^2} + h\right)\phi = AC\sum_{n=1}^{+\infty} n^{-h}\varphi^n\,, \quad (4.42)$$

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2} + h\right)\tilde{\phi}(k)dk = AC\sum_{n=1}^{+\infty} n^{-h}\phi^n\,, \quad (4.43)$$

respectively.
Now, we consider five values of h, which seem to be the most interesting, regarding the Lagrangian (4.40): $h = 0$, $h = \pm 1$, and $h = \pm 2$. For $h = -2$, the corresponding equation of motion now read:

$$\zeta\left(\frac{\Box}{2m^2} - 2\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2} - 2\right)\tilde{\phi}(k)dk = \frac{\phi(\phi+1)}{(1-\phi)^3}. \quad (4.44)$$

This equation has two trivial solutions: $\phi(x) = 0$ and $\phi(x) = -1$. Solution $\phi(x) = -1$ can be also shown taking $\tilde{\phi}(k) = -\delta(k)(2\pi)^D$ and $\zeta(-2) = 0$ in (4.44).
For $h = -1$, the corresponding equation of motion is:

$$\zeta\left(\frac{\Box}{2m^2} - 1\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2} - 1\right)\tilde{\phi}(k)dk = \frac{\phi}{(1-\phi)^2}. \quad (4.45)$$

where $\zeta(-1) = -\frac{1}{12}$.
The equation of motion (4.45) has a constant trivial solution only for $\phi(x) = 0$.
For $h = 0$, the equation of motion is

$$\zeta\left(\frac{\Box}{2m^2}\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (4.46)$$

It has two solutions: $\phi = 0$ and $\phi = 3$. The solution $\phi = 3$ follows from the Taylor expansion of the Riemann zeta function operator

$$\zeta\left(\frac{\Box}{2m^2}\right) = \zeta(0) + \sum_{n\geq 1}\frac{\zeta^{(n)}(0)}{n!}\left(\frac{\Box}{2m^2}\right)^n, \quad (4.47)$$

as well as from $\tilde{\phi}(k) = (2\pi)^D 3\delta(k)$.
For $h = 1$, the equation of motion is:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+1\right)\tilde{\phi}(k)dk = -\frac{1}{2}\ln(1-\phi)^2\,, \quad (4.48)$$

where $\zeta(1)=\infty$ gives $V_1(\phi)=\infty$.
In conclusion, for $h=2$, we have the following equation of motion:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+2\right)\tilde{\phi}(k)dk = -\int_0^{\phi}\frac{\ln(1-w)^2}{2w}dw\,. \quad (4.49)$$

Since holds equality

$$-\int_0^1\frac{\ln(1-w)}{w}dw = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2)$$

one has trivial solution $\phi=1$ in (4.49).

Now, we want to analyze the following case: $C_n = \frac{n^2-1}{n^2}$. In this case, from the Lagrangian (4.33), we obtain:

$$L = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\left\{\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)\right\}\phi+\frac{\phi^2}{1-\phi}\right]. \quad (4.50)$$

The corresponding potential is:

$$V(\phi) = -\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2\,. \quad (4.51)$$

We note that 7 and 31 are prime natural numbers, i.e. $6n\pm1$ with $n=1$ and 5, with 1 and 5 that are Fibonacci's numbers. Furthermore the number 24 is related to the Ramanujan function that has 24 "modes" that correspond to the physical vibrations of a bosonic string. Thence, we obtain:

$$V(\phi) = -\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2 \Rightarrow \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\,. \quad (4.51b)$$

The equation of motion is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)\right]\phi = \frac{\phi\left[(\phi-1)^2+1\right]}{(\phi-1)^2}\,. \quad (4.52)$$

Its weak field approximation is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-2\right]\phi = 0\,, \quad (4.53)$$

which implies condition on the mass spectrum

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right)=2. \quad (4.54)$$

From (4.54) it follows one solution for $M^2>0$ at $M^2\approx 2.79m^2$ and many tachyon solutions when $M^2<-38m^2$.
We note that the number 2.79 in connected with the ϕ and Φ, i.e. the "aureo" numbers. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\left(\frac{\sqrt{5}-1}{2}\right)\cong 2.78$$

With regard the extension by ordinary Lagrangian, we have the Lagrangian, potential, equation of motion and mass spectrum condition that, when $C_n=\frac{n^2-1}{n^2}$, are:

$$L=\frac{m^D}{g^2}\left[\frac{\phi}{2}\left\{\frac{\Box}{m^2}-\zeta\left(\frac{\Box}{2m^2}-1\right)-\zeta\left(\frac{\Box}{2m^2}\right)-1\right\}\phi+\frac{\phi^2}{2}\ln\phi^2+\frac{\phi^2}{1-\phi}\right], \quad (4.55)$$

$$V(\phi)=\frac{m^D}{g^2}\frac{\phi^2}{2}\left[\zeta(-1)+\zeta(0)+1-\ln\phi^2-\frac{1}{1-\phi}\right], \quad (4.56)$$

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-\frac{\Box}{m^2}+1\right]\phi=\phi\ln\phi^2+\phi+\frac{2\phi-\phi^2}{(1-\phi)^2}, \quad (4.57)$$

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right)=\frac{M^2}{m^2}. \quad (4.58)$$

In addition to many tachyon solutions, equation (4.58) has two solutions with positive mass: $M^2\approx 2.67m^2$ and $M^2\approx 4.66m^2$.
We note, also here, that the numbers 2.67 and 4.66 are related to the "aureo" numbers. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\frac{1}{2\cdot 5}\left(\frac{\sqrt{5}-1}{2}\right)\cong 2.6798,$$

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\left(\frac{\sqrt{5}+1}{2}\right)+\frac{1}{2\cdot 2}\left(\frac{\sqrt{5}+1}{2}\right)\cong 4.64057.$$

5. Mathematical connections

In this section we want to show some interesting mathematical connections that we have obtained between various equations regarding the **Sections 1**, **2** and **4**. Before of this, we want to describe about the Ramanujan's sum for the mathematical connections concerning some equations of **Section 1**.
In mathematics, Ramanujan's sum, named for Srinivasa Ramanujan and usually denoted $c_q(m)$, is defined to be

$$c_q(m) = \sum_{h^*(q)} e^{2\pi i hm/q} = \sum_{h^*(q)} \exp\left(2\pi i \frac{hm}{q}\right), \quad (5.1)$$

where h runs through the residues *relatively prime* to q, which is important in the representation of numbers by the sums of squares. If $(q,q')=1$ (i.e., q and q' are *relatively prime*), then

$$c_{qq'}(m) = c_q(m)c_{q'}(m), \quad (5.2)$$

hence, the Ramanujan's sum is multiplicative. For argument 1, we have:

$$c_b(1) = \mu(b), \quad (5.3)$$

where μ is the Mobius Function and, for general m, we have:

$$c_b(m) = \mu\left(\frac{b}{(b,m)}\right)\frac{\phi(b)}{\phi\left(\frac{b}{(b,m)}\right)}. \quad (5.4)$$

S. Ramanujan in his paper "Modular equations and approximations to π" (1914), gives various series concerning $1/\pi$ and related to the Ramanujan's sum. Now, we show some interesting equations concerning this important argument. We have:

$$\pi = \frac{12}{\sqrt{130}}\log\left[\frac{\left(2+\sqrt{5}\right)\left(3+\sqrt{13}\right)}{\sqrt{2}}\right], \quad (5.5) \quad \text{hence} \quad 12 = \frac{\pi\sqrt{130}}{\log\left[\frac{\left(2+\sqrt{5}\right)\left(3+\sqrt{13}\right)}{\sqrt{2}}\right]}. \quad (5.6)$$

$$\pi = \frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right], \quad (5.7) \quad \text{hence}$$

$$24 = \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (5.8)$$

$$1 - \frac{3}{\pi\sqrt{n}} - 24\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \ldots\right) = \left(\frac{K}{\pi}\right)^2 A(k), \quad (5.9) \quad \text{hence}$$

$$24 = \left[1 - \left(\frac{K}{\pi}\right)^2 A(k) - \frac{3}{\pi\sqrt{n}}\right] \cdot \frac{1}{\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \ldots\right)}. \quad (5.10)$$

We remember that the number 24 (hence also 12 = 24/2), correspond to the Ramanujan function that has 24 "modes" that correspond to the physical vibrations of a bosonic string.
Now, from the eq. (1.7b), we obtain the following mathematical connection:

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R + \frac{2}{\ell^2}\right) \Rightarrow c_L = c_R = 3\ell/2G = 24k \Rightarrow$$

$$\Rightarrow \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow \left[1 - \left(\frac{K}{\pi}\right)^2 A(k) - \frac{3}{\pi\sqrt{n}}\right] \cdot \frac{1}{\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \ldots\right)}. \quad (5.11)$$

Also the eqs. (1.11), (1.12), (1.15) and (1.74)-(1.77) can be connected with eqs. (5.8) and (5.10). Hence, we obtain:

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \Rightarrow \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow \left[1 - \left(\frac{K}{\pi}\right)^2 A(k) - \frac{3}{\pi\sqrt{n}}\right] \cdot \frac{1}{\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \ldots\right)}. \quad (5.12)$$

$$Z_{0,1}(\tau) = \frac{1}{|\eta(\tau)|^2}|\bar{q}q|^{-(k-1/24)}|1-q|^2 \Rightarrow$$

$$\Rightarrow \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow \left[1 - \left(\frac{K}{\pi}\right)^2 A(k) - \frac{3}{\pi\sqrt{n}}\right] \cdot \frac{1}{\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \ldots\right)}. \quad (5.13)$$

$$Z_{0,1}(\tau) = \left|q^{-k}\prod_{n=2}^{\infty}(1-q^n)^{-1}\right|^2 = \frac{|\bar{q}q|^{-k+1/24}|1-q|^2}{|\eta(\tau)|^2} \Rightarrow$$

$$\Rightarrow \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow \left[1 - \left(\frac{K}{\pi}\right)^2 A(k) - \frac{3}{\pi\sqrt{n}}\right] \cdot \frac{1}{\left(\frac{1}{e^{2\pi\sqrt{n}}-1} + \frac{2}{e^{4\pi\sqrt{n}}-1} + \ldots\right)}. \quad (5.14)$$

$$G(\tau) = \sum\left(|q|^{-k^*+3/24}\left|1+q^{1/2}\right|^2 \frac{|\eta(\tau/2)|^2}{|\eta(\tau)|^4}\right)\Big|_{\gamma} \Rightarrow$$

$$\Rightarrow \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow \left[1-\left(\frac{K}{\pi}\right)^2 A(k)-\frac{3}{\pi\sqrt{n}}\right]\cdot\frac{1}{\left(\frac{1}{e^{2\pi\sqrt{n}}-1}+\frac{2}{e^{4\pi\sqrt{n}}-1}+\ldots\right)}. \quad (5.15)$$

$$G(\tau)=\frac{|\eta(\tau/2)|^2}{y^{1/2}|\eta(\tau)|^4}\sum\left(y^{1/2}|q|^{-k^*+3/24}\left|1-q^{1/2}\right|^2\right)_\gamma =$$

$$=\frac{|\eta(\tau/2)|^2}{y^{1/2}|\eta(\tau)|^4}\left(\hat{E}(k^*-3/24,0)+\hat{E}(k^*+1-3/24,0)+\hat{E}(k^*+1/2-3/24,1/2)+\hat{E}(k^*+1/2-3/24,-1/2)\right)\Rightarrow$$

$$\Rightarrow \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow \left[1-\left(\frac{K}{\pi}\right)^2 A(k)-\frac{3}{\pi\sqrt{n}}\right]\cdot\frac{1}{\left(\frac{1}{e^{2\pi\sqrt{n}}-1}+\frac{2}{e^{4\pi\sqrt{n}}-1}+\ldots\right)}. \quad (5.16)$$

With regard the Ramanujan's sum, we have the following mathematical connections between the eqs. (1.30h) and (1.36) and (5.1), (5.8) and (5.10):

$$S(\hat{n},0;c)=\sum_{d\in(Z/cZ)^*} e^{2\pi i d/c}=\mu(c)\Rightarrow c_q(m)=\sum_{h^*(q)} e^{2\pi i hm/q}=\sum_{h^*(q)}\exp\left(2\pi i\frac{hm}{q}\right)\Rightarrow$$

$$\Rightarrow \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow \left[1-\left(\frac{K}{\pi}\right)^2 A(k)-\frac{3}{\pi\sqrt{n}}\right]\cdot\frac{1}{\left(\frac{1}{e^{2\pi\sqrt{n}}-1}+\frac{2}{e^{4\pi\sqrt{n}}-1}+\ldots\right)}. \quad (5.17)$$

$$S(\hat{n},0;c)=\sum_{d\in(Z/cZ)^*} e^{2\pi i\hat{n}d/c}=\sum_{\delta|\hat{n}}\mu(\delta)\Rightarrow c_q(m)=\sum_{h^*(q)} e^{2\pi i hm/q}=\sum_{h^*(q)}\exp\left(2\pi i\frac{hm}{q}\right)\Rightarrow$$

$$\Rightarrow \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow \left[1-\left(\frac{K}{\pi}\right)^2 A(k)-\frac{3}{\pi\sqrt{n}}\right]\cdot\frac{1}{\left(\frac{1}{e^{2\pi\sqrt{n}}-1}+\frac{2}{e^{4\pi\sqrt{n}}-1}+\ldots\right)}. \quad (5.18)$$

With regard the Selberg zeta function, we have the following mathematical connections between eqs. (1.27), (1.28), (1.29), (1.30c), (1.32), (1.33), (1.34), (1.42), (1.85), (1.87) of **Section 1** and various equations of **Section 2** and **Section 4**. With regard the **Section 2**, we have some mathematical connections with the eqs. (2.68), (2.69), (2.76), (2.87), (2.90b), (2.107b), (2.113b) and (2.119b). Indeed, we have obtained that:

$$\varphi(s)=\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow\frac{2\pi^{m+1/2}}{m\Gamma(m+1/2)\zeta(2m+1)}T_m(\kappa). \quad (5.19)$$

$$\varphi(s)=\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow\sum_{c>0}c^{-2(m+s)}S(0,0,2c)=\sum_{c>0}c^{-2(m+s)}\phi(2c)=\frac{2^{2(m+s)}}{2^{2(m+s)}-1}\frac{\zeta(2(m+s)-1)}{\zeta(2(m+s))}. \quad (5.20)$$

$$\begin{aligned}&\varphi(s)=\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow\\ &\Rightarrow\cos\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{m-1}{2}+s\right)}{m!\Gamma(m+s)}2F_1\left(\frac{m-1}{2}+s,-\frac{m}{2}+s;\frac{1}{2};\kappa^2\right)+\\ &+m\kappa\sin\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+s\right)}{m!\Gamma(m+s)}2F_1\left(\frac{1-m}{2},\frac{m}{2}+s;\frac{3}{2};\kappa^2\right)\Rightarrow\\ &\Rightarrow\frac{\sin\pi s}{\pi}\int_0^\infty\frac{d\lambda}{[\lambda(\lambda+2c)]^s}\cdot 2\sum_{\gamma\in\Gamma}\sum_{k=1}^\infty\frac{l_\gamma}{1-e^{-kl_\gamma}}e^{-kl_\gamma(\lambda+c+1/2)}\left[\cosh\left(\frac{m+1}{2}kl_\gamma\right)-\chi_\gamma^k\cosh\left(\frac{m}{2}kl_\gamma\right)\right]=\\ &=\frac{\sin\pi s}{\pi}\int_0^\infty\frac{d\lambda}{[\lambda(\lambda+2c)]^s}\frac{d}{d\lambda}\ln\left[\frac{Z_0\left(\frac{m}{2}+\lambda+c+1\right)Z_0\left(\lambda+c-\frac{m}{2}\right)}{Z_1\left(\frac{m+1}{2}+\lambda+c\right)Z_1\left(\frac{m-1}{2}+\lambda+c\right)}\right]. \quad (5.21)\end{aligned}$$

$$\begin{aligned}&\varphi(s)=\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow\\ &\Rightarrow\cos\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{m-1}{2}+s\right)}{m!\Gamma(m+s)}2F_1\left(\frac{m-1}{2}+s,-\frac{m}{2}+s;\frac{1}{2};\kappa^2\right)+\\ &+m\kappa\sin\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+s\right)}{m!\Gamma(m+s)}2F_1\left(\frac{1-m}{2},\frac{m}{2}+s;\frac{3}{2};\kappa^2\right)\Rightarrow\\ &\Rightarrow(g-1)\sum_{k=0}^m\ln(c^2-k^2)+\int_0^\infty d\lambda\frac{d}{d\lambda}\ln\left[\frac{Z_0\left(\lambda+\frac{m}{2}+c+1\right)Z_0\left(\lambda-\frac{m}{2}+c\right)}{Z_1\left(\lambda+\frac{m+1}{2}+c\right)Z_1\left(\lambda+\frac{1-m}{2}+c\right)}\right]=\\ &=(g-1)\sum_{k=0}^m\ln(c^2-k^2)-\ln\left[\frac{Z_0\left(c+1+\frac{m}{2}\right)Z_0\left(c-\frac{m}{2}\right)}{Z_1\left(c+\frac{m+1}{2}\right)Z_1\left(c+\frac{1-m}{2}\right)}\right]. \quad (5.22)\end{aligned}$$

$$\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} \Rightarrow \sqrt{\pi}\frac{\zeta(1+2s)\Gamma(s+1/2)}{\zeta(2+2s)\Gamma(1+s)} + O(y^{-1}) \Rightarrow$$

$$\Rightarrow \int_{sM_g} d\mu_{SWP}\left[s\det'\left(-\square_0^2\right)\right]^{-5/2}\left[s\det\left(-\square_2^2\right)\right]^{1/2} = \left(\frac{1}{2\pi^4}\right)^{g-1}\int_{sM_g} d\mu_{SWP}(-1)^{5/2\Delta n_0^{(0)}}\left(\frac{Z_0(1)}{\tilde{Z}_1\left(\frac{1}{2}\right)}\right)^{-5}\frac{Z_0(2)}{Z_1\left(\frac{3}{2}\right)}\left(\frac{Z_1(1)}{Z_1(0)}\right)^2$$

(5.23).

$$\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} \Rightarrow \sqrt{\pi}\frac{\zeta(1+2s)\Gamma(s+1/2)}{\zeta(2+2s)\Gamma(1+s)} + O(y^{-1}) \Rightarrow$$

$$\Rightarrow \Delta(s)\exp\left\{-A\dim V\int_0^{s-\frac{1}{2}} t\tan\pi t dt\right\}\left[\frac{\Gamma\left(\frac{3}{2}-s\right)}{\Gamma\left(s+\frac{1}{2}\right)}\right]^{\kappa_0} Z(s)\times$$

$$\times\exp\left\{\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{tr_V\left[U^k(R)\right]}{\nu\sin(2k\pi/\nu)}\sum_{l=1}^{\infty}\left(2l\frac{k\pi}{\nu}\right)\ln\left|\frac{(s+l)(s+l-1)}{(s-l)(s-l-1)}\right| + (1-2s)\right.$$

$$\left.\left[\kappa_0\ln 2 + \sum_{\alpha=1}^{\nu}\sum_{l=1+k_\alpha}^{\dim V}\ln\left|1-e^{2\pi i\theta_{l\alpha}}\right|\right] - i\arg\Delta\left(\frac{1}{2}\right)\right\}. \quad (5.24)$$

$$\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} \Rightarrow \sqrt{\pi}\frac{\zeta(1+2s)\Gamma(s+1/2)}{\zeta(2+2s)\Gamma(1+s)} + O(y^{-1}) \Rightarrow$$

$$\Rightarrow \text{const.}\, e^{4\left[\kappa_0+\kappa_-\ln\left|s\det(1U(S))\right|\right]}\times\left(\frac{1}{\Delta(s)\Delta(s+1)}\right)^{\kappa_-}\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\kappa_--\kappa_0}\left(\frac{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(\frac{3}{2}-s\right)}\right)^{\tilde{\kappa}_0}$$

$$\exp\left\{-2\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{str\left[U^k(R)\right]}{\nu\sin(2k\pi/\nu)}\sum_{l=1}^{\infty}\sin\left(\frac{2lk\pi}{\nu}\right)\times\right.$$

$$\left.\times\left[2\left(1-2\chi_R^k\cos\frac{k\pi}{\nu}\right)\ln\left|\frac{s+l}{s-l}\right| + \ln\left|\frac{(s+l-1)(s+l+1)}{(s-l+1)(s-l-1)}\right|\right]\right\}. \quad (5.25)$$

In conclusion, we have various mathematical connections with some equations of **Section 4**. It is possible to obtain connections between eqs. (4.28)-(4.31), (4.43)-(4.46), (4.48)-(4.49), and eqs. (5.21)-(5.24). Indeed, as example most importants, we have that:

$$\varphi(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} \Rightarrow$$

$$\Rightarrow \cos\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{m-1}{2}+s\right)}{m!\Gamma(m+s)}\,2F_1\left(\frac{m-1}{2}+s,-\frac{m}{2}+s;\frac{1}{2};\kappa^2\right)+$$

$$+m\kappa\sin\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+s\right)}{m!\Gamma(m+s)}\,2F_1\left(\frac{1-m}{2},\frac{m}{2}+s;\frac{3}{2};\kappa^2\right)\Rightarrow$$

$$\Rightarrow \frac{\sin\pi s}{\pi}\int_0^\infty \frac{d\lambda}{[\lambda(\lambda+2c)]^s}\cdot 2\sum_{\gamma\in\Gamma}\sum_{k=1}^\infty \frac{l_\gamma}{1-e^{-kl_\gamma}} e^{-kl_\gamma(\lambda+c+1/2)}\left[\cosh\left(\frac{m+1}{2}kl_\gamma\right)-\chi_\gamma^k\cosh\left(\frac{m}{2}kl_\gamma\right)\right]=$$

$$=\frac{\sin\pi s}{\pi}\int_0^\infty\frac{d\lambda}{[\lambda(\lambda+2c)]^s}\frac{d}{d\lambda}\ln\left[\frac{Z_0\left(\frac{m}{2}+\lambda+c+1\right)Z_0\left(\lambda+c-\frac{m}{2}\right)}{Z_1\left(\frac{m+1}{2}+\lambda+c\right)Z_1\left(\frac{m-1}{2}+\lambda+c\right)}\right]\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}. \qquad (5.26)$$

$$\varphi(s)=\sqrt{\pi}\,\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow$$

$$\Rightarrow \cos\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{m-1}{2}+s\right)}{m!\Gamma(m+s)}\,2F_1\left(\frac{m-1}{2}+s,-\frac{m}{2}+s;\frac{1}{2};\kappa^2\right)+$$

$$+m\kappa\sin\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+s\right)}{m!\Gamma(m+s)}\,2F_1\left(\frac{1-m}{2},\frac{m}{2}+s;\frac{3}{2};\kappa^2\right)\Rightarrow$$

$$\Rightarrow \frac{\sin\pi s}{\pi}\int_0^\infty \frac{d\lambda}{[\lambda(\lambda+2c)]^s}\cdot 2\sum_{\gamma\in\Gamma}\sum_{k=1}^\infty \frac{l_\gamma}{1-e^{-kl_\gamma}} e^{-kl_\gamma(\lambda+c+1/2)}\left[\cosh\left(\frac{m+1}{2}kl_\gamma\right)-\chi_\gamma^k\cosh\left(\frac{m}{2}kl_\gamma\right)\right]=$$

$$=\frac{\sin\pi s}{\pi}\int_0^\infty\frac{d\lambda}{[\lambda(\lambda+2c)]^s}\frac{d}{d\lambda}\ln\left[\frac{Z_0\left(\frac{m}{2}+\lambda+c+1\right)Z_0\left(\lambda+c-\frac{m}{2}\right)}{Z_1\left(\frac{m+1}{2}+\lambda+c\right)Z_1\left(\frac{m-1}{2}+\lambda+c\right)}\right]\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}. \qquad (5.27)$$

$$\varphi(s)=\sqrt{\pi}\,\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow$$

$$\Rightarrow \cos\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{m-1}{2}+s\right)}{m!\Gamma(m+s)}\,2F_1\left(\frac{m-1}{2}+s,-\frac{m}{2}+s;\frac{1}{2};\kappa^2\right)+$$

$$+m\kappa\sin\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+s\right)}{m!\Gamma(m+s)}\,2F_1\left(\frac{1-m}{2},\frac{m}{2}+s;\frac{3}{2};\kappa^2\right)\Rightarrow$$

$$\Rightarrow \frac{\sin \pi s}{\pi}\int_0^\infty \frac{d\lambda}{[\lambda(\lambda+2c)]^s}\cdot 2\sum_{\gamma\in\Gamma}\sum_{k=1}^\infty \frac{l_\gamma}{1-e^{-kl_\gamma}} e^{-kl_\gamma(\lambda+c+1/2)}\left[\cosh\left(\frac{m+1}{2}kl_\gamma\right)-\chi_\gamma^k \cosh\left(\frac{m}{2}kl_\gamma\right)\right]=$$

$$=\frac{\sin \pi s}{\pi}\int_0^\infty \frac{d\lambda}{[\lambda(\lambda+2c)]^s}\frac{d}{d\lambda}\ln\left[\frac{Z_0\left(\frac{m}{2}+\lambda+c+1\right)Z_0\left(\lambda+c-\frac{m}{2}\right)}{Z_1\left(\frac{m+1}{2}+\lambda+c\right)Z_1\left(\frac{m-1}{2}+\lambda+c\right)}\right]\Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-1\right)\tilde{\phi}(k)dk=\frac{\phi}{(1-\phi)^2}. \quad (5.28)$$

$$\varphi(s)=\sqrt{\pi}\,\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow$$

$$\Rightarrow \cos\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{m-1}{2}+s\right)}{m!\Gamma(m+s)}2F_1\left(\frac{m-1}{2}+s,-\frac{m}{2}+s;\frac{1}{2};\kappa^2\right)+$$

$$+m\kappa\sin\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+s\right)}{m!\Gamma(m+s)}2F_1\left(\frac{1-m}{2},\frac{m}{2}+s;\frac{3}{2};\kappa^2\right)\Rightarrow$$

$$\Rightarrow (g-1)\sum_{k=0}^m \ln(c^2-k^2)+\int_0^\infty d\lambda\frac{d}{d\lambda}\ln\left[\frac{Z_0\left(\lambda+\frac{m}{2}+c+1\right)Z_0\left(\lambda-\frac{m}{2}+c\right)}{Z_1\left(\lambda+\frac{m+1}{2}+c\right)Z_1\left(\lambda+\frac{1-m}{2}+c\right)}\right]=$$

$$=(g-1)\sum_{k=0}^m \ln(c^2-k^2)-\ln\left[\frac{Z_0\left(c+1+\frac{m}{2}\right)Z_0\left(c-\frac{m}{2}\right)}{Z_1\left(c+\frac{m+1}{2}\right)Z_1\left(c+\frac{1-m}{2}\right)}\right]\Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}. \quad (5.29)$$

$$\varphi(s)=\sqrt{\pi}\,\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow$$

$$\Rightarrow \cos\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{m-1}{2}+s\right)}{m!\Gamma(m+s)}2F_1\left(\frac{m-1}{2}+s,-\frac{m}{2}+s;\frac{1}{2};\kappa^2\right)+$$

$$+m\kappa\sin\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+s\right)}{m!\Gamma(m+s)}2F_1\left(\frac{1-m}{2},\frac{m}{2}+s;\frac{3}{2};\kappa^2\right)\Rightarrow$$

$$\Rightarrow (g-1)\sum_{k=0}^m \ln(c^2-k^2)+\int_0^\infty d\lambda\frac{d}{d\lambda}\ln\left[\frac{Z_0\left(\lambda+\frac{m}{2}+c+1\right)Z_0\left(\lambda-\frac{m}{2}+c\right)}{Z_1\left(\lambda+\frac{m+1}{2}+c\right)Z_1\left(\lambda+\frac{1-m}{2}+c\right)}\right]=$$

$$= (g-1)\sum_{k=0}^{m} \ln\left(c^2 - k^2\right) - \ln\left[\frac{Z_0\left(c+1+\frac{m}{2}\right)Z_0\left(c-\frac{m}{2}\right)}{Z_1\left(c+\frac{m+1}{2}\right)Z_1\left(c+\frac{1-m}{2}\right)}\right] \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (5.30)$$

$$\varphi(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} \Rightarrow$$

$$\Rightarrow \cos\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m \Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{m-1}{2}+s\right)}{m!\Gamma(m+s)} 2F_1\left(\frac{m-1}{2}+s, -\frac{m}{2}+s; \frac{1}{2}; \kappa^2\right) +$$

$$+ m\kappa \sin\left(\frac{m\pi}{2}\right)\frac{(2\pi)^m \Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+s\right)}{m!\Gamma(m+s)} 2F_1\left(\frac{1-m}{2}, \frac{m}{2}+s; \frac{3}{2}; \kappa^2\right) \Rightarrow$$

$$\Rightarrow (g-1)\sum_{k=0}^{m} \ln\left(c^2 - k^2\right) + \int_0^\infty d\lambda \frac{d}{d\lambda}\ln\left[\frac{Z_0\left(\lambda+\frac{m}{2}+c+1\right)Z_0\left(\lambda-\frac{m}{2}+c\right)}{Z_1\left(\lambda+\frac{m+1}{2}+c\right)Z_1\left(\lambda+\frac{1-m}{2}+c\right)}\right] =$$

$$= (g-1)\sum_{k=0}^{m} \ln\left(c^2 - k^2\right) - \ln\left[\frac{Z_0\left(c+1+\frac{m}{2}\right)Z_0\left(c-\frac{m}{2}\right)}{Z_1\left(c+\frac{m+1}{2}\right)Z_1\left(c+\frac{1-m}{2}\right)}\right] \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-1\right)\tilde{\phi}(k)dk = \frac{\phi}{(1-\phi)^2}. \quad (5.31)$$

$$\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} \Rightarrow \sqrt{\pi}\frac{\zeta(1+2s)\Gamma(s+1/2)}{\zeta(2+2s)\Gamma(1+s)} + \mathrm{O}\left(y^{-1}\right) \Rightarrow$$

$$\Rightarrow \int_{sM_g} d\mu_{SWP}\left[s\det'\left(-\square_0^2\right)\right]^{-5/2}\left[s\det\left(-\square_2^2\right)\right]^{1/2} = \left(\frac{1}{2\pi^4}\right)^{g-1}\int_{sM_g} d\mu_{SWP}(-1)^{5/2\Delta n_0^{(0)}}\left(\frac{Z_0(1)}{\tilde{Z}_1\left(\frac{1}{2}\right)}\right)^{-5}\frac{Z_0(2)}{Z_1\left(\frac{3}{2}\right)}\left(\frac{Z_1(1)}{Z_1(0)}\right)^2 \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (5.32)$$

$$\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow\sqrt{\pi}\frac{\zeta(1+2s)\Gamma(s+1/2)}{\zeta(2+2s)\Gamma(1+s)}+O\left(y^{-1}\right)\Rightarrow$$

$$\Rightarrow\int_{sM_g}d\mu_{SWP}\left[s\det'\left(-\square_0^2\right)\right]^{-5/2}\left[s\det\left(-\square_2^2\right)\right]^{1/2}=\left(\frac{1}{2\pi^4}\right)^{g-1}\int_{sM_g}d\mu_{SWP}(-1)^{5/2\Delta n_0^{(0)}}\left(\frac{Z_0(1)}{\tilde{Z}_1\left(\frac{1}{2}\right)}\right)^{-5}\frac{Z_0(2)}{Z_1\left(\frac{3}{2}\right)}\left(\frac{Z_1(1)}{Z_1(0)}\right)^2\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}. \quad (5.33)$$

$$\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow\sqrt{\pi}\frac{\zeta(1+2s)\Gamma(s+1/2)}{\zeta(2+2s)\Gamma(1+s)}+O\left(y^{-1}\right)\Rightarrow$$

$$\Rightarrow\int_{sM_g}d\mu_{SWP}\left[s\det'\left(-\square_0^2\right)\right]^{-5/2}\left[s\det\left(-\square_2^2\right)\right]^{1/2}=\left(\frac{1}{2\pi^4}\right)^{g-1}\int_{sM_g}d\mu_{SWP}(-1)^{5/2\Delta n_0^{(0)}}\left(\frac{Z_0(1)}{\tilde{Z}_1\left(\frac{1}{2}\right)}\right)^{-5}\frac{Z_0(2)}{Z_1\left(\frac{3}{2}\right)}\left(\frac{Z_1(1)}{Z_1(0)}\right)^2\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-1\right)\tilde{\phi}(k)dk=\frac{\phi}{(1-\phi)^2}. \quad (5.34)$$

$$\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow\sqrt{\pi}\frac{\zeta(1+2s)\Gamma(s+1/2)}{\zeta(2+2s)\Gamma(1+s)}+O\left(y^{-1}\right)\Rightarrow$$

$$\Rightarrow\Delta(s)\exp\left\{-A\dim V\int_0^{s-\frac{1}{2}}t\tan\pi t\,dt\right\}\left[\frac{\Gamma\left(\frac{3}{2}-s\right)}{\Gamma\left(s+\frac{1}{2}\right)}\right]^{\kappa_0}Z(s)\times$$

$$\times\exp\left\{\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{tr_V\left[U^k(R)\right]}{\nu\sin(2k\pi/\nu)}\sum_{l=1}^{\infty}\left(2l\frac{k\pi}{\nu}\right)\ln\left|\frac{(s+l)(s+l-1)}{(s-l)(s-l-1)}\right|+(1-2s)\right.$$

$$\left.\left[\kappa_0\ln 2+\sum_{\alpha=1}^{\nu}\sum_{l=1+k_\alpha}^{\dim V}\ln\left|1-e^{2\pi i\theta_{l\alpha}}\right|\right]-i\arg\Delta\left(\frac{1}{2}\right)\right\}\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}. \quad (5.35)$$

$$\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow\sqrt{\pi}\frac{\zeta(1+2s)\Gamma(s+1/2)}{\zeta(2+2s)\Gamma(1+s)}+O\left(y^{-1}\right)\Rightarrow$$

$$\Rightarrow\Delta(s)\exp\left\{-A\dim V\int_0^{s-\frac{1}{2}}t\tan\pi t\,dt\right\}\left[\frac{\Gamma\left(\frac{3}{2}-s\right)}{\Gamma\left(s+\frac{1}{2}\right)}\right]^{\kappa_0}Z(s)\times$$

$$\times\exp\left\{\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{tr_V\left[U^k(R)\right]}{\nu\sin(2k\pi/\nu)}\sum_{l=1}^{\infty}\left(2l\frac{k\pi}{\nu}\right)\ln\left|\frac{(s+l)(s+l-1)}{(s-l)(s-l-1)}\right|+(1-2s)\right.$$

$$\left.\left[\kappa_0\ln 2+\sum_{\alpha=1}^{\nu}\sum_{l=1+k_\alpha}^{\dim V}\ln\left|1-e^{2\pi i\theta_{l\alpha}}\right|\right]-i\arg\Delta\left(\frac{1}{2}\right)\right\}\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}. \quad (5.36)$$

$$\sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\Rightarrow\sqrt{\pi}\frac{\zeta(1+2s)\Gamma(s+1/2)}{\zeta(2+2s)\Gamma(1+s)}+O\left(y^{-1}\right)\Rightarrow$$

$$\Rightarrow\Delta(s)\exp\left\{-A\dim V\int_0^{s-\frac{1}{2}}t\tan\pi t dt\right\}\left[\frac{\Gamma\left(\frac{3}{2}-s\right)}{\Gamma\left(s+\frac{1}{2}\right)}\right]^{\kappa_0}Z(s)\times$$

$$\times\exp\left\{\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{tr_V\left[U^k(R)\right]}{\nu\sin(2k\pi/\nu)}\sum_{l=1}^{\infty}\left(2l\frac{k\pi}{\nu}\right)\ln\left|\frac{(s+l)(s+l-1)}{(s-l)(s-l-1)}\right|+(1-2s)\right.$$

$$\left.\left[\kappa_0\ln 2+\sum_{\alpha=1}^{\nu}\sum_{l=1+k_\alpha}^{\dim V}\ln\left|1-e^{2\pi i\theta_{l\alpha}}\right|\right]-i\arg\Delta\left(\frac{1}{2}\right)\right\}\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-1\right)\tilde{\phi}(k)dk=\frac{\phi}{(1-\phi)^2}. \quad (5.37)$$

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http://www.secamlocal.ex.ac.uk/people/staff/mrwatkin/zeta/physics.htm

References

[1] Alexander Maloney and Edward Witten: "*Quantum Gravity Partition Functions in Three Dimensions*" – arXiv:0712.0155v1 [hep-th] – 2 Dec 2007.

[2] S. Carlip: "*Conformal Field Theory, (2+1)-Dimensional Gravity, and the BTZ Black Hole*" - arXiv:gr-qc/0503022v4 – 6 Jun 2005.

[3] Christian Grosche: "*Selberg Supertrace Formula for Super Riemann Surfaces, Analytic*

Properties of Selberg Super Zeta-Functions and Multiloop Contributions for the Fermionic String" – Commun. Math. Phys. 133, 433-485 (1990).

[4] C. Grosche: "*Selberg Super-Trace Formula for Super Riemann Surfaces II: Elliptic and Parabolic Conjugacy Classes, and Selberg Super-Zeta Functions*" – Commun. Math. Phys. 151, 1-37 (1993).

[5] Michael B. Green, John H. Schwarz and P. C. West: "*Anomaly-Free Chiral Theories in Six Dimensions*" – Calt-68-1210 – 28 Nov 1984.

[6] I. V. Volovich: "*Baby Universes and the Dimensionality of Space-time*" – NBI-HE-88-79 December 1988.

[7] Sidney Coleman: "*Why There is Nothing Rather than Something: A Theory of the Cosmological Constant*" – HUTP-88/A022 – 4/88 – 88-9-5 (1988).

[8] I. Ya. Aref'eva, B. G. Dragovich and I. V. Volovich: "*Extra Time-Like Dimensions Lead To a Vanishing Cosmological Constant*" – Physics Letters B – Vol.177 18 Sep 1986.

[9] Z. Berezhiani, M. Chaichian, A.B. Kobakhidze and Z. –H. Yu: "*Vanishing of cosmological constant and fully localized gravity in a Brane World with extra time(s)*" – arXiv.hep-th/ 0102207 – 10 Aug 2001.

[10] Branko Dragovich: "*Adelic strings and noncommutativity*" – arXiv:hep-th/0105103v1- 11 May 2001.

[11] Branko Dragovich: "*Adeles in Mathematical Physics*" – arXiv:0707.3876v1 [hep-th] – 26 Jul 2007.

[12] Branko Dragovich: "*Zeta Strings*" – arXiv:hep-th/0703008v1 – 1 Mar 2007.

[13] Branko Dragovich: "*Zeta Nonlocal Scalr Fields*" – arXiv:0804.4114v1 – [hep-th] – 25 Apr 2008.

[14] Branko Dragovich: "*Some Lagrangians with Zeta Function Nonlocality*" – arXiv:0805.0403 v1 – [hep-th] – 4 May 2008.

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