On the possible mathematical connections between the Hartle-Hawking no boundary proposal concerning the Randall-Sundrum cosmological scenario, Hartle-Hawking wave-function in the mini-superspace sector of physical superstring theory, p-adic Hartle-Hawking wave function and some sectors of Number Theory.

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Abstract

In this paper we have described the Hartle-Hawking no boundary proposal concerning the Randall-Sundrum cosmological scenario, nonlocal braneworld action in the two-brane Randall-Sundrum model, Hartle-Hawking wave-function in the mini-superspace sector of physical superstring theory, p-adic models in the Hartle-Hawking proposal and p-adic and adelic wave functions of the universe. Furthermore, we have showed some possible mathematical connections between some equations of these arguments and, in conclusion, we have also described some mathematical connections between some equations of arguments above mentioned and some equations concerning the Riemann zeta function, the Ramanujan's modular equations and the Palumbo-Nardelli model.
In the **section 1**, we have described the Hartle-Hawking "no boundary" proposal applied to Randall-Sundrum cosmological scenario. In the **section 2**, we have described nonlocal braneworld action in the two-brane Randall-Sundrum model. In the **section 3**, we have described the compactifications of type IIB strings on a Calabi-Yau three-fold and Hartle-Hawking wave-function in the mini-superspace sector of physical superstring theory. In the **section 4**, we have described the p-Adic models in the Hartle-Hawking proposal. In the **section 5**, we have described the p-Adic and Adelic wave functions of the Universe. In the **section 6**, we have described some equations concerning the Riemann zeta function, specifically, the Goldston-Montgomery Theorem, the study of the behaviour of the argument of the Riemann function $\zeta(s)$ with the condition that s lies on the critical line $s=\frac{1}{2}+it$, where t is real, the P-N Model (Palumbo-Nardelli model) and the Ramanujan identities.
In conclusion, in the **section 7**, we have described some possible mathematical connections between some equations of arguments above discussed and some equations concerning the Riemann zeta-function, the Ramanujan's modular equations and the Palumbo-Nardelli model.

1. Hartle-Hawking "No Boundary" proposal applied to Randall-Sundrum cosmological scenario. Randall-Sundrum from AdS/CFT, CFT in the Domain Wall. [1]

The AdS/CFT correspondence relates IIB supergravity theory in $AdS_5 \times S^5$ to a $\mathrm{N} = 4U(N)$ superconformal field theory. If g_{YM} is the coupling constant of this theory then the't Hooft parameter is defined to be $\lambda = g_{YM}^2 N$. The CFT parameters are related to the supergravity parameters by

$$l = \lambda^{1/4} l_s, \quad (1.1) \qquad \frac{l^3}{G} = \frac{2N^2}{\pi}, \quad (1.2)$$

where l_s is the string length, l the AdS radius and G the *five* dimensional Newton constant. Note that λ and N must be large in order for stringy effects to be small. The CFT lives on the conformal boundary of AdS_5. The correspondence takes the following form:

$$Z[\mathbf{h}] \equiv \int d[\mathbf{g}] \exp(-S_{grav}[\mathbf{g}]) = \int d[\phi] \exp(-S_{CFT}[\phi;\mathbf{h}]) \equiv \exp(-W_{CFT}[\mathbf{h}]), \quad (1.3)$$

here $Z[\mathbf{h}]$ denotes the supergravity partition function in AdS_5. This is given by a path integral over all metrics in AdS_5 which induce a given conformal equivalence class of metrics $\mathbf{h}$ on the conformal boundary of AdS_5. A problem with equation (1.3) as it stands is that the usual gravitational action in AdS is divergent, rendering the path integral ill-defined. A procedure for solving this problem is the following: first one brings the boundary into a finite radius, next one adds a finite number of counterterms to the action in order to render it finite as the boundary is moved back off to infinity. These counterterms can be expressed solely in terms of the geometry of the boundary. The total gravitational action for AdS_{d+1} becomes

$$S_{grav} = S_{EH} + S_{GH} + S_1 + S_2 + ... \quad (1.4)$$

The first term is the usual Einstein-Hilbert action with a negative cosmological constant:

$$S_{EH} = -\frac{1}{16\pi G} \int d^{d+1}x \sqrt{g} \left[R + \frac{d(d-1)}{l^2} \right] \quad (1.5)$$

the overall minus sign arises because we are considering a Euclidean theory. The second term in the action is the Gibbons-Hawking boundary term, which is necessary for a well-defined variational problem:

$$S_{GH} = -\frac{1}{8\pi G} \int d^d x \sqrt{h} K, \quad (1.6)$$

where K is the trace of the extrinsic curvature of the boundary and h the determinant of the induced metric. The first two counterterms are given by the following expressions:

$$S_1 = \frac{d-1}{8\pi G l} \int d^d x \sqrt{h}, \quad (1.7) \qquad S_2 = \frac{l}{16\pi G(d-2)} \int d^d x \sqrt{h} R, \quad (1.8)$$

where R now refers to the Ricci scalar of the boundary metric. The third counterterm is

$$S_3 = \frac{l^3}{16\pi G(d-2)^2(d-4)}\int d^d x\sqrt{h}\left[R_{ij}R^{ij} - \frac{d}{4(d-1)}R^2\right], \quad (1.9)$$

where R_{ij} is the Ricci tensor of the boundary metric and boundary indices i, j are raised and lowered with the boundary metric h_{ij}. This expression is ill-defined for d = 4, which is the case of most interest to us. We can now use the AdS/CFT correspondence to explain the behaviour discovered by Randall and Sundrum. The (Euclidean) RS model has the following action:

$$S_{RS} = S_{EH} + S_{GH} + 2S_1 + S_m. \quad (1.10)$$

Here $2S_1$ is the action of a domain wall with tension $(d-1)/(4\pi Gl)$. The final term is the action for any matter present on the domain wall. We want to study quantum fluctuations of the metric on the domain wall. Let $\mathbf{g_0}$ denote the five dimensional background metric we have just described and $\mathbf{h_0}$ the metric it induces on the wall. Let $\mathbf{h}$ denote a metric perturbation on the wall. If we wish to calculate correlates of $\mathbf{h}$ on the domain wall then we are interested in a path integral of the form

$$\langle h_{ij}(x)h_{i'j'}(x')\rangle = \int d[\mathbf{h}]Z[\mathbf{h}]h_{ij}(x)h_{i'j'}(x'), \quad (1.11)$$

where

$$Z[\mathbf{h}] = \int_{B_1\cup B_2} d[\delta\mathbf{g}]d[\phi]\exp(-S_{RS}[\mathbf{g_0}+\delta\mathbf{g}]) = \exp(-2S_1[\mathbf{h_0}+\mathbf{h}])\times$$
$$\times\int_{B_1\cup B_2} d[\delta\mathbf{g}]d[\phi]\exp(-S_{EH}[\mathbf{g_0}+\delta\mathbf{g}] - S_{GH}[\mathbf{g_0}+\delta\mathbf{g}] - S_m[\phi;\mathbf{h_0}+\mathbf{h}]), \quad (1.12)$$

$\delta\mathbf{g}$ denotes a metric perturbation in the bulk that approaches $\mathbf{h}$ on the boundary and ϕ denotes the matter fields on the domain wall. The integrals in the two balls are independent so we can replace the path integral by

$$Z[\mathbf{h}] = \exp(-2S_1[\mathbf{h_0}+\mathbf{h}])\left(\int_B d[\delta\mathbf{g}]\exp(-S_{EH}[\mathbf{g_0}+\delta\mathbf{g}] - S_{GH}[\mathbf{g_0}+\delta\mathbf{g}])\right)^2\times$$
$$\times\int d[\phi]\exp(-S_m[\phi;\mathbf{h_0}+\mathbf{h}]), \quad (1.13)$$

where B denotes either ball. We now take d = 4 and use the AdS/CFT correspondence (1.3) to replace the path integral over $\delta\mathbf{g}$ by the generating functional for a conformal field theory:

$$\int_B d[\delta\mathbf{g}]\exp(-S_{EH}[\mathbf{g_0}+\delta\mathbf{g}] - S_{GH}[\mathbf{g_0}+\delta\mathbf{g}]) =$$
$$\exp(-W_{RS}[\mathbf{h_0}+\mathbf{h}] + S_1[\mathbf{h_0}+\mathbf{h}] + S_2[\mathbf{h_0}+\mathbf{h}] + S_3[\mathbf{h_0}+\mathbf{h}]). \quad (1.14)$$

This is the RS (Randall-Sundrum) CFT since it arises as the dual of the RS geometry.
Now we will consider the RS analogue of Starobinsky's model by putting a CFT on the domain wall. Our five dimensional (Euclidean) action is the following:

$$S = S_{EH} + S_{GH} + 2S_1 + W_{CFT}. \quad (1.15)$$

We seek a solution in which two balls of AdS_5 are separated by a spherical domain wall. Inside each ball, the metric can be written

$$ds^2 = l^2\left(dy^2 + \sinh^2 y d\Omega_d^2\right), \quad (1.16)$$

with $0 \le y \le y_0$. The domain wall is at $y = y_0$ and has radius

$$R = l \sinh y_0 . \quad (1.17)$$

The effective tension of the domain wall is given by the Israel equations as

$$\sigma_{eff} = \frac{3}{4\pi G l} \coth y_0 . \quad (1.18)$$

The actual tension of the domain wall is

$$\sigma = \frac{3}{4\pi G l} . \quad (1.19)$$

We therefore need a contribution to the effective tension from the CFT. This is provided by the conformal anomaly, which takes the value

$$\langle T \rangle = -\frac{3N^2}{8\pi^2 R^4} . \quad (1.20)$$

This contributes an effective tension $-\langle T \rangle / 4$. We can now obtain an equation for the radius of the domain wall:

$$\frac{R^3}{l^3}\sqrt{\frac{R^2}{l^2}+1} = \frac{N^2 G}{8\pi l^3} + \frac{R^4}{l^4} . \quad (1.21)$$

It is easy to see that this has a unique positive solution for R.
The AdS/CFT correspondence can be used to give the generating functional of the CFT on the perturbed sphere:

$$W_{CFT} = S_{EH} + S_{GH} + S_1 + S_2 + \ldots . \quad (1.22)$$

We shall give the terms on the right hand side for d = 4. The Einstein-Hilbert action with cosmological constant is

$$S_{EH} = -\frac{1}{16\pi \overline{G}} \int d^5 x \sqrt{g}\left(R + \frac{12}{\overline{l}^2}\right), \quad (1.23)$$

and perturbing this gives

$$S_{bulk} = -\frac{1}{16\pi \overline{G}} \int d^5 x \sqrt{g}\left(-\frac{8}{\overline{l}^2} + \frac{1}{4} h^{\mu\nu}\nabla^2 h_{\mu\nu} + \frac{1}{2\overline{l}^2} h^{\mu\nu} h_{\mu\nu}\right)$$
$$-\frac{1}{16\pi \overline{G}} \int d^4 x \sqrt{\gamma}\left(-\frac{1}{2} n^\mu h^{\nu\rho}\nabla_\nu h_{\mu\rho} + \frac{3}{4} h_{\nu\rho} n^\mu \nabla_\mu h^{\nu\rho}\right), \quad (1.24)$$

where Greek indices are five dimensional and we are raising and lowering with the unperturbed five dimensional metric. $n = l dy$ is the unit normal to the boundary and ∇ is the covariant derivative defined with the unperturbed bulk metric. $\gamma_{ij} = R^2 \hat{\gamma}_{ij}$ is the unperturbed boundary metric. Evaluating on shell gives

$$S_{EH} = \frac{\bar{l}^3}{2\pi\bar{G}} \int d^4x\sqrt{\hat{\gamma}} \int_0^{y_0} dy \sinh^4 y - \frac{\bar{l}^3}{16\pi\bar{G}} \int d^4x\sqrt{\hat{\gamma}} \left(\frac{3}{4\bar{l}^4} h^{ij}\partial_y h_{ij} - \frac{\coth}{\bar{l}^4} h^{ij}h_{ij} \right), \quad (1.25)$$

where we are now raising and lowering with $\hat{\gamma}_{ij}$. The Gibbons-Hawking term is

$$S_{GH} = -\frac{\bar{l}^3}{2\pi\bar{G}} \int d^4x\sqrt{\hat{\gamma}} \left(\sinh^3 y_0 \cosh y_0 - \frac{1}{8\bar{l}^4} h^{ij}\partial_y h_{ij} \right). \quad (1.26)$$

The first counter term is

$$S_1 = \frac{3}{8\pi\hat{G}\bar{l}} \int d^4x\sqrt{\gamma} = \frac{3\bar{l}^3}{8\pi\bar{G}} \int d^4x\sqrt{\hat{\gamma}} \left(\sinh^4 y_0 - \frac{1}{4\bar{l}^4} h^{ij}h_{ij} \right). \quad (1.27)$$

The second counter term is

$$S_2 = \frac{\bar{l}}{32\pi\bar{G}} \int d^4x\sqrt{\gamma}R = \frac{\bar{l}^3}{32\pi\bar{G}} \int d^4x\sqrt{\hat{\gamma}} \left(12\sinh^2 y_0 - \frac{2}{\bar{l}^4 \sinh^2 y_0} h^{ij}h_{ij} + \frac{1}{4\bar{l}^4 \sinh^2 y_0} h^{ij}\hat{\nabla}^2 h_{ij} \right). \quad (1.28)$$

Thus with only two counter terms we would have

$$W_{CFT} = \frac{3N^2\Omega_4}{8\pi^2} \log\frac{R}{\bar{l}} - \frac{\bar{l}^3}{16\pi\bar{G}} \int d^4x\sqrt{\hat{\gamma}} \left(-\frac{1}{4\bar{l}^4} h^{ij}\partial_y h_{ij} + \frac{1}{\bar{l}^4} h^{ij}h_{ij} \left(\frac{3}{2} - \sqrt{1+\frac{\bar{l}^2}{R^2}} \right) + \frac{1}{\bar{l}^2R^2} h^{ij}h_{ij} - \frac{1}{8\bar{l}^2R^2} h^{ij}\hat{\nabla}^2 h_{ij} \right) \quad (1.29)$$

Ω_4 is the area of a unit four-sphere and we have used the following equation: $\frac{\bar{l}^3}{\bar{G}} = \frac{2N^2}{\pi}$. The expansion of $\partial_y h_{ij}$ at $y = y_0$ is obtained from

$$\partial_y h_{ij} = \sum_p \frac{f_p'(y_0)}{f_p(y_0)} H_{ij}^{(p)}(x) \int d^4x' \sqrt{\hat{\gamma}} h^{kl}(x') H_{kl}^{(p)}(x') \quad (1.30)$$

and

$$\frac{f_p'(y_0)}{f_p(y_0)} = 2 + \frac{\bar{l}^2}{2R^2}(p+1)(p+2) + p(p+1)(p+2)(p+3)\frac{\bar{l}^4}{4R^4}\log(\bar{l}/R) + \frac{\bar{l}^4}{8R^4}\big[p^4 + 2p^3$$
$$-5p^2 - 10p - 2 - p(p-1)(p+2)(p+3)(\psi(1)+\psi(2)-\psi(p/2+2)-\psi(p/2+5/2))\big] + \mathrm{O}\left(\frac{\bar{l}^6}{R^6}\log(\bar{l}/R)\right) \quad (1.31)$$

The psi function is defined by $\psi(z) = \Gamma'(z)/\Gamma(z)$. Substituting into the action we find that the divergences as $\bar{l} \to 0$ cancel at order $R^4/\bar{l}^4$ and $R^2/\bar{l}^2$. The term of order $\bar{l}^4/R^4$ in the above expansion makes a contribution to the finite part of the action:

$$W_{CFT} = \frac{3N^2\Omega_4}{8\pi^2}\log\frac{R}{\bar{l}} + \frac{N^2}{256\pi^2 R^4}\sum_p \left(\int d^4x' \sqrt{\hat{\gamma}} h^{kl}(x') H^{(p)}_{kl}(x')\right)^2$$
$$\left(2p(p+1)(p+2)(p+3)\log(\bar{l}/R) + \Psi(p)\right), \quad (1.32)$$

where

$$\Psi(p) = p(p+1)(p+2)(p+3)\left[\psi(p/2+5/2) + \psi(p/2+2) - \psi(2) - \psi(1)\right] + p^4 + 2p^3 - 5p^2 - 10p - 6 \quad (1.33)$$

To cancel the logarithmic divergences as $\bar{l} \to 0$, we have to introduce a length scale ρ defined by $\bar{l} = \varepsilon\rho$ and add a counter term proportional to $\log\varepsilon$ to cancel the divergence as ε tends to zero. The counter term is

$$S_3 = -\frac{\bar{l}^3}{64\pi\bar{G}}\log\varepsilon \int d^4x \sqrt{\gamma}\left(\gamma^{ik}\gamma^{jl}R_{ij}R_{kl} - \frac{1}{3}R^2\right)$$
$$= -\frac{\bar{l}^3}{64\pi\bar{G}}\log\varepsilon \int d^4x \sqrt{\hat{\gamma}}\left(-12 + \frac{1}{R^4}\left[2h^{ij}h_{ij} - \frac{3}{2}h^{ij}\hat{\nabla}^2 h_{ij} + \frac{1}{4}h^{ij}\hat{\nabla}^4 h_{ij}\right]\right). \quad (1.34)$$

This term does indeed cancel the logarithmic divergence, leaving us with

$$W_{CFT} = \frac{3N^2\Omega_4}{8\pi^2}\log\frac{R}{\rho} + \frac{N^2}{256\pi^2 R^4}\sum_p \left(\int d^4x' \sqrt{\hat{\gamma}} h^{kl}(x')\right)^2 \left(2p(p+1)(p+2)(p+3)\log(\rho/R) + \Psi(p)\right). \quad (1.35)$$

Now, recall that our five dimensional action is

$$S = S_{EH} + S_{GH} + 2S_1 + W_{CFT}. \quad (1.36)$$

In order to calculated correlators of the metric, we need to evaluate the path integral

$$Z[\mathbf{h}] = \int_{B_1 \cup B_2} d[\delta\mathbf{g}]\exp(-S) =$$
$$= \exp\left(-2S_1[\mathbf{h_0} + \mathbf{h}] - W_{CFT}[\mathbf{h_0} + \mathbf{h}]\right)\left(\int_B d[\delta\mathbf{g}]\exp\left(-S_{EH}[\mathbf{g_0} + \delta\mathbf{g}] - S_{GH}[\mathbf{g_0} + \delta\mathbf{g}]\right)\right)^2. \quad (1.37)$$

Here $\mathbf{g_0}$ and $\mathbf{h_0}$ refer to the unperturbed background metrics in the bulk and on the wall respectively and $\mathbf{h}$ denotes the metric perturbation on the wall. Replacing $\bar{l}$ and $\bar{G}$ with l and G, from equation (1.27) we obtain

$$S_1[\mathbf{h_0} + \mathbf{h}] = \frac{3l^3}{8\pi G}\int d^4x \sqrt{\hat{g}}\left(\sinh^4 y_0 - \frac{1}{4l^4}\right), \quad (1.38)$$

where y_0 is defined by $R = l\sinh y_0$. The path integral over $\delta\mathbf{g}$ is performed by splitting it into a classical and quantum part:

$$\delta\mathbf{g} = \mathbf{h} + \mathbf{h'}, \quad (1.39)$$

where the boundary perturbation $\mathbf{h}$ is extended into the bulk using the linearized Einstein equations and the requirement of finite Euclidean action. $\mathbf{h'}$ denotes a quantum fluctuation that vanishes at the domain wall. The gravitational action splits into separate contributions from the classical and quantum parts:

$$S_{EH} + S_{GH} = S_0[\mathbf{h}] + S'[\mathbf{h'}], \quad (1.40)$$

where S_0 can be read off from the equations (1.25) and (1.26) as

$$S_0 = -\frac{3l^3\Omega_4}{2\pi G}\int_0^{y_0} dy \sinh^2 y_0 \cosh^2 y_0 + \frac{l^3}{16\pi G}\int d^4x\sqrt{\hat{\gamma}}\left(\frac{1}{4l^4}h^{ij}\partial_y h_{ij} + \frac{\coth y_0}{l^4}h^{ij}h_{ij}\right). \quad (1.41)$$

We shall not need the explicit form for S' since the path integral over $\mathbf{h'}$ just contributes a factor of some determinant Z_0 to $Z[\mathbf{h}]$. We obtain

$$Z[\mathbf{h}] = Z_0 \exp(-2S_0[\mathbf{h_0}+\mathbf{h}] - 2S_1[\mathbf{h_0}+\mathbf{h}] - W_{CFT}[\mathbf{h_0}+\mathbf{h}]). \quad (1.42)$$

The exponent is given by

$$\begin{aligned} 2S_0 + 2S_1 + W_{CFT} = {} & -\frac{3l^3\Omega_4}{\pi G}\int_0^{y_0} dy \sinh^2 y \cosh^2 y + \frac{3\Omega_4 R^4}{4\pi G l} + \frac{3N^2\Omega_4}{8\pi^2}\log\frac{R}{\rho} \\ & + \frac{1}{l^4}\sum_p \left(\int d^4x'\sqrt{\hat{\gamma}}h^{kl}(x')H_{kl}^{(p)}(x')\right)^2 \left[\frac{l^3}{32\pi G}\left(\frac{f_p'(y_0)}{f_p(y_0)} + 4\coth y_0 - 6\right)\right. \\ & \left. + \frac{N^2}{256\pi^2\sinh^4 y_0}\left(2p(p+1)(p+2)(p+3)\log(\rho/R) + \Psi(p)\right)\right]. \quad (1.43) \end{aligned}$$

The (Euclidean) graviton correlator can be read off from the action as

$$\langle h_{ij}(x)h_{i'j'}(x')\rangle = \frac{128\pi^2 R^4}{N^2}\sum_{p=2}^{\infty} W_{iji'j'}^{(p)}(x,x')F(p,y_0)^{-1} \quad (1.44)$$

where we have eliminated l^3/G using the equation (1.21). The function $F(p, y_0)$ is given by

$$F(p,y_0) = e^{y_0}\sinh y_0\left(\frac{f_p'(y_0)}{f_p(y_0)} + 4\coth y_0 - 6\right) + \Psi(p), \quad (1.45)$$

and the bitensor $W_{iji'j'}^{(p)}(x,x')$ is defined as

$$W_{iji'j'}^{(p)}(x,x') = \sum_{k,l,m,\ldots} H_{ij}^{(p)}(x)H_{i'j'}^{(p)}(x'), \quad (1.46)$$

with the sum running over all the suppressed labels $k, l, m, \ldots$ of the tensor harmonics.
Now, we consider the radius R of the domain wall given by equation (1.21). It is convenient to write this in terms of the rank N_{RS} of the RS CFT (given by $l^3/G = 2N_{RS}^2/\pi$)

$$\frac{R^3}{l^3}\sqrt{\frac{R^2}{l^2}+1} = \frac{N^2}{16N_{RS}^2} + \frac{R^4}{l^4}. \quad (1.47)$$

If we assume $N >> N_{RS} >> 1$ then the solution is

$$\frac{R}{l} = \frac{N}{2\sqrt{2}N_{RS}}\left[1+\frac{N_{RS}^2}{N^2}+\mathrm{O}\left(N_{RS}^4/N^4\right)\right]. \quad (1.48)$$

Note that this implies $R >> l$, i.e., the domain wall is large compared with the anti-de Sitter length scale. Now let's turn to a four dimensional description in which we are considering a four sphere with no interior. The only matter present is the CFT. The metric is simply

$$ds^2 = R_4^2\hat{\gamma}_{ij}dx^i dx^j\,, \quad (1.49)$$

where R_4 remains to be determined. The action is the four dimensional Einstein-Hilbert action (without cosmological constant) together with W_{CFT}. There is no Gibbons-Hawking term because there is no boundary. Without a metric perturbation, the action is simply

$$S = -\frac{1}{16\pi G_4}\int d^4x\sqrt{\gamma}R + W_{CFT} = -\frac{3\Omega_4 R_4^2}{4\pi G_4}+\frac{3N^2\Omega_4}{8\pi^2}\log\frac{R_4}{\rho}. \quad (1.50)$$

where G_4 is the four dimensional Newton constant. Varying R_4 gives

$$R_4^2 = \frac{N^2 G_4}{4\pi}, \quad (1.51)$$

and N is large hence R_4 is much greater than the four dimensional Planck length.
Now we can to include the metric perturbation. The perturbed four dimensional Einstein-Hilbert action is

$$S_{EH}^{(4)} = -\frac{1}{16\pi G_4}\int d^4x\sqrt{\hat{\gamma}}\left(12R_4^2 - \frac{2}{R_4^2}h^{ij}h_{ij}+\frac{1}{4R_4^2}h^{ij}\hat{\nabla}^2 h_{ij}\right). \quad (1.52)$$

Adding the perturbed CFT gives

$$S = -\frac{3N^2\Omega_4}{16\pi^2}+\frac{3N^2\Omega_4}{8\pi^2}\log\frac{R_4}{\rho}+\sum_p\left(\int d^4x'\sqrt{\hat{\gamma}}h^{kl}(x')H_{kl}^{(p)}(x')\right)^2\left[\frac{1}{64\pi G_4 R_4^2}\left(p^2+3p+6\right)\right.$$
$$\left.+\frac{N^2}{256\pi^2 R_4^4}\left(2p(p+1)(p+2)(p+3)\log(\rho/R_4)+\Psi(p)\right)\right]. \quad (1.53)$$

Setting $\rho = R_4$ we find that the graviton correlator for a four dimensional universe containing the CFT is

$$\left\langle h_{ij}(x)h_{i'j'}(x')\right\rangle = 8N^2G_4^2\sum_{p=2}^{\infty}W_{iji'j'}^{(p)}(x,x')\left[p^2+3p+6+\Psi(p)\right]^{-1}. \quad (1.54)$$

Hence, we have computed the graviton correlator using the Hartle-Hawking "No Boundary" proposal.

We shall show how the Euclidean correlator calculated above is analytically continued to give a correlator for Lorentzian signature.
We begin by continuing the graviton correlator (equation 1.44) obtained via the five dimensional calculation. The analytic continuation of the correlator for four dimensional gravity (equation 1.54) is completely analogous. In terms of the new label p', the Euclidean correlator 1.44 between two points on the wall is given by

$$\left\langle h_{ij}(\Omega) h_{i'j'}(\Omega')\right\rangle = \frac{128\pi^2 R^4}{N^2} \sum_{p'=7i/2}^{i\infty} W_{iji'j'}^{(p')}(\mu) G(p', y_0)^{-1} \qquad (1.55)$$

where

$$G(p', y_0) = F(-ip'-3/2, y_0) = e^{y_0} \sinh y_0 \left[\frac{g'_{p'}(y_0)}{g_{p'}(y_0)} + 4\coth y_0 - 6\right]$$
$$+ \left(p'^4 - 4ip'^3 + p'^2/2 - 5ip' - 63/16 + (p'^2+1/4)(p'^2+9/4)\left[\psi(-ip'/2+5/4) + \psi(-ip'/2+7/4)\right.\right.$$
$$\left.\left. - \psi(1) - \psi(2)\right]\right),$$
$$(1.56)$$

with $g_{p'}(y) = Q^2_{-ip'-1/2}(\coth y)$. The function $G(p', y_0)$ is real and positive for all values of p' in the sum and for arbitrary $y_0 \geq 0$. We have the Euclidean correlator defined as an infinite sum.
Now we write the sum in equation (1.55) as an integral along a contour C_1 encircling the points $p' = 7i/2, 9i/2, \ldots ni/2$, where n tends to infinity. This yields

$$\left\langle h_{ij}(\Omega) h_{i'j'}(\Omega')\right\rangle = \frac{-i64\pi^2 R^4}{N^2} \int_{C_1} dp' \tanh p'\pi W_{iji'j'}^{(p')}(\mu) G(p', y_0)^{-1}. \qquad (1.57)$$

When the contribution from the closing of the contour in the upper half p'-plane vanishes, the final result for the Euclidean correlator reads

$$\left\langle h_{ij}(\Omega) h_{i'j'}(\Omega')\right\rangle = \frac{-i64\pi^2 R^4}{N^2} \left[\int_{-\infty}^{+\infty} dp' \tanh p'\pi W_{iji'j'}^{(p')}(\mu) G(p', y_0)^{-1}\right.$$
$$\left. + 2\pi \sum_{k=1}^{2} \tan \Lambda_k \pi W_{iji'j'}^{(i\Lambda_k)}(\mu) \mathbf{Res}\left(G(p', y_0)^{-1}; i\Lambda_k\right)\right]. \qquad (1.58)$$

The analytic continuation from a four sphere into Lorentzian closed de Sitter space is given by setting the polar angle $\Omega = \pi/2 - it$. We may take $\mu = \Omega$, and μ then continues to $\pi/2 - it$.
In conclusion, we note that the Lorentzian tensor Feynman (time-ordered) correlator is

$$\left\langle h_{ij}(x) h_{i'j'}(x')\right\rangle = \frac{128\pi^2 R^4}{N^2} \left[\int_0^{+\infty} dp' \tanh p'\pi W_{iji'j'}^{L(p')}(\mu) R\left(G(p', y_0)^{-1}\right)\right.$$
$$\left. + \pi \sum_{k=1}^{2} \tan \Lambda_k \pi W_{iji'j'}^{L(i\Lambda_k)}(\mu) \mathbf{Res}\left(G(p', y_0)^{-1}; i\Lambda_k\right)\right] + i\frac{128\pi^2 R^4}{N^2} \left[\int_0^{+\infty} dp' W_{iji'j'}^{L(p')}(\mu) R\left(G(p', y_0)^{-1}\right)\right.$$
$$\left. - \pi \sum_{k=1}^{2} W_{iji'j'}^{L(i\Lambda_k)}(\mu) \mathbf{Res}\left(G(p', y_0)^{-1}; i\Lambda_k\right)\right]. \qquad (1.59)$$

In this integral the bitensor $W^{L(p')}_{iji'j'}(\mu(x,x'))$ may be written as the sum of the degenerate rank-two tensor harmonics on closed de Sitter space with eigenvalue $\lambda_{p'} = (p'^2 + 17/4)$ of the Laplacian.
We can understand the effect of the strongly coupled CFT on the microwave fluctuation spectrum by comparing the result (1.59) with the transverse traceless part of the graviton propagator in four-dimensional de Sitter spacetime. On the four-sphere, this is easily obtained by varying the Einstein-Hilbert action with a cosmological constant. In terms of the bitensor, this yields

$$\langle h_{ij}(\Omega) h_{i'j'}(\Omega')\rangle = 32\pi G_4 R^2 \sum_{p'=7i/2}^{i\infty} \frac{W^{(p')}_{iji'j'}(\mu(\Omega,\Omega'))}{\lambda_{p'} - 2}, \quad (1.60)$$

which continues to

$$\langle h_{ij}(x) h_{i'j'}(x')\rangle = 32\pi G_4 R^2 \int_0^{+\infty} \frac{dp'}{\lambda_{p'} - 2} W^{L(p')}_{iji'j'}(\mu(x,x')). \quad (1.61)$$

We note that this can be compared with equation (1.59).

2. Nonlocal braneworld action in the two-brane Randall-Sundrum model. [2]

In the following definition

$$\exp(iS_{eff}[\phi]) = \int D\Phi \exp(iS[\Phi])\big|_{\Phi(\Sigma)=\phi}, \quad (2.1)$$

the effective action by construction depends on the four-dimensional fields associated with brane(s)[1]. The number of these fields equals the number of branes, geometrically each field being carried by one of the branes in the system. In the generalized Randall-Sundrum setup, the braneworld effective action is generated by the path integral of the type (2.1),

$$\int DG \exp(iS[G,g,\phi])\big|_{{}^4G(\Sigma)=g} = \exp(iS_{eff}[g,\phi]), \quad (2.1a)$$

where the integration over bulk metrics runs subject to fixed induced metrics on the branes – the arguments of $S_{eff}[g,\phi]$. Here $S[G,g,\phi]$ is the action of the five-dimensional gravitational field with the metric $G = G_{AB}(x,y)$, $A = (\mu,5)$, $\mu = 0,1,2,3,$ propagating in the bulk spacetime $(x^A = (x,y), x = x^\mu, x^5 = y)$, and matter fields ϕ are confined to the branes Σ_I - four-dimensional timelike surfaces embedded in the bulk,

$$S[G,g,\phi] = S_5[G] + \sum_I \int_{\Sigma_I} d^4x \left(L_m(\phi,\partial\phi,g) - g^{1/2}\sigma_I + \frac{1}{8\pi G_5}[K] \right), \quad (2.2)$$

$$S_5[G] = \frac{1}{16\pi G_5} \int_{M^5} d^5x G^{1/2} \left({}^5R(G) - 2\Lambda_5 \right). \quad (2.3)$$

The branes are enumerated by the index I and carry induced metrics $g = g_{\mu\nu}(x)$ and matter field Lagrangians $L_m(\phi,\partial\phi,g)$. The bulk part of the action contains the five-dimensional gravitational

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[1] The scope of this formula is very large, because it arises in very different contexts. In particolar, **its Euclidean version** $(iS \to -S_{Euclid})$ **underlies the construction of the no-boundary wavefunction in quantum cosmology**. (Hartle-Hawking Phys. Rev. D 28, 2960 (1983))

and cosmological constants, G_5 and Λ_5, while the brane parts have four-dimensional cosmological constants σ_I. The bulk cosmological constant Λ_5 is negative and, therefore, is capable of generating the AdS geometry, while the brane cosmological constants play the role of brane tensions σ_I and, depending on the model, can be of either sign. The Einstein-Hilbert bulk action (2.3) is accompanied by the brane "Gibbons-Hawking" terms containing the jump of the extrinsic curvature trace $[K]$ associated with both sides of each brane. In the tree-level approximation the path integral (2.1a) is dominated by the stationary point of the action (2.2). Its variation is given as a sum of five- and four-dimensional integrals,

$$\delta S[G,g,\phi] = -\frac{1}{16\pi G_5}\int d^4x dy G^{1/2}\left({}^5R^{AB} - \frac{1}{2}{}^5RG^{AB} + \Lambda_5 G^{AB}\right)\delta G_{AB}(x,y)$$
$$+\sum_I \int_{\Sigma_I} d^4x g^{1/2}\left(-\frac{1}{16\pi G_5}\left[K^{\mu\nu} - g^{\mu\nu}K\right] + \frac{1}{2}\left(T^{\mu\nu} - g^{\mu\nu}\sigma\right)\right)\delta g_{\mu\nu}(x), \quad (2.4)$$

where $[K^{\mu\nu} - g^{\mu\nu}K]$ denotes the jump of the extrinsic curvature terms across the brane, and $T^{\mu\nu}(x)$ is the corresponding four-dimensional stress-energy tensor of matter fields on the branes,

$$T^{\mu\nu}(x) = \frac{2}{g^{1/2}}\frac{\delta S_m[g,\phi]}{\delta g_{\mu\nu}(x)}, \quad (2.5) \qquad S_m[g,\phi] = \sum_I \int_{\Sigma_I} d^4x L_m(\phi, \partial\phi, g), \quad (2.6)$$

hence

$$T^{\mu\nu}(x) = \frac{2}{g^{1/2}}\frac{1}{\delta g_{\mu\nu}(x)}\delta\sum_I \int_{\Sigma_I} d^4x L_m(\phi, \partial\phi, g). \quad (2.6a)$$

The action is stationary when the integrands of both integrals in (2.4) vanish, which gives rise to Einstein equations in the bulk,

$$\frac{\delta S[G,g,\phi]}{\delta G_{AB}(x,y)} \equiv -\frac{1}{16\pi G_5}G^{1/2}\left({}^5R^{AB} - \frac{1}{2}G^{AB\,5}R + \Lambda_5 G^{AB}\right) = 0, \quad (2.7)$$

which are subject to (generalized) Neumann type boundary conditions – the well-known Israel junction conditions –

$$\frac{\delta S[G,g,\phi]}{\delta g_{\mu\nu}(x)} \equiv -\frac{1}{16\pi G_5}g^{1/2}\left[K^{\mu\nu} - g^{\mu\nu}K\right] + \frac{1}{2}g^{1/2}\left(T^{\mu\nu} - g^{\mu\nu}\sigma\right) = 0, \quad (2.8)$$

or to Dirichlet type boundary conditions corresponding to fixed (induced) metrics on the branes, with $\delta g_{\mu\nu} = 0$ in the variation (2.4),

$${}^4G_{\mu\nu}\big|_\Sigma = g_{\mu\nu}(x). \quad (2.9)$$

The solution of the latter, Dirichlet, problem is obviously a functional of brane metrics, $G_{AB} = G_{AB}\left[g_{\mu\nu}(x)\right]$, and it enters the tree-level approximation for the path integral (2.1a). $S_{eff}[g,\phi]$ in this approximation reduces to the original action (2.2)-(2.3) calculated on this solution $G_{AB}\left[g_{\mu\nu}(x)\right]$, $S_{eff}[g,\phi] = S[G[g], g, \phi] + O(\hbar)$. With this definition, the matter part of effective action coincides with the original action Eq. (2.6)

$$S_{eff}[g,\phi] = S_4[g] + S_m[g,\phi], \quad (2.10)$$

while all non-trivial dependence on g arising from the functional integration is contained in $S_4[g]$. Given the action (2.10) as a result of solving the Dirichlet problem (2.7), (2.9), one can further apply the variational procedure, with respect to the induced metric $g_{\mu\nu}$, to get the effective equations

$$\frac{\delta S_{eff}[g,\phi]}{\delta g_{\mu\nu}(x)} = \frac{\delta S_4[g]}{\delta g_{\mu\nu}(x)} + \frac{1}{2} g^{1/2} T^{\mu\nu}(x) = 0, \quad (2.11)$$

which are equivalent to the Israel junction conditions – a part of the full system of the bulk-brane equations of motion (2.7), (2.8).

The action of the two-brane Randall-Sundrum model is given by Eq. (2.2) in which the index $I = \pm$ enumerates two branes with tensions $\sigma_\pm$. The fifth dimension has the topology of a circle labelled by the coordinate y, $-d < y \leq d$, with an orbifold Z_2-identification of points y and $-y$. The branes are located at antipodal fixed points of the orbifold, $y = y_\pm$, $y_+ = 0$, $|y_-| = d$. When they are empty, $L_m(\phi, \partial\phi, g_{\mu\nu}) = 0$, and their tensions are opposite in sign and fine-tuned to the values of Λ_5 and G_5,

$$\Lambda_5 = -\frac{6}{l^2}, \quad \sigma_+ = -\sigma_- = \frac{3}{4\pi G_5 l}, \quad (2.12)$$

this model admits a solution with an AdS metric in the bulk (l is its curvature radius),

$$ds^2 = dy^2 + e^{-2|y|/l} \eta_{\mu\nu} dx^\mu dx^\nu, \quad (2.13)$$

$0 = y_+ \leq |y| \leq y_- = d$, and with a flat induced metric $\eta_{\mu\nu}$ on both branes. The metric on the negative tension brane is rescaled by the warp factor $\exp(-2d/l)$ providing a possible solution for the hierarchy problem. With the fine tuning (2.12) this solution exists for arbitrary brane separation d - two flat branes stay in equilibrium. Their flatness is the result of compensation between the bulk cosmological constant and brane tensions.

Now consider the Randall-Sundrum model with small matter sources for metric perturbations $h_{AB}(x, y)$ on the background of this solution,

$$ds^2 = dy^2 + e^{-2|y|/l} \eta_{\mu\nu} dx^\mu dx^\nu + h_{AB}(x, y) dx^A dx^B, \quad (2.14)$$

such that this five-dimensional metric induces on the branes two four-dimensional metrics of the form

$$g^\pm_{\mu\nu}(x) = a^2_\pm \eta_{\mu\nu} + h^\pm_{\mu\nu}(x). \quad (2.15)$$

Here the scale factors $a_\pm = a(y_\pm)$ can be expressed in terms of the interbrane distance

$$a_+ = 1, \quad a_- = e^{-2d/l} \equiv a, \quad (2.16)$$

and $h^{\pm}_{\mu\nu}(x)$ are the perturbations by which the brane metrics $g^{\pm}_{\mu\nu}(x)$ differ from the (conformally) flat metrics of the Randall-Sundrum solution (2.13). The main result is the braneworld effective action (2.10) calculated for the boundary conditions (2.9) of this perturbed form (2.15). The braneworld effective action is invariant under the four-dimensional diffeomorphisms acting on the branes. In the linearized approximation they reduce to the transformations of metric perturbations,

$$h^{\pm}_{\mu\nu} \to h^{\pm}_{\mu\nu} + f^{\pm}_{\mu,\nu} + f^{\pm}_{\nu,\mu} \qquad (2.17)$$

with two independent local vector field parameters $f^{\pm}_{\mu} = f^{\pm}_{\mu}(x)$. Therefore, rather than in terms of metric perturbations themselves, the action in question is expressible in terms of the tensor invariants of these transformations – linearized Ricci tensors of $h_{\mu\nu} = h^{\pm}_{\mu\nu}(x)$,

$$R_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} + h^{\lambda}_{\nu,\lambda\mu} + h^{\lambda}_{\mu,\lambda\nu} - h_{,\mu\nu}\right), \qquad (2.18)$$

on flat four-dimensional backgrounds of both branes. (Strictly speaking, $R^{-}_{\mu\nu}$ is the linearized Ricci tensor of the artificial metric $\eta_{\mu\nu} + h^{-}_{\mu\nu}$. It differs from the linearized Ricci curvature of the second brane, $R_{\mu\nu}\left(a^2\eta + h^{-}\right) = R^{-}_{\mu\nu}/a^2$, by a factor of a^2). Commas denote partial derivatives, raising and lowering of braneworld indices here and everywhere is performed with the aid of the flat four-dimensional metric $\eta_{\mu\nu}$, $h^{\lambda}_{\nu} \equiv \eta^{\lambda\sigma} h_{\sigma\nu}$, $h \equiv \eta^{\mu\nu} h_{\mu\nu}$, $R = \eta^{\mu\nu} R_{\mu\nu}$, and $\Box$ denotes the flat spacetime d'Alambertian

$$\Box = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}. \qquad (2.19)$$

Hence, we have to describe the variables which determine the embedding of branes into the bulk. Due to metric perturbations the branes no longer stay at fixed values of the fifth coordinate. Up to four-dimensional diffeomorphism (2.17), their embedding variables consist of two four-dimensional scalar fields – the radions $\psi^{\pm}(x)$ – and the braneworld action can depend on these scalars. Their geometrically invariant meaning is revealed in a special coordinate system where the bulk metric perturbations $h_{AB}(x, y)$ of Eq. (2.14) satsfy the so called Randall-Sundrum gauge conditions, $h_{A5} = 0$, $h^{\ ;\nu}_{\mu\nu} = h^{\mu}_{\mu} = 0$. In this coordinate system the brane embeddings are defined by the equations

$$\Sigma_{\pm} : y = y_{\pm} + \frac{l}{a^2_{\pm}}\psi^{\pm}(x), \qquad y_{+} = 0, \qquad y_{-} = d. \qquad (2.20)$$

In the approximation linear in perturbation fields and vector gauge parameters, these radion fields are invariant under the action of diffeomorphism (2.17). The answer for the braneworld effective action, is given in terms of the invariant fields, $\left(R^{\pm}_{\mu\nu}(x), \psi^{\pm}(x)\right)$, by the following spacetime integral of a 2×2 quadratic form,

$$S_4\left[g^{\pm}_{\mu\nu}, \psi^{\pm}\right] = \frac{1}{16\pi G_4}\int d^4x \left[\mathbf{R}^{T}_{\mu\nu}\,\frac{2\mathbf{F}(\Box)}{l^2\Box^2}\,\mathbf{R}^{\mu\nu} + \frac{1}{6}\mathbf{R}^{T}\,\frac{\mathbf{K}(\Box) - 6\mathbf{F}(\Box)}{l^2\Box^2}\,\mathbf{R}\right.$$
$$\left. - 3\left(\Box\Psi + \frac{1}{6}\mathbf{R}\right)^{T} \frac{\mathbf{K}(\Box)}{l^2\Box^2}\left(\Box\Psi + \frac{1}{6}\mathbf{R}\right)\right]. \qquad (2.21)$$

Here G_4 is an effective four-dimensional gravitational coupling constant,

$$G_4 = \frac{G_5}{l}, \quad (2.22)$$

$\left(\mathbf{R}^{\mu\nu}, \Psi\right)$ and $\left(\mathbf{R}^T_{\mu\nu}, \Psi^T\right)$ are the two-dimensional columns

$$\mathbf{R}_{\mu\nu} = \begin{bmatrix} R^+_{\mu\nu}(x) \\ R^-_{\mu\nu}(x) \end{bmatrix}, \quad \Psi = \begin{bmatrix} \psi^+(x) \\ \psi^-(x) \end{bmatrix} \quad (2.23)$$

and rows

$$\mathbf{R}^T_{\mu\nu} = \left[R^+_{\mu\nu}(x) R^-_{\mu\nu}(x)\right], \quad \Psi^T = \left[\psi^+(x)\psi^-(x)\right], \quad (2.24)$$

of two sets of curvature perturbations and radion fields, associated with two branes (T denotes the matrix and vector transposition).
The quadratic approximation for the action and its nonlocal formfactors obviously determines the spectrum of excitations in the theory. Now we show that in the graviton sector this spectrum corresponds to the tower of Kaluza-Klein modes well-known from a conventional Kaluza-Klein setup. The graviton sector arises when one decomposes metric perturbations on both branes into irreducible components – transverse-traceless tensor, vector and scalar parts,

$$h^\pm_{\mu\nu} = \gamma^\pm_{\mu\nu} + \varphi^\pm \eta_{\mu\nu} + f_{\mu,\nu} + f_{\nu,\mu}, \quad \gamma^\nu_{\mu\nu} = \eta^{\mu\nu}\gamma_{\mu\nu} = 0. \quad (2.25)$$

On substituting this decomposition in the linearized curvatures of (2.21) one finds that the vector parts do not contribute to the action, and the latter reduces to the sum of the graviton and scalar sectors,

$$S_4\left[g^\pm_{\mu\nu}, \psi^\pm\right] = S_{graviton}\left[\gamma^\pm_{\mu\nu}\right] + S_{scalar}\left[\varphi^\pm, \psi^\pm\right]. \quad (2.26)$$

The graviton part in entirely determined by the operator $\mathbf{F}(\Box)$ and reads

$$S_{graviton}\left[\gamma^\pm_{\mu\nu}\right] = \frac{1}{16\pi G_4}\int d^4x \frac{1}{2}\left[\gamma^+_{\mu\nu}\gamma^-_{\mu\nu}\right]\frac{\mathbf{F}(\Box)}{l^2}\begin{bmatrix} \gamma^{\mu\nu}_+ \\ \gamma^{\mu\nu}_- \end{bmatrix}, \quad (2.27)$$

while the scalar sector consists of the radion fields of Eq. (2.23) and the doublets of the trace (or conformal) parts of the metric perturbations $\varphi^\pm$,

$$\Phi = \begin{bmatrix} \varphi^+(x) \\ \varphi^-(x) \end{bmatrix}, \quad \Phi^T = \left[\varphi^+(x)\varphi^-(x)\right]. \quad (2.28)$$

Their action diagonalizes in terms of the conformal modes and the (redefined) radion modes $2\Psi - \Phi$,

$$S_{scalar}\left[\varphi^\pm, \psi^\pm\right] = \frac{3}{32\pi G_4}\int d^4x\left(-\varphi^+\Box\varphi^+ + \frac{1}{a^2}\varphi^-\Box\varphi^-\right) - \frac{3}{16\pi G_4}\int d^4x(2\Psi - \Phi)^T\frac{\mathbf{K}(\Box)}{l^2}(2\Psi - \Phi). \quad (2.29)$$

Now we consider in the low-energy approximation on the positive-tension brane the case of large brane separation, when $a << 1$ and

$$l\sqrt{\Box} << 1\,, \qquad \frac{l\sqrt{\Box}}{a} >> 1\,. \qquad (2.30)$$

This range of coordinate distances $1/\sqrt{\Box}$ corresponds to the long-distance approximation on the Σ_+-brane. Now one should use the asymptotic expressions of large arguments of the Bessel functions $\left(J_\nu^-, Y_\nu^-\right)$, $\nu = 1,2$,

$$J_\nu^- \cong \sqrt{\frac{2a}{\pi l \Box^{1/2}}}\cos\left(\frac{l\sqrt{\Box}}{a} - \frac{\pi}{4} - \frac{\pi\nu}{2}\right), \quad Y_\nu^- \cong \sqrt{\frac{2a}{\pi l \Box^{1/2}}}\sin\left(\frac{l\sqrt{\Box}}{a} - \frac{\pi}{4} - \frac{\pi\nu}{2}\right), \quad (2.31)$$

and the small-argument expansions for $\left(J_\nu^+, Y_\nu^+\right)$. Then, in the leading order the operator $\mathbf{F}(\Box)$ reads

$$\mathbf{F}(\Box) \cong \begin{bmatrix} \dfrac{l^2\Box}{2} & \dfrac{l^2\Box}{2J_2^-} \\ \dfrac{l^2\Box}{2J_2^-} & -\dfrac{J_1^-}{J_2^-}\dfrac{l}{a}\sqrt{\Box} \end{bmatrix}. \quad (2.32)$$

In contrast to the case of small brane separation, the short-distance corrections to this matrix operator contain a nonlocal $\Box^2 \ln\Box$-term. We present it for the $F_{++}(\Box)$-element,

$$F_{++}(\Box) = \frac{l^2\Box}{2} + \frac{\left(l^2\Box\right)^2}{2}k_2(\Box) + O\left[\left(l^2\Box\right)^3\right], \quad (2.33) \qquad k_2(\Box) = \frac{1}{4}\left(\ln\frac{4}{l^2\Box} - 2\mathbf{C} + \pi\frac{Y_2^-}{J_2^-}\right). \quad (2.34)$$

This is a manifestation of the well-known phenomenon of AdS/CFT-correspondence when typical quantum field theoretical effects in four-dimensional theory can be generated from the classical theory in the bulk.

If we take the usual viewpoint of the braneworld framework, that our visible world is one of the branes embedded in a higher-dimensional bulk, then the fields living on other branes are not directly observable. In this case the effective dynamics should be formulated in terms of fields on the visible brane. In the two-brane Randall-Sundrum model this is equivalent to constructing the reduced action – an action with on-shell reduction for the invisible fields in terms of the visible ones. We perform the reduction of the action to the Σ_+-fields separately in the graviton and scalar sectors. In the graviton sector (2.27) the on-shell reduction – the exclusion of $\gamma_{\mu\nu^-}^-$ perturbations in terms of $\gamma_{\mu\nu}^+ = \gamma_{\mu\nu}$ – corresponds to the replacement of the original action by the new one,

$$S_{graviton}\left[\gamma_{\mu\nu}^{\pm}\right] \Rightarrow S_{graviton}^{red}\left[g_{\mu\nu}\right] = \frac{1}{16\pi G_4}\int d^4x\sqrt{g}\,\gamma_{\mu\nu}^+\frac{F_{red}(\Box)}{2l^2}\gamma_+^{\mu\nu}\,, \quad (2.35)$$

with the original kernel $\mathbf{F}(\Box)$ going over to the new one-component kernel $F_{red}(\Box)$ according to the following simple prescription

$$\mathbf{F}(\Box) \Rightarrow F_{red}(\Box) = F_{++}(\Box) - F_{+-}(\Box)\frac{1}{F_{--}(\Box)}F_{-+}(\Box). \quad (2.36)$$

It is useful to rewrite (2.35) back to the covariant form in terms of (linearized) Ricci curvatures on a single visible brane,

$$S^{red}_{graviton}\left[g_{\mu\nu}\right] = \frac{1}{8\pi G_4}\int d^4x\sqrt{g}\left(R_{\mu\nu}\frac{F_{red}(\Box)}{l^2\Box^2}R^{\mu\nu} - \frac{1}{3}R\frac{F_{red}(\Box)}{l^2\Box^2}R\right). \quad (2.37)$$

A similar reduction in the scalar sector implies omitting in the first integral of (2.29) the negative-tension term and replacing the 2×2 quadratic form in the second integral by the quadratic form in ψ^+ with the reduced operator

$$K_{red}(\Box) = K_{++}(\Box) - K_{+-}(\Box)\frac{1}{K_{--}(\Box)}K_{-+}(\Box) = \frac{\det \mathbf{K}(\Box)}{K_{--}(\Box)}. \quad (2.38)$$

Thence, we express the conformal mode in terms of the (linearized) Ricci scalar $\varphi^+ = -(1/3\Box)R$, and denote the radion by $\psi^+ = \psi$. Then the combination of the reduced scalar sector together with the graviton part (2.37) yields the reduced action in its covariant form

$$S_{red}\left[g_{\mu\nu},\psi\right] = \frac{1}{16\pi G_4}\int d^4x\sqrt{g}\left[R_{\mu\nu}\frac{2F_{red}}{l^2\Box^2}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) - \frac{1}{6}R\left(\frac{1}{\Box} - \frac{2F_{red}}{l^2\Box^2}\right)R\right.$$
$$\left. - 6l^2\left(\Box\psi + \frac{R}{6}\right)\frac{2K_{red}}{l^2\Box^2}\left(\Box\psi + \frac{R}{6}\right)\right]. \quad (2.39)$$

In the regime of small or finite brane separation $l\sqrt{\Box} << 1$, $\frac{l\sqrt{\Box}}{a} << 1$, the calculation of the reduced operator (2.36) gives the following result,

$$F_{red}(\Box) = \frac{l^2\Box}{2}\left(1-a^2\right) + \frac{\left(l^2\Box\right)^2}{2}\kappa_1(a) + O\left[\left(l^2\Box\right)^3\right], \quad (2.40) \qquad \kappa_1(a) = \frac{1}{4}\left[\ln\frac{1}{a^2} - \left(1-a^2\right) - \frac{1}{2}\left(1-a^2\right)^2\right]. \quad (2.41)$$

Similarly, in view of $\det \mathbf{K}(\Box) = \frac{4\ln a}{1-a^4}\left(l^2\Box\right)^2 + O\left[\left(l^2\Box\right)^3\right]$, the reduced operator in the radion sector (2.38) is at least quadratic in $\Box$,

$$K_{red}(\Box) = \kappa_2(a)\left(l^2\Box\right)^2 + O\left[\left(l^2\Box\right)^3\right], \quad (2.42) \qquad \kappa_2(a) = \frac{1}{4}\ln\frac{1}{a^2}, \quad (2.43)$$

so that the low-energy radion turns out to be a dipole ghost.
The reduced braneworld action in the low-energy regime of finite interbrane distance is:

$$S_{red}\left[g_{\mu\nu},\psi\right] = \frac{1}{16\pi G_4}\int d^4x\sqrt{g}\left[\left(1-a^2\right)R - \frac{a^2}{6}R\frac{1}{\Box}R - 6l^2\kappa(a)\left(\Box\psi + \frac{R}{6}\right)^2 + \frac{l^2}{2}\kappa_1(a)C^2_{\mu\nu\alpha\beta}\right]. \quad (2.44)$$

Furthermore, the following reparametrization from ψ to the new field φ,

$$\varphi = \sqrt{\frac{3}{4\pi G_4}}\left[a\left(1-\frac{1}{6\Box}R\right)-l\sqrt{\frac{\kappa_2(a)}{-\Box}}\left(\Box\psi+\frac{1}{6}R\right)\right] \quad (2.45)$$

converts the action (2.44) to the local form

$$S_{red}\left[g_{\mu\nu},\varphi\right]=\int d^4x\sqrt{g}\left[\left(\frac{1}{16\pi G_4}-\frac{1}{12}\varphi^2\right)R+\frac{1}{2}\varphi\Box\varphi+\frac{l^2}{32\pi G_4}\kappa_1(a)C^2_{\mu\nu\alpha\beta}\right]. \quad (2.46)$$

The field φ introduced here by the formal transformation (2.45) directly arose as a local redefinition of the radion field relating the Randall-Sundrum coordinates to the Gaussian normal coordinates associated with the positive tension brane.

Large interbrane distance and Hartle boundary conditions

In the limit $a \to 0$ the nonlocal and correspondingly non-minimal terms of (2.44) and (2.46) vanish and the low-energy model seems to reproduce the Einstein theory. However, this limit corresponds to another energy regime (2.30) in which one should use the expressions (2.32) – (2.34) in order to obtain the reduced operator (2.36). Then the latter, up to quadratic in $\Box$ terms inclusive, reads as

$$F_{red}(\Box)=\frac{l^2\Box}{2}+\frac{(l^2\Box)^2}{8}\left(\ln\frac{4}{l^2\Box}-2\mathbf{C}\right)+\frac{(l^2\Box)^2}{2}\left(\frac{\pi}{4}\frac{Y_2^-}{J_2^-}+\frac{a}{2l\sqrt{\Box}J_1^-J_2^-}\right). \quad (2.47)$$

As in (2.33) it involves the logarithmic nonlocality (2.34) in $\Box^2$-terms. Moreover, the last term here simplifies to the ratio of the first order Bessel functions Y_1^-/J_1^-, so that $F_{red}(\Box)$ takes a form very similar to that of the large interbrane separation (2.33),

$$F_{red}(\Box)=\frac{l^2\Box}{2}+\frac{(l^2\Box)^2}{8}\left[\ln\frac{4}{l^2\Box}-2\mathbf{C}+\pi\frac{Y_1^-}{J_1^-}\right]+O\left[(l^2\Box)^3\right]. \quad (2.48)$$

The calculation of the radion operator (2.38) with $\mathbf{K}(\Box)$ following from (2.32) for $l^2\Box/a^2 >> 1$ results in

$$K_{red}(\Box)=(l^2\Box)^2 k_2(\Box), \quad (2.49)$$

where $k_2(\Box)$ is defined by (2.34). Thus, $F_{red}(\Box)$ and $K_{red}(\Box)$ are given by the following two nonlocal operators,

$$k_\nu(\Box)=\frac{1}{4}\left[\ln\frac{4}{l^2\Box}-2\mathbf{C}+\pi\frac{Y_\nu^-}{J_\nu^-}\right], \quad \nu=1,2, \quad (2.50)$$

and the reduced (one-brane) action finally reads

$$S_{red}\left[g_{\mu\nu},\psi\right]=\frac{1}{16\pi G_4}\int d^4x\sqrt{g}\left[R+\frac{l^2}{2}C_{\mu\nu\alpha\beta}k_1(\Box)C^{\mu\nu\alpha\beta}-6l^2\left(\Box\psi+\frac{R}{6}\right)k_2(\Box)\left(\Box\psi+\frac{R}{6}\right)\right]. \quad (2.51)$$

Here terms quadratic in curvature represent short distance corrections with form factors whose logarithmic parts have an interpretation in terms of the AdS/CFT-correspondence. With the usual Wick rotation prescription $\Box\to\Box+i\varepsilon$ these ratios tend to

$$\frac{Y_\nu^-}{J_\nu^-}\cong\tan\left(\frac{l}{a}\sqrt{\Box+i\varepsilon}-\frac{\pi}{4}-\frac{\pi\nu}{2}\right)\to i\,,\quad a\to 0\,,\quad (2.52)$$

and both form factors (2.50) for $\Box<0$ (Euclidean or spacelike momenta) become real and can be expressed in terms of one Euclidean form factor as

$$k_\nu\left(\Box+i\varepsilon\right)\Big|_{a\to 0}=k(\Box)\equiv\frac{1}{4}\left(\ln\frac{4}{l^2(-\Box)}-\mathbf{C}\right),\quad (2.53)\quad \text{and hence}\quad k(\Box)+\frac{1}{4}\mathbf{C}\equiv\frac{1}{4}\ln\frac{4}{l^2(-\Box)}.\quad (2.54)$$

This Wick rotation after moving the second brane to the AdS horizon impose a special choice of vacuum or special boundary conditions at the AdS horizon. The Hartle boundary conditions corresponding to this type of analytic continuation imply that the basis function $u_-(z)$ is given by the Hankel function, $u_-(z)=H_2^{(1)}\left(z\sqrt{\Box}\right)=J_2\left(z\sqrt{\Box}\right)+iY_2\left(z\sqrt{\Box}\right)$, and thus corresponds to ingoing waves at the horizon. This is equivalent to the replacement $Y_1^-,Y_2^-\to 1, J_1^-,J_2^-\to -i$, in (2.50) and, thus, justifies the Wick rotation of the above type. Hartle boundary conditions and the Euclidean form factor (2.53) naturally arise when the Lorentzian AdS spacetime is viewed as the analytic continuation from the Euclidean AdS (EAdS) via Wick rotation in the complex plane of time.

3. Compactifications of Type IIB Strings on a Calabi-Yau three-fold and Hartle-Hawking wave-function in the mini-superspace sector of physical superstring theory.

A. *Compactifications of Type IIB Strings on a Calabi-Yau three-fold*. [3]

Although the self-duality of the five-form field strength in type IIB string theory implies that the latter cannot be described by a supersymmetric 10-dimensional action, the bosonic fields can be described by a non-self-dual action in which the equation of motion for the five-form field strength is replaced by its Bianchi identity. This is consistent with self-duality, but does not imply it. When self-duality is imposed as compactification condition, the non-self-dual action yields the correct compactified theory. In the Einstein frame, the action is,

$$S=\int d^{10}\hat{x}\sqrt{-\hat{g}}\left\{\frac{1}{2}\hat{R}-\frac{1}{8}Tr\left(\partial_{\hat{\mu}}\hat{\mathrm{M}}\partial^{\hat{\mu}}\hat{\mathrm{M}}^{-1}\right)+\frac{3}{8}\hat{\mathrm{H}}^T_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{\mathrm{M}}\hat{\mathrm{H}}^{\hat{\mu}\hat{\nu}\hat{\rho}}+\frac{5}{12}\hat{F}^2_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}+\right.$$
$$\left.+\frac{1}{192}\varepsilon^{ij}\hat{\varepsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\lambda}\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}D_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\hat{H}^{(i)}_{\hat{\tau}\hat{\lambda}\hat{\alpha}}\hat{H}^{(j)}_{\hat{\beta}\hat{\gamma}\hat{\delta}}\right\}\quad (3.1)$$

The field definitions are

$$\hat{\mathrm{M}} = \frac{1}{\mathrm{Im}\,\hat{\lambda}}\begin{pmatrix} \left|\hat{\lambda}\right|^2 & -\mathrm{Re}\,\hat{\lambda} \\ -\mathrm{Re}\,\hat{\lambda} & 1 \end{pmatrix}; \quad \hat{\lambda} = \hat{l} + ie^{-\hat{\varphi}}, \quad (3.2a)$$

$$\hat{\mathrm{H}}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \begin{pmatrix} \hat{H}^{(1)}_{\hat{\mu}\hat{\nu}\hat{\rho}} \\ H^{(2)}_{\hat{\mu}\hat{\nu}\hat{\rho}} \end{pmatrix}; \quad \hat{H}^{(i)}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \partial_{[\hat{\mu}}\hat{B}^{(i)}_{\hat{\nu}\hat{\rho}]}, \quad (3.2b)$$

$$\hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}} = \partial_{[\hat{\mu}}\hat{D}_{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}]} + \frac{3}{4}\varepsilon^{ij}\hat{B}^{(i)}_{[\hat{\mu}\hat{\nu}}\partial_{\hat{\rho}}\hat{B}^{(j)}_{\hat{\sigma}\hat{\tau}]}. \quad (3.2c)$$

Also, $\hat{\varepsilon}_{0\ldots9} = \sqrt{-g}$. It is in the final term of equation (3.1) that $D=4$ vectors (from $\hat{D}$) interact with the $D=4$ scalars from the three-form field strengths. It is convenient to rewrite this (up to an overall constant) as

$$\varepsilon^{ij}\int \hat{F}\wedge\hat{H}^{(i)}\wedge\hat{B}^{(j)}. \quad (3.3)$$

To compactify to four dimensions use

$$\hat{F} = F^{\Lambda}\wedge\alpha_{\Lambda} - G_{\Lambda}\wedge\beta^{\Lambda} + \ldots, \quad (3.4)$$

where $(\alpha_{\Lambda},\beta^{\Lambda}), \Lambda = \{0,1,\ldots,h_{21}\}$ are some choice of symplectic basis for $H^3(CY)$, $F^{\Lambda}_{\mu\nu}$ are the 4-dimensional vector field strengths and $G_{\Lambda\mu\nu}$ are the magnetic field strengths and the dual relationship between $F^{\Lambda}_{\mu\nu}$ and $G_{\Lambda\mu\nu}$ is due to self-duality of $\hat{F}$; the terms that have been left out of equation (3.4) are those which will not contribute to the integral in equation (3.3). The 3-form field strengths are given Calabi-Yau expectation values via

$$<\hat{H}^{(1)}> = \nu^{(1)}_{e\Lambda}\beta^{\Lambda} - \nu^{(1)\Lambda}_{m}\alpha_{\Lambda}, \quad (3.5a) \qquad <\hat{H}^{(2)}> = \nu^{(2)}_{e\Lambda}\beta^{\Lambda} - \nu^{(2)\Lambda}_{m}\alpha_{\Lambda}, \quad (3.5b)$$

where the $\nu_{m(e)}$ are constants that have been prematurely identified as values of the magnetic (electric) charges. Using equations (3.5), integration of equation (3.3) over the Calabi-Yau gives

$$\varepsilon^{ij}\int\left(\nu^{(i)}_{e\Lambda}F^{\Lambda}\wedge B^{(j)} - \nu^{(i)\Lambda}_{m}G_{\Lambda}\wedge B^{(j)}\right). \quad (3.6)$$

Writing F^{Λ} and G_{Λ} in terms of electric and magnetic vector potentials A^{Λ}_{μ} and $\tilde{A}_{\Lambda\mu}$, gives, after an integration by parts, (again up to a constant)

$$\varepsilon^{ij}\int d^4x\sqrt{-g}\left(\nu^{(i)}_{e\Lambda}A^{\Lambda}_{\mu}H^{(j)\mu} - \nu^{(i)\Lambda}_{m}\hat{A}_{\Lambda\mu}H^{(j)\mu}\right), \quad (3.7)$$

where

$$H^{(i)\mu} = \frac{1}{6}\varepsilon^{\nu\sigma\tau\mu}\partial_{\nu}B^{(i)}_{\sigma\tau}. \quad (3.8)$$

The result is that to lowest order in the coupling constant, with, for simplicity, the fields corresponding to the h_{21} data set to zero,

$$H_\mu^{(1)} = \frac{2}{3} e^{\frac{5}{4}\tilde{K} - \frac{1}{4}K} \partial_\mu \operatorname{Im} S\,, \quad (3.9a) \qquad H_\mu^{(2)} = \frac{2\sqrt{2}}{3} e^{\frac{\tilde{K}}{4} + \frac{3K}{4}} \partial_\mu \operatorname{Im} C_0\,; \quad (3.9b)$$

also, the string coupling constant is

$$e^{\hat{\varphi}} = \sqrt{2} e^{\frac{\tilde{K}}{2} - \frac{K}{2}}. \quad (3.9c)$$

Here S and C_0 are the $N=1$ superfields which form the dilaton hypermultiplet; the four dimensional dilaton has been generalized to

$$\phi = \frac{1}{2} e^{-\tilde{K}}. \quad (3.10)$$

Substituting equations (3.9) into equation (3.7) gives, after a Weyl rescaling $g_{\mu\nu} \to \sqrt{2} e^{\frac{3\tilde{K}}{4} + \frac{K}{4}} g_{\mu\nu}$ (to go to the $D=4$ Einstein metric),

$$\frac{2\sqrt{2}}{3} \int d^4x \sqrt{-g} \Big\{ \sqrt{2} e^{\tilde{K}+K} \nu_{e\Lambda}^{(1)} A_\mu^\Lambda \partial^\mu \operatorname{Im} C_0 - \sqrt{2} e^{\tilde{K}+K} \nu_m^{(1)\Lambda} \tilde{A}_{\mu\Lambda} \partial^\mu \operatorname{Im} C_0 - e^{2\tilde{K}} \nu_{e\Lambda}^{(2)} A_\mu^\Lambda \partial^\mu \operatorname{Im} S + e^{2\tilde{K}} \nu_m^{(2)\Lambda} \tilde{A}_{\Lambda\mu} \partial^\mu \operatorname{Im} S \Big\}. \quad (3.11)$$

This can be recognized as the interaction terms of the vector potentials with charged fields e^S and e^{C_0}. Hence, completing the square with the kinetic terms for the hypermultiplets gives (with an appropriate numerical rescaling of $\nu_{e(m)\Lambda}^{(i)}$)

$$S = \int d^4x \sqrt{-g} \Big\{ 8 e^{\tilde{K}+K} \left(\nu_{e\Lambda}^{(1)} A_\mu^\Lambda - \nu_m^{(1)\Lambda} \tilde{A}_{\Lambda\mu} + \partial_\mu \operatorname{Im} C_0 \right)^2 + 2 e^{2\tilde{K}} \left(\nu_{e\Lambda}^{(2)} A_\mu^\Lambda - \nu_m^{(2)\Lambda} \tilde{A}_{\Lambda\mu} + \partial_\mu \operatorname{Im} S \right)^2 + \ldots \Big\}. \quad (3.12)$$

From this equation, it is seen that $\operatorname{Im} C_0$ carries electric (magnetic) charges $\nu_{e\Lambda}^{(1)}$ $\left(\nu_m^{(1)\Lambda}\right)$ and that $\operatorname{Im} S$ carries electric (magnetic) charges $\nu_{e\Lambda}^{(2)}$ $\left(\nu_m^{(2)\Lambda}\right)$.

With regard the compactification of IIB on a Calabi-Yau, now, the attention is restricted to an $h_{11} = 1$, $h_{21} = 0$ Calabi-Yau. The (uncomplexified) moduli space therefore is one-dimensional, and corresponds to the choice of metric; specifically a conformal factor e^σ. Furthermore, as RR fields are suppressed in string perturbation theory, and because only the structure of the dilaton multiplet is of interest, it will be convenient to take

$$\hat{l} = 0; \quad \hat{B}_{i\bar{j}} = 0; \quad D_{\hat{\mu}\hat{\nu}\hat{\sigma}\hat{\tau}} = 0. \quad (3.13)$$

The equations of motion are usually written in terms of the fields

$$\hat{\psi} = \frac{1 + i\hat{\lambda}}{1 - i\hat{\lambda}} = \frac{1 - e^{-\hat{\varphi}}}{1 + e^{-\hat{\varphi}}}, \quad (3.14a) \qquad \hat{P}_{\hat{\mu}} = \frac{\partial_{\hat{\mu}} \hat{\psi}}{1 - \hat{\psi}^* \hat{\psi}}, \quad (3.14b)$$

$$\hat{Q}_{\hat{\mu}} = \frac{\operatorname{Im}\left(\hat{\psi} \hat{\partial}_{\hat{\mu}} \hat{\psi}^*\right)}{1 - \hat{\psi}^* \hat{\psi}}, \quad (3.14c) \qquad \hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \frac{\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} - \hat{\psi} \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}^*}{\left(1 - \hat{\psi}^* \hat{\psi}\right)^{\frac{1}{2}}}; \quad \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}^{(1)} + i \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}^{(2)}. \quad (3.14d)$$

The equation of motion that will be most interesting is

$$\left(\nabla_{\hat\mu} - i\hat Q_{\hat\mu}\right)\hat G^{\hat\mu}_{\hat\nu\hat\rho} = \hat P_{\hat\mu}\hat G^{*\hat\mu}_{\hat\nu\hat\rho}. \quad (3.15)$$

Equation (3.15) is satisfied trivially on the Calabi-Yau. After performing a 4-dimensional Weyl rescaling $g_{\mu\nu} \to e^{-3\sigma} g_{\mu\nu}$, equation (3.15) becomes (on the spacetime)

$$\left(\nabla_\mu + \frac{\psi^*\partial_\mu\psi - \psi\partial_\mu\psi^*}{2\left(1-\psi^*\psi\right)}\right) e^{3\sigma} G^\mu_{\nu\rho} = \frac{\partial_\mu\psi}{1-\psi^*\psi} e^{3\sigma} G^{*\mu}_{\nu\rho}. \quad (3.16)$$

Subtracting ψ times the complex conjugate of equation (3.16), from equation (3.16), gives

$$\left(1-\psi^*\psi\right)^{\frac{1}{2}} \nabla_\mu \left[e^{3\sigma} \frac{G^\mu_{\nu\rho} - \psi G^{*\mu}_{\nu\rho}}{\left(1-\psi^*\psi\right)^{\frac{1}{2}}} \right], \quad (3.17)$$

which is satisfied by introducing a complex scalar field D such that

$$\partial_\mu D = e^{3\sigma} \frac{G_\mu - \psi G^*_\mu}{\left(1-\psi^*\psi\right)^{\frac{1}{2}}}, \quad (3.18)$$

where, as in equation (3.8)

$$G_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho}{}^{\sigma} G_\sigma. \quad (3.19)$$

The other equation of motion that is used is

$$\hat R_{\hat\mu\hat\nu} = 2\hat P_{(\hat\mu}\hat P^*_{\hat\nu)} + \frac{9}{4}\hat G_{(\hat\mu}{}^{\hat\sigma\hat\tau}\hat G^*_{\hat\nu)\hat\sigma\hat\tau} - \frac{3}{16}\hat g_{\hat\mu\hat\nu}\hat G^{\hat\nu\hat\sigma\hat\tau}\hat G^*_{\hat\nu\hat\sigma\hat\tau}. \quad (3.20)$$

By substituting the Calabi-Yau part of this equation into the space-time part of the equation, the four-dimensional action

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2}R + \left|P_\mu\right|^2 + 3\left(\partial_\mu\sigma\right)^2 + \frac{9}{4}\left|G_\mu\right|^2\right\} \quad (3.21)$$

can be deduced. Alternatively, this can be found, almost by inspection, via dimensional reduction of the NSD action of equation (3.1). There is a space-time dependent conformal factor of e^σ in the Calabi-Yau metric; hence $\sqrt{-\hat g} = e^{3\sigma}\sqrt{-g}$ and so to remain in the Einstein frame required the Weyl rescaling of the four dimensional metric $g_{\mu\nu} \to e^{-3\sigma} g_{\mu\nu}$. This is the same Weyl rescaling used in the derivation of equation (3.17) and the reason for it. To obtain the standard quaternionic geometry, we make the field redefinitions,

$$Z = -ie^{\sigma+\frac{1}{2}\varphi}; \quad (3.22a) \qquad C_0 = i\frac{3\sqrt{2}}{4}\operatorname{Im} D; \quad (3.22b)$$

$$\phi = e^{3\sigma-\frac{1}{2}\varphi}; \quad (3.22c) \qquad \phi = \operatorname{Re} D; \quad (3.22d) \qquad \text{and} \qquad S = \phi + i\tilde\phi. \quad (3.22e)$$

Define also

$$K = -\ln\left(-\frac{i}{8}\left(Z - \overline{Z}\right)\right); \quad (3.23a) \quad \widetilde{K} = -\ln\left[S + \overline{S}\right]; \quad (3.23b) \quad \mathrm{N}_{00} = R_{00} = \frac{i}{32}\left(Z - \overline{Z}\right)^3; \quad (3.23c)$$

$$D_\mu C_0 = \partial_\mu C_0; \quad (3.23d); \text{ and } \quad D_\mu S = \partial_\mu S. \quad (3.23e)$$

Then, the scalar part of the action of equation (3.21) becomes

$$S = \int d^4x\sqrt{-g}\left\{K_{Z\overline{Z}}\partial_\mu Z\partial^\mu \overline{Z} + \widetilde{K}_{S\overline{S}} D_\mu S D^\mu \overline{S} + \widetilde{K}_{S\overline{C}_0} D_\mu S D^\mu \overline{C}_0 + \widetilde{K}_{C_0\overline{S}} D_\mu C_0 D^\mu \overline{S} + \widetilde{K}_{C_0\overline{C}_0} D_\mu C_0 D^\mu \overline{C}_0\right\}, \quad (3.24)$$

where the subscripts on K, $\widetilde{K}$ denote differentiation. We note that C_0 is pure imaginary.

B. *Hartle-Hawking Wave-function in the mini-superspace sector of physical superstring theory.* [4]

Suppose we consider the compactification of type IIB superstrings to two dimensions on a Calabi-Yau threefold M times a 2-sphere $M \times S^2$. Turn on 5-form fluxes for the RR[2] 5-form field strength to be

$$F_5 = F_3 \wedge \omega, \quad (3.25)$$

where ω is a unit volume form on S^2, and F_3 is a 3-form on M. Choosing an integral basis of magnetic/electric $H^3(M)$ as $\{\alpha_I, \beta^J\}_{I=0,\ldots h^{2,1}}$, we write

$$F_3 = \sum_I \left(p^I \alpha_I + q_I \beta^I\right). \quad (3.26)$$

It is possible to write down a superpotential whose extremization leads to the condition for (2, 2) supersymmetry in d = 2:

$$W = \int_{M\times S^2} F_5 \wedge \Omega, \quad (3.27)$$

hence

$$W = \int_{M\times S^2} \sum_I \left(p^I \alpha_I + q_I \beta^I\right) \wedge \omega \wedge \Omega, \quad (3.27b)$$

where Ω is the holomorphic 3-form on the Calabi-Yau three-fold. To deduce this superpotential, we note that this superpotential is consistent with the tension (BPS mass in 1 dimension) of the domain wall D3 brane which wraps a 3-cycle in the Calabi-Yau and changes the F_5 flux.
The condition for extremization of W and preserving supersymmetry is

[2] We remember that the RR (Ramond-Ramond) states in type I and type II superstring theories, are the bosonic closed string states whose left- and right-moving parts are fermionic. These include p-form potentials C_p, with p taking all odd values in the IIA string and all even values in the IIB string.

$$DW = 0 . \quad (3.28)$$

The complex structure of the Calabi-Yau are field variables and we can look for extrema of W with respect to their variation. We denote variation of Ω in arbitrary direction of $H^{2,1}(X)$ by $\delta\Omega$. The supersymmetry condition (3.28) is equivalent to

$$\int_{M\times S^2} \delta\Omega \wedge F_5 = \int_M \delta\Omega \wedge F_3 = 0 . \quad (3.28b)$$

Since F_3 is also real, this implies that $F_3 \in H^{3,0} + H^{0,3}$. Using the fact that there is only one element in $H^{3,0}$ represented by Ω, and using the reality condition for F_3 we deduce that

$$F_3 = \mathrm{Re}(C\Omega),$$

for some complex number C. In other words

$$p^I = \mathrm{Re}(CX^I), \quad q_I = \mathrm{Re}(CF_I), \quad (3.29)$$

where

$$X^I = \int_{A_I} \Omega, \quad F_I = \int_{B^I} \Omega,$$

hence

$$W = \int_{M\times S^2} \sum_I \left[\mathrm{Re}\left(C\int_{A^I}\Omega\right)\alpha_I + \mathrm{Re}\left(C\int_{B^I}\Omega\right)\beta^I \right] \wedge \omega \wedge \Omega , \quad (3.29b)$$

and (A_I, B^I) are 3-cycles on M that are dual to the 3-forms (α_I, β^I). With the complex structure of the Calabi-Yau satisfying (3.29), supersymmetry is preserved with a suitable choice of metric in 2 spacetime dimensions, i.e. the AdS_2 metric. For such a complex structure, the superpotential W is not zero, but it is proportional to $\int_M \Omega \wedge \overline{\Omega}$. In addition, the size of S^2 is also determined by the supersymmetry condition, and we find that

$$\mathrm{Area}\left(S^2\right) = \pi C\overline{C} \int_M \Omega \wedge \overline{\Omega} . \quad (3.30)$$

This is exactly the content of the attractor mechanism. To case it in the standard description of the black hole attractor, we consider D3 branes wrapping q_I times on A_I and p^I times on B^I. This gives rise to a supersymmetric black hole in four dimensions, whose BPS[3] mass M_{BPS} is given by

$$M_{BPS}^2 = K^{-1}|W|^2 , \quad (3.31)$$

where the exponentiated Kahler potential K is given by

[3] We remember that the BPS (Bogomolny-Prasad-Sommerfield) state is a state that is invariant under a non-trivial subalgebra of the full supersymmetry algebra. Such states always carry conserved charges, and the supersymmetry algebra determines the mass of the state exactly in terms of its charges.

$$K = i\left(\overline{X}^I F_I - X^I \overline{F}_I\right) = -2\,\mathrm{Im}\,\tau_{IJ} X^I \overline{X}^J\,, \quad (3.32)$$

and W is the superpotential (3.27), which can also be expressed as

$$W = q_I \int_{A_I} \Omega - p^I \int_{B^I} \Omega = q_I X^I - p^I F_I\,. \quad (3.33)$$

Furthermore, the variation of the action

$$S = -\frac{\pi}{4}\left(K\left(X, \overline{X}\right) + 2iW(X) - 2i\overline{W}\left(\overline{X}\right)\right) = -\frac{\pi}{4}\left(\int \Omega \wedge \overline{\Omega} + \int \left(\Omega + \overline{\Omega}\right) \wedge F_3\right), \quad (3.34)$$

in arbitrary directions of $H^3(M)$, reproduces the attractor equations (3.29).
Consider type IIB superstring compactified on the Calabi-Yau 3-fold M times $S^2 \times S^1$. There is a natural Euclidean solution to the classical equations of motion which develops from this spacelike section. It is the geometry $M \times S^2 \times H_2 / \mathbf{Z}$, where H_2 is the hyperbolic disk, i.e. the Euclideanized AdS_2, with the metric

$$ds^2 = d\rho^2 + e^{2\rho} d\tau^2\,, \quad (3.35)$$

and the $\mathbf{Z}$ quotient periodically identifies $\tau \approx \tau + \beta$. If we view ρ as an Euclideanized time, the geometry $M \times S^2 \times H_2 / \mathbf{Z}$ describes an Euclidean time evolution of type II string compactified on $M \times S^2 \times S^1$. This is how we were originally led to the metric (3.35).
Let us consider a natural notion of a "mini-superspace" where we view ρ as an Euclideanized time. Among relevant light modes are the complex moduli of the Calabi-Yau three-fold denoted by $z^i\left(i = 1,\ldots,h^{2,1}\right)$, which are in vector multiplets in four dimensions. The gravity multiplet also produces some scalar fields upon compactification on $S^2 \times S^1$. One is the radius R of S^2. Another scalar field is related to how the S^1 is fibered over S^2. The radius R and the chemical potential φ naturally combine with the complex structure moduli z of the Calabi-Yau to make a "large moduli space" with coordinates $X^I\left(I = 0,1,\ldots,h^{2,1}\right)$. More explicitly, choose any holomorphic section $X_0^I(z)$ over the complex structure moduli space and define

$$X^I = i2\,\mathrm{R}e^{i\varphi}\left(\frac{\overline{W}(X_0)}{K\left(X_0, \overline{X}_0\right)W(X_0)}\right)^{1/2} X_0^I\,. \quad (3.36)$$

Upon compactification on S^1, each gauge field becomes equivalent to a pair of massless scalar fields – one is the Wilson line of the gauge field along the S^1, and the other is the dual magnetic potential around the S^1,

$$\phi^I = \oint_{S^1} A^I\,, \quad \tilde{\phi}_I = \oint_{S^1} \tilde{A}_I\,, \quad (3.37)$$

where $\tilde{A}_I$ is the dual of A^I in four dimensions. One can also think of $\tilde{\phi}_I$ as the dual of the massless gauge field in three dimensions. By definition, they couple to the charges $\left(p^I, q_I\right)$ of the black hole as

$$e^{i\sum_I \left(q_I \phi^I + p^I \tilde{\phi}_I\right)}\,. \quad (3.38)$$

Combining with X^I defined in (3.36), a set of four scalar fields $\left(X^I, \bar{X}^I, \phi^I, \tilde{\phi}_I\right)$ for each I gives bosonic components of supermultiplets. The dependence of the wave-function on $\left(\phi^I, \tilde{\phi}_I\right)$ is also simple. Since ϕ^I is dual to $\tilde{A}_I$ and $\tilde{\phi}_I$ is dual to A^I in three dimensions, we have

$$G_{IJ} \frac{d\phi^J}{d\rho} = \int_{S^2} \tilde{F}_I = q_I \,, \quad G^{IJ} \frac{d\tilde{\phi}_J}{d\rho} = \int_{S^2} F^I = p^I \,, \quad (3.39)$$

where G_{IJ} is the metric in the kinetic term for the gauge fields, and $F^I = dA^I$, $\tilde{F}_I = d\tilde{A}_I$. These equations means that, when we quantize the theory along the ρ direction, $(\phi^I, \tilde{\phi}_I)$ are canonically conjugate to (q_I, p^I). Therefore, the wave-function depends on $\left(\phi, \tilde{\phi}\right)$ as

$$\Phi_{p,q}\left(X, \bar{X}; \phi, \tilde{\phi}\right) = e^{i\sum_I \left(q_I \phi^I + p^I \tilde{\phi}_I\right)} \Psi_{p,q}\left(X, \bar{X}\right), \quad (3.40)$$

if it is an eigenstate of the fluxe quantum numbers (p,q). This $\left(\phi, \tilde{\phi}\right)$ dependence is also expected from the fact that $\left(\phi, \tilde{\phi}\right)$ are electric and magnetic static potentials for the black hole charges (3.38). Now we consider the original Hartle-Hawking wave-function for a three-sphere S^3. The saddle point computation of the wave-function on S^3 can be viewed as filling it with a 4-dimensional ball with the S^3 as its boundary, and this leads to the action S_E in the Euclidean ball,

$$S_E \approx -\frac{1}{\Lambda},$$

where Λ is the cosmological constant, and the mini-superspace wave-function behaves as

$$\Psi \approx \exp\left(-\frac{1}{\Lambda}\right).$$

In the present context $\Lambda < 0$, and $-\frac{1}{\Lambda} \approx Area\left(S^2\right) \approx S_{entropy}$, so we may expect that

$$\int dX d\bar{X} \left|\Psi_{p,q}\left(X, \bar{X}\right)\right|^2 \approx \exp\left(S_{entropy}\right), \quad (3.41)$$

namely the wave-function is normalized by the exponential of the entropy. It is natural since the string partition function on the full space $M \times S^2 \times H_2 / \mathbf{Z}$ should give the black hole entropy. Thus, at least semi-classically we expect (3.41) to hold. In view of our discussion following (2.11) a natural guess for the probability measure is

$$\left|\psi_{p,q}\left(X, \bar{X}\right)\right|^2 \approx \exp\left[-\frac{\pi}{4} K - \frac{\pi}{2} i\left(W - \bar{W}\right)\right], \quad (3.42)$$

hence

$$\int dX d\bar{X} \exp\left[-\frac{\pi}{4} K - \frac{\pi}{2} i\left(W - \bar{W}\right)\right] \approx \exp\left(S_{entropy}\right), \quad (3.41b)$$

where $W(X)$ depends on the flux determined by (p,q) as in (3.33). Indeed, the right-hand side is peaked at the attractor value and its value is given by the exponential of the entropy. To see this in terms of the physical variables, the complex structure moduli z^i and the radius R of the S^2, we can substitute (3.36) into (3.42) and find

$$\left|\Psi_{p,q}\right|^2 \approx \exp\left[-\pi\left(R^2 - M_{BPS}R\right)\right]. \quad (3.43)$$

Extremizing this with respect to the complex structure moduli z^i gives the attractor equation and extremizing with respect to R gives

$$\left|\Psi_{p,q}\right|^2\Big|_{extremum} \approx \exp\left(\frac{\pi}{4}M_{BPS}^2\right) = \exp\left(S_{entropy}\right),$$

reproducing the expected result. The eq. (3.42) captures essential aspects of the wave-function. This same wave function will give a semi-classical approximation to the topological string partition function, which we will argue gives the exact answer for the Hartle-Hawking wave function including all string loop corrections.

The wave-function $\Psi_{p,q}\left(X,\bar{X}\right)$ should satisfy the Wheeler-De Witt equation. In mini-superspace the WDW equation corresponds to the quantization of the attractor flow for a black hole with charges p^I and q_I. Consider a ten-dimensional Euclidean metric of the form,

$$ds^2 = e^{2U+2\rho}d\tau^2 + e^{-2U}d\rho^2 + e^{-2U}d\Omega^2 + ds_{CY}^2, \quad (3.44)$$

where τ is the Euclideanized time direction compactified on S^1, ρ is the radius coordinate, $d\Omega^2$ is the metric on a two-sphere of unit radius, and ds_{CY}^2 is the metric on the internal Calabi-Yau three-fold. Note that e^{-U} is the radius of the S^2, and the AdS_2 geometry is realized when U is constant. Since we are interested in BPS configurations and since the supercharges preserved by the background square to become the translation along the τ direction, we assume that the scale factor e^U and the complex moduli $z^i\left(i=1,\ldots,h^{2,1}\right)$ of the Calabi-Yau three-fold are independent of τ. In this case, we have a one-dimensional system along the ρ direction described by the effective action,

$$S_{eff} = \frac{1}{2}\int_{-\infty}^{\infty} d\rho e^{\rho}\left[\left(\frac{dU}{d\rho}+1\right)^2 + g_{i\bar{j}}\frac{dz^i}{d\rho}\frac{d\bar{z}^{\bar{j}}}{d\rho} + e^{2U}\left(M_{BPS}^2(z,\bar{z}) + 4g^{i\bar{j}}\partial_i M_{BPS}\bar{\partial}_{\bar{j}}M_{BPS}\right)\right], \quad (3.45)$$

where

$$M_{BPS} = \sqrt{\frac{W(z)\bar{W}(\bar{z})}{K(z,\bar{z})}}. \quad (3.46)$$

We regard ρ as the Euclidean time of the system, which flows from $\rho=+\infty$ to $-\infty$. Since the effective action (3.45) can be written as

$$S_{eff} = \frac{1}{2}\int_{-\infty}^{\infty} d\rho e^{\rho}\left[\left(\frac{dU}{d\rho}+1-e^U M_{BPS}\right)^2 + g_{i\bar{j}}\left(\frac{dz^i}{d\rho} - 2e^U g^{i\bar{m}}\bar{\partial}_{\bar{m}}M_{BPS}\right)\right.$$

$$\left(\frac{d\bar{z}^{\bar{j}}}{d\rho} - 2e^U g^{\bar{j}n}\partial_n M_{BPS}\right)\Bigg] + (\text{total derivative}), \quad (3.46b)$$

the BPS equations are

$$\frac{dU}{d\rho} = -1 + e^U M_{BPS}(z,\bar{z}), \quad \frac{dz^i}{d\rho} = 2e^U g^{i\bar{j}}\bar{\partial}_{\bar{j}} M_{BPS}(z,\bar{z}). \quad (3.47)$$

The signs on the right-hand side of these equations are chosen so that they are compatible with the initial condition at $\rho \to \infty$, which we regard as the infinite past in the Euclidean time.
The equations (3.47) can be combined into a single equation on the large moduli space. To write down such an equation, we start with a holomorphic section $X_0^I(z)(I = 0,1,..., h^{2,1})$ over the moduli space of complex structure. They make projective coordinates of the moduli space, and as such there is a freedom to rescale these coordinates. We define the exponentiated Kahler potential K_0 and the superpotential W_0 for these coordinates as

$$K_0 = -2\,\mathrm{Im}\,\tau_{IJ} X_0^I \bar{X}_0^J, \quad W_0 = q_I X_0^I - p^I F_I(X_0).$$

We then combine the scale factor e^U in the metric (3.44) and the complex moduli z^i into a single set of coordinates X^I defined by

$$X^I = 2ie^{-U}\left(\frac{\overline{W_0}}{K_0 W_0}\right)^{1/2} X_0^I. \quad (3.48)$$

Note that the right-hand side of (3.48) is invariant under rescaling of X_0^I. Moreover

$$K(X,\bar{X}) = -2\,\mathrm{Im}\,\tau_{IJ} X^I \bar{X}^J = \left(2e^{-U}\right)^2$$

is the diameter squared of the S^2. Thus, the large moduli space parametrized by X^I combines the complex moduli z^i and the radius of the S^2.
Using X^I, the attractor flow equations (3.47) can be written as a single equation

$$\frac{dX^I}{d\rho} = X^I + \left(\frac{i}{\mathrm{Im}\,\tau}\right)^{IJ} \bar{\partial}_J \bar{W}(\bar{X}). \quad (3.49)$$

Let us show that (3.49) is equivalent to (3.47). If we multiply $\bar{X}^J\,\mathrm{Im}\,\tau_{JI}$ to both sides of (3.49), the left-hand side becomes

$$\bar{X}^I\,\mathrm{Im}\,\tau_{IJ}\frac{dX^J}{d\rho} = 2e^{-2U}\frac{dU}{d\rho} - \frac{e^{-2U}}{M_{BPS}}\left(\frac{dz^i}{d\rho}\partial_i M_{BPS} - \frac{d\bar{z}^{\bar{i}}}{d\rho}\bar{\partial}_{\bar{i}} M_{BPS}\right).$$

On the other hand, the right hand-side becomes

$$X^I \operatorname{Im}\tau_{IJ}\overline{X}^J + i\overline{X}^I \partial_I \overline{W}\left(\overline{X}_0\right) = -\frac{1}{2}K\left(X,\overline{X}\right) + i\overline{W}\left(\overline{X}\right) = -2e^{-2U} + 2e^{-U}M_{BPS}.$$

Combining them together, we obtain

$$\frac{dU}{d\rho} - \frac{1}{2M_{BPS}}\left(\frac{dz^i}{d\rho}\partial_i M_{BPS} - \frac{d\overline{z}^{\bar{i}}}{d\rho}\overline{\partial}_{\bar{i}} M_{BPS}\right) = -1 + e^U M_{BPS}.$$

The real part of this equation is precisely the first of (3.47):

$$\frac{dU}{d\rho} = -1 + e^U M_{BPS}. \quad (3.50)$$

The imaginary part gives

$$\frac{dz^i}{d\rho}\partial_i M_{BPS} = \frac{d\overline{z}^{\bar{i}}}{d\rho}\overline{\partial}_{\bar{i}} M_{BPS}. \quad (3.51)$$

Similarly multiplying $\partial_i X_0^J \operatorname{Im}\tau_{IJ}$ to both sides of (3.49) and using (3.50) and (3.51), we find

$$\frac{dz^i}{d\rho} = 2e^U g^{i\bar{j}}\overline{\partial}_{\bar{j}} M_{BPS}. \quad (3.52)$$

Moreover, (3.52) implies (3.51) since both sides of (3.51) are now equal to $-2e^U g^{i\bar{j}}\partial_i M_{BPS}\overline{\partial}_{\bar{j}} M_{BPS}$. Therefore, (3.49) for X^I defined by (3.48) is equivalent to the standard BPS equations (3.47).
In (3.49), a general BPS solution can be easily expressed. Taking the real and imaginary parts of this equation, one finds

$$\operatorname{Re}\left(X^I - \frac{dX^I}{d\rho}\right) = p^I, \quad \operatorname{Re}\left(F_I - \frac{dF_I}{d\rho}\right) = q_I.$$

A general solution to this is then

$$\operatorname{Re} X^I = p^I + c^I e^\rho, \quad \operatorname{Re} F_I = q_I + d_I e^\rho,$$

where $\left(c^I, d_I\right)$ are integration constants specified by the initial condition at the infinite past $\rho = \infty$. Whatever initial condition one chooses there, X^I at the infinite future $\rho \to -\infty$ are fixed to be at the attractor value,

$$\operatorname{Re}\left(X^I\right) \to p^I, \quad \operatorname{Re}\left(F_I\right) \to q_I.$$

It is useful to write the BPS equation (3.49) as

$$\operatorname{Im}\left(\tau_{IJ}\right)\frac{dX^I}{d\rho} = -\frac{1}{2}\overline{\partial}_I K + i\partial_I \overline{W}. \quad (3.53)$$

Now, we will use this equation to obtain the supersymmetric version of the WDW equation. The supersymmetric WDW equation is equivalent to the quantum version of the BPS equation (3.49) and its complex conjugate. To understand the quantum version of the BPS equation, we note that the metric in the X space implied by the effective action (3.45) is almost given by $\mathrm{Im}(\tau_{IJ})$ since

$$X^I \,\mathrm{Im}(\tau_{IJ})\overline{X}^J = -2R^2, \quad D_i X^I \,\mathrm{Im}(\tau_{IJ})\overline{X}^J = X^I \,\mathrm{Im}(\tau_{IJ})\overline{D}_{\bar{j}}\overline{X}^J = 0, \quad D_i X^I \,\mathrm{Im}(\tau_{IJ})\overline{D}_{\bar{j}}\overline{X}^J = 2R^2 g_{i\bar{j}}, \tag{3.54}$$

where

$$D_i X^I = K\partial_i\left(K^{-1}X^I\right), \quad \overline{D}_{\bar{j}}\overline{X}^J = K\overline{\partial}_{\bar{j}}\left(K^{-1}\overline{X}^J\right),$$

and $g_{i\bar{j}} = \partial_i \overline{\partial}_{\bar{j}} \ln K$. We note that $\mathrm{Im}\,\tau_{IJ}$ has one negative sign in the direction of $H^{(3,0)}$ whereas it gives the standard positive definite metric in the $H^{(2,1)}$ direction. Flipping the sign in the $H^{(3,0)}$ direction gives what is denoted by $\mathrm{Im}\,\mathrm{N}_{IJ}$ in the supergravity literature, which is the metric derived from the effective action (3.45). In the semi-classical approximation, flipping of the sign of the metric can be done by a suitable contour deformation in a functional integral. Thus, we will use $\mathrm{Im}\,\tau_{IJ}$ as our metric and the corresponding quantization rule is

$$\frac{\pi}{4}\mathrm{Im}(\tau_{IJ})\frac{dX^J}{d\rho} \to -\frac{\partial}{\partial \overline{X}_J}.$$

Given this rule, the quantum version of the BPS equation (3.49) is

$$\left(\frac{\partial}{\partial \overline{X}^I} - \frac{\pi}{8}\overline{\partial}_I K + i\frac{\pi}{4}\overline{\partial}_I \overline{W}\right)\Psi_{p,q} = 0, \tag{3.55}$$

and similarly for the complex conjugate equation

$$\left(\frac{\partial}{\partial X^I} - \frac{\pi}{8}\partial_I K - i\frac{\pi}{4}\partial_I W\right)\Psi_{p,q} = 0. \tag{3.56}$$

We denote the operator appearing in the constraint (3.55) $\overline{C}_I$ and the one in (3.56) by C_I, so that in terms of the state $\left|\Psi_{p,q}\right\rangle$ the constraints are $\overline{C}_I\left|\Psi_{p,q}\right\rangle = 0$ and $C_I\left|\Psi_{p,q}\right\rangle = 0$ respectively. Imposing both constraints is sufficient to determine the entire wave function. One finds in this way

$$\Psi_{p,q}\left(X,\overline{X}\right) = \exp\left[\frac{\pi}{8}K\left(X,\overline{X}\right) + \frac{\pi}{4}\left(iW(X) - i\overline{W}\left(\overline{X}\right)\right)\right].$$

One can use the BPS constraints also in a way that leads to a description of the covariant supersymmetric gradient flow, also known as the attractor flow, in terms of a holomorphic wave function $\psi_{p,q}(X)$. This wave function can be obtained by imposing first the constraint (3.55). This reduces $\Psi\left(X,\overline{X}\right)$ essentially to a holomorphic function. The second condition (3.56) then fixes $\psi_{p,q}(X)$ to be given by

$$\psi_{p,q}(X) = e^{i\frac{\pi}{2}W_{p,q}(X)}. \tag{3.57}$$

We note, in particular, that $\psi_{0,0}(X)=1$. In the remaining part of this chapter we will provide evidence that this wave function coincides with the semi-classical approximation obtained from the topological string partition function, which we propose to be the exact Hartle-Hawking wave function including all string loop corrections.

We abbreviate the operators appearing in the constraints (3.55) and (3.56) by $\overline{C}_I$ and C_I respectively. We want to impose $\overline{C}_I$ on the ket state $|\Psi\rangle$ and its conjugate $\overline{C}_J^\dagger$ on the bra state $\langle\Psi|$. Notice that $\overline{C}_J^\dagger$ differs from C_J in the sign of the derivative. The Dirac bracket is defined as

$$\left[X^I,\overline{X}^J\right]_{Dirac}=\left[X^I,\overline{X}^J\right]-\sum_{K,L}\left[X^I,\overline{C}_K^\dagger\right]\frac{1}{\left[\overline{C}_K^\dagger,\overline{C}_L\right]}\left[\overline{C}_L,\overline{X}^J\right],$$

where the denominator should be read as the inverse matrix. Here we only wrote commutators that we know are non-vanishing. For the constraints one finds

$$\left[\overline{C}_K,\overline{C}_L^\dagger\right]=-\frac{\pi}{4}\overline{\partial}_K\partial_L K=\frac{\pi}{2}\operatorname{Im}\tau_{KL},$$

while the commutators of the constraints with the coordinates give

$$\left[X^I,\overline{C}_K^\dagger\right]=\delta^I{}_K,\qquad \left[\overline{C}_L,\overline{X}^J\right]=\delta_L{}^J.$$

Inserting this in to the definition for the Dirac bracket leads to the following commutation relations for X^I and $\overline{X}^J$

$$\left[X^I,\overline{X}^J\right]_{Dirac}=\frac{2}{\pi}\left(\frac{1}{\operatorname{Im}\tau}\right)^{IJ}. \quad (3.58)$$

The Hilbert space for (3.58) can be represented by holomorphic wave-functions $\psi(X^I)$ of X^I, with the inner product defined by

$$\langle\psi_1|\psi_2\rangle=\int dXd\overline{X}e^{\frac{\pi}{4}K}\,\overline{\psi}_1(\overline{X})\psi_2(X). \quad (3.59)$$

The relation between the wave function in the real and complex polarization can be found at the semi-classical level by applying standard canonical transformation techniques. In classical mechanics canonical transformations can be described with the help of a generating function. In our case this generating function should depend on one of the real and one of the complex coordinates. Since we are interested in transforming wave functions $\psi(X)$ to $\psi(\chi)$ the appropriate choice is to use a function $S(X,\chi)$ of the real coordinate χ^I and the complex coordinate X^I. It is determined by requiring that the following canonical transformation

$$\chi^I=\operatorname{Re}(X^I),\qquad \eta_I=\operatorname{Re}(F_I), \quad (3.59b)$$

takes the form

$$\eta_I=\frac{1}{i\pi}\frac{\partial S(X,\chi)}{\partial\chi^I},\quad \overline{X}^I=\frac{2}{i\pi}\left(\frac{1}{\operatorname{Im}\tau}\right)^{IJ}\frac{\partial S(X,\chi)}{\partial X^J}.$$

After a little algebra one finds

$$S(X,\chi)=\frac{i\pi}{2}\left(\chi^I F_I(X)-\frac{1}{2}F_0(X)+\frac{1}{2}\overline{F}_0(2\chi-X)\right). \quad (3.60)$$

This leads to the equations

$$\eta_I=\frac{1}{2}F_I(X)+\frac{1}{2}\overline{F}_I(2\chi-X), \qquad \overline{X}^I=2\chi^I-X^I,$$

which are equivalent to (3.59b). The relation between the semi-classical wave function $\psi(X)=e^{g(X)}$ and the corresponding wave function $\psi(\chi)=e^{f(\chi)}$ in the real polarization is

$$\psi(X)=\int d\chi e^{\frac{i}{\pi}S(X,\chi)}\psi(\chi), \quad (3.61)$$

where it is understood that the right-hand side is computed in the saddle point approximation.
Now we want propose an exact such wave function which agrees in the semi-classical limit with the wave function discussed before. We will argue that the state with no flux can be identified with the topological string wave function:

$$|\psi_{0,0}\rangle=|\psi_{top}\rangle.$$

That this relation could hold presupposes that topological string partition function also corresponds to a wave function associated to quantizing $H^3(M)$. Moreover this is in agreement with the fact that at least semi-classically the topological string partition function describes the Hartle-Hawking wave function in the real polarization. We argue that the semi-classical result

$$|\psi_{p,q}\rangle=O_{p,q}|\psi_{0,0}\rangle$$

also holds with the expected factor

$$O_{p,q}=\exp\left[-\frac{i\pi}{4}W_{p,q}+c.c.\right] \quad (3.62)$$

interpreted as an operator acting on the topological string Hilbert space.
Semi-classically, the entropy of the BPS black hole obtained by wrapping D3 branes around cycles of M is given by the area (3.30) of the horizon. The resulting quantum corrected entropy formula can be concisely expressed as

$$S_{BH}(q,p)=\mathcal{F}(p,\phi)+\sum_I q_I\phi^I, \quad (3.63)$$

where

$$\mathcal{F}(p,\phi)=F_{top}(X)+\overline{F}_{top}(\overline{X}), \quad X=p+\frac{i}{\pi}\phi. \quad (3.64)$$

and

$$F_{top}(X)=\sum_{g=0}^{\infty}F_g(X)$$

is the full topological string partition function.
Moreover the quantum corrected attractor equations also take the simple form:

$$q_I = -\frac{\partial}{\partial \phi^I}\mathcal{F}(p,\phi). \quad (3.65)$$

At the attractor point, the string perturbation expansion is an asymptotic expansion for large black hole charges. Since (3.63) takes the Legendre transformation from ϕ to q, the number of states $\Omega(p,q)$ of the black hole with finite charges p^I, q_I is given by Laplace transformation of the topological string partition function

$$\Omega(p,q) = \int d\phi e^{-q_I\phi^I + \mathcal{F}(p,q)} = \int d\phi e^{-q_I\phi^I}\left| e^{F_{top}\left(p+\frac{i}{\pi}\phi\right)}\right|^2. \quad (3.66)$$

More precisely, the conjecture states that $\Omega(p,q)$ given by (3.66) is the Witten index for the quantum Hilbert space of the black hole.
We note also that the expression (3.66) for the number of states can be written in a nice way as a Wigner function. Namely, by taking the contour of the ϕ integral in (3.66) along the imaginary axis as $\phi = -i\pi\chi$, one gets

$$\Omega(p,q) = \int d\chi e^{i\pi q\chi}\overline{\psi}_{top}(p-\chi)\psi_{top}(p+\chi), \quad (3.67)$$

where

$$\psi_{top}(\chi) = e^{F_{top}(\chi)} \quad (3.68)$$

is the exact topological string partition function.
Now we can relate this result to the normalization of the Hartle-Hawking wave function.

Let $\left|\psi_{p,q}\right\rangle$ denote the state we obtain upon doing the path-integral on the right in a fixed flux sector. The above consideration leads to the statement that

$$\Omega(p,q) = \left\langle \psi_{p,q} \middle| \psi_{p,q}\right\rangle.$$

Now let us write the expression (3.66) in invariant form

$$\Omega(p,q) = \left\langle \overline{\psi}_{top}\middle| e^{i\pi(q\chi - p\eta)}\middle|\psi_{top}\right\rangle,$$

where χ and η are to be regarded as operators, and the state $\overline{\psi}$ is defined by the wave function $\psi^*(-\chi)$. Here, the state on the left is not simply the complex conjugate but it contains a minus sign due to the time reversal in the attractor flow equation. Next we note that the attractor relations (3.59b) imply that $q_I\chi^I - p^I\eta_I = \mathrm{Re}\, W(X)$, and hence

$$\Omega(p,q)=\left\langle \overline{\psi}_{top}\left| e^{\frac{-\pi}{2}i\left(W(X)-\overline{W}(\overline{X})\right)}\right|\psi_{top}\right\rangle .$$

It follows that if we identify

$$\left|\psi_{p,q}\right\rangle = e^{-i\frac{\pi}{4}\left(W(X)-\overline{W}(\overline{X})\right)}\left|\psi_{top}\right\rangle , \quad (3.69)$$

or in the wave function form as

$$\psi_{p,q}\left(\phi^I\right)=e^{-\frac{1}{2}q_I\phi^I+F_{top}\left(p^I+\frac{i}{\pi}\phi^I\right)},$$

then

$$\Omega(p,q)=\left\langle \psi_{p,q}\middle|\psi_{p,q}\right\rangle=\int d\phi^I\left|\psi_{p,q}\left(\phi^I\right)\right|^2 \quad (3.69b)$$

exactly as expected. Moreover the form of the wave function (3.69) is exactly consistent with the semi-classical reasoning which led to (3.62). The fact that the wave functions for both the IIA and IIB side would lead to the same state is clear once we recall that the internal part of the Calabi-Yau and thus the mini-superspace is identical for both cases where a D2 brane IIA instanton is playing the role of D3 brane of IIB. We find this a highly non-trivial evidence for our conjecture for the exact Hartle-Hawking wave function.

The relation between the wave function $\psi_{top}(X)$ and the exact topological string partition function $\psi_{top}(\chi)$

$$\psi_{top}(X)=\int d\chi e^{\frac{i}{\pi}S(X,\chi)}\psi_{top}(\chi) \quad (3.70)$$

is a semi-classical formula. It describes the loop corrections to the Hartle-Hawking wave function to all orders in perturbation theory.

The partition function of topological string theory can be computed perturbatively around a given background by writing $X^I=Z^I+x^I$, and treating the perturbation x^I as coupling constants on the worldsheet. The coordinates x^I used by the perturbative topological string are a linearization of the "curved" X^I coordinates. The relation with (χ,η) is

$$\chi^I=\mathrm{Re}\left(Z^I+x^I\right) \qquad \eta_I=\mathrm{Re}\left(F_I(Z)+\tau_{IJ}x^J\right), \quad (3.71)$$

where $\tau_{IJ}=\partial_I\partial_J F_0(Z)$ is determined by the background. These are just the attractor equations (3.59b) linearized around $X^I=Z^I$. In this way the topological string avoids the normal ordering problems but at the cost of a background dependence. The partition function $\psi_{top}\left(x;Z,\overline{Z}\right)$ is related to the background independent wave function $\psi_{top}(\chi)$ via the Bargman transform

$$\psi_{top}\left(x;Z,\overline{Z}\right)=\left|\det \mathrm{Im}\,\tau\right|^{\frac{1}{2}}\int d\chi e^{iS\left(x,\chi;Z,\overline{Z}\right)}\psi_{top}(\chi), \quad (3.72)$$

with

$$S(x,\chi;Z,\bar{Z}) = \frac{\pi}{4}\bar{\tau}_{IJ}(\bar{Z})\chi^I\chi^J + \pi\chi_I(x^I + Z^I) + \frac{\pi}{4}(x^I + Z^I)(x_I + Z_I),$$

where indices are lowered with $\mathrm{Im}\,\tau_{IJ}(Z)$. This expression is a linearization of the generating function $S(X,\chi)$ defined in (3.60). The topological partition function $\psi(x;Z,\bar{Z})$ thus gives in a certain sense a linearized description of the Hartle-Hawking wave function $\psi_{top}(X)$.

4. p-Adic Models in the Hartle-Hawking proposal. [5]

Ordinary and p-adic quantum mechanics can be unified in the form of adelic quantum mechanics

$$(L_2(\mathrm{A}), W(z), U(t)). \quad (4.1)$$

$L_2(\mathrm{A})$ is the Hilbert-space on A, $W(z)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(\mathrm{A})$ and $U(t)$ is a unitary representation of the evolution operator on $L_2(\mathrm{A})$. The evolution operator $U(t)$ is defined by

$$U(t)\psi(x) = \int_{\mathrm{A}} \mathrm{K}_t(x,y)\psi(y)dy = \prod_v \int_{Q_v} \mathrm{K}_t^{(v)}(x_v,y_v)\psi^{(v)}(y_v)dy_v . \quad (4.2)$$

The eigenvalue problem for $U(t)$ reads

$$U(t)\psi_{\alpha\beta}(x) = \chi(E_\alpha t)\psi_{\alpha\beta}(x), \quad (4.3)$$

where $\psi_{\alpha\beta}$ are adelic eigenfunctions, $E_\alpha = (E_\infty, E_2, \ldots, E_p, \ldots)$ is the corresponding adelic energy, indices α and β denote energy levels and their degeneration. Any adelic eigenfunction has the form

$$\Psi_S(x) = \Psi_\infty(x_\infty)\prod_{p\in S}\Psi_p(x_p)\prod_{p\notin S}\Omega(|x_p|_p), \quad x\in \mathrm{A}, \quad (4.4)$$

where $\Psi_\infty \in L_2(R)$, $\Psi_p \in L_2(Q_p)$ are ordinary and p-adic eigenfunctions, respectively. The Ω-function, that is defined from the following formula

$$\Omega(|x|_p) = 1, \quad |x|_p \le 1; \qquad \Omega(|x|_p) = 0, \quad |x|_p > 1, \quad (4.4b)$$

is an element of the Hilbert space $L_2(Q_p)$, and provides convergence of the infinite product (4.4).
A suitable way to calculate p-adic propagator $\mathrm{K}_p(x'',t'';x',t')$ is to use Feynman's path integral method, i.e.

$$\mathrm{K}_p(x'',t'';x',t') = \int_{x',t'}^{x'',t''} \chi_p\left(-\frac{1}{h}\int_{t'}^{t''} L(\dot{q},q,t)dt\right)Dq . \quad (4.5)$$

For quadratic Lagrangians it has been evaluated in the same way for real and p-adic cases, and the following exact general expression is obtained:

$$\mathrm{K}_v(x'',t'';x',t')=\lambda_v\left(-\frac{1}{2h}\frac{\partial^2\bar{S}}{\partial x''\partial x'}\right)\left|\frac{1}{h}\frac{\partial^2\bar{S}}{\partial x''\partial x'}\right|_v^{\frac{1}{2}}\chi_v\left(-\frac{1}{h}\bar{S}(x'',t'';x',t')\right). \quad (4.6)$$

With regard the Hartle-Hawking proposal for the wave function of the universe, the p-adic wave function is given by the integral

$$\Psi_p(q^\alpha)=\int_{G_p}dN\mathrm{K}_p(q^\alpha,N;0,0), \quad (4.7)$$

where, according to the adelic structure of N, $G_p=Z_p$ (i.e. $|N|_p\le 1$) for every or almost every p.

Models of the de Sitter type

Models of the de Sitter type are models with cosmological constant Λ and without matter fields. We consider two minisuperspace models of this type, with D = 4 and D = 3 space-time dimensions. The corresponding real Einstein-Hilbert action is

$$S=\frac{1}{16\pi G}\int_M d^Dx\sqrt{-g}(R-2\Lambda)+\frac{1}{8\pi G}\int_{\partial M}d^{D-1}x\sqrt{h}K, \quad (4.8)$$

where R is the scalar curvature of D-dimensional manifold M, Λ is the cosmological constant, and K is the trace of the extrinsic curvature K_{ij} on the boundary ∂M. The metric for this model is of the Robertson-Walker type

$$ds^2=\sigma^{D-2}\left[-N^2dt^2+a^2(t)d\Omega_{D-1}^2\right]. \quad (4.9)$$

In this expression $d\Omega_{D-1}^2$ denotes the metric on the unit $(D-1)$-sphere, $\sigma^{D-2}=8\pi G/\left[V^{D-1}(D-1)(D-2)\right]$, where V^{D-1} is the volume of the unit $(D-1)$-sphere.
In the real D = 3 case, the model is related to the multiple-sphere configuration and wormhole solutions. v-adic classical action for this model is

$$\bar{S}_v(a'',N;a',0)=\frac{1}{2\sqrt{\lambda}}\left[N\sqrt{\lambda}+\lambda\left(\frac{2a''a'}{\sinh(N\sqrt{\lambda})}-\frac{a'^2+a''^2}{\tanh(N\sqrt{\lambda})}\right)\right]. \quad (4.10)$$

Let us note that λ, $(\lambda=\Lambda G^2)$, denotes the rescaled cosmological constant Λ. Using (4.6) for the propagator of this model we have

$$\mathrm{K}_v(a'',N;a',0)=\lambda_v\left(-\frac{2\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})}\right)\left|\frac{\sqrt{\lambda}}{\sinh(N\sqrt{\lambda})}\right|_v^{1/2}\chi_v\left(-\bar{S}_v(a'',N;a',0)\right). \quad (4.11)$$

The p-adic Hartle-Hawking wave function is

$$\Psi_p(a)=\int_{|N|_p\le 1}dN\frac{\lambda_p(-2N)}{|N|_p^{1/2}}\chi_p\left(-\frac{N}{2}+\frac{\sqrt{\lambda}\coth(N\sqrt{\lambda})}{2}a^2\right), \quad (4.12)$$

which after p-adic integration becomes

$$\Psi_p(a) = \Omega(|a|_p), \qquad |\lambda|_p \le p^{-2}, \qquad p \ne 2,$$
$$\Psi_p(a) = \frac{1}{2}\Omega(|a|_2), \qquad |\lambda|_2 \le 2^{-4}, \qquad p = 2. \quad (4.13)$$

The de Sitter model in D = 4 space-time dimensions may be described by the metric

$$ds^2 = \sigma^2\left(-\frac{N^2}{q(t)}dt^2 + q(t)d\Omega_3^2\right), \quad \sigma^2 = \frac{2G}{3\pi}, \quad (4.14)$$

i.e.

$$ds^2 = \frac{2G}{3\pi}\left(-\frac{N^2}{q(t)}dt^2 + q(t)d\Omega_3^2\right), \quad (4.14b)$$

and the corresponding action $S_v[q] = \frac{1}{2}\int_{t'}^{t''} dtN\left(-\frac{\dot{q}^2}{4N^2} - \lambda q + 1\right)$, where $\lambda = 2\Lambda G/(9\pi)$.

For $N = 1$, the equation of motion $\ddot{q} = 2\lambda$ has solution $q(t) = \lambda t^2 + \left(\frac{q''-q'}{T} - \lambda T\right)t + q'$, where $q'' = q(t'')$, $q' = q(t')$ and $T = t''-t'$. Note that this classical solution resembles motion of a particle in a constant field and defines an algebraic manifold. The choice of metric in the form (4.14) yields quadratic v-adic classical action

$$\bar{S}_v(q'',T;q',0) = \frac{\lambda^2 T^3}{24} - [\lambda(q'+q'') - 2]\frac{T}{4} - \frac{(q''-q')^2}{8T}. \quad (4.15)$$

According to (4.6), the corresponding propagator is

$$\mathrm{K}_v(q'',T|q',0) = \frac{\lambda_v(-8T)}{|4T|_v^{1/2}}\chi_v(-\bar{S}_v(q'',T|q',0)). \quad (4.16)$$

We obtain the p-adic Hartle-Hawking wave function by the integral

$$\Psi_p(q) = \int_{|T|_p \le 1} dT\,\frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2 T^3}{24} + (\lambda q - 2)\frac{T}{4} + \frac{q^2}{8T}\right), \quad (4.17)$$

and as a result we get also $\Omega(|q|_p)$ function with the condition $\lambda = 4\cdot 3\cdot l$, $l \in Z_p$. The above Ω-functions allow adelic wave functions of the form (4.4) for both D = 3 and D = 4 cases. Since $|\lambda|_p \le p^{-2}$ in (4.13) for all $p \ne 2$, it means that λ cannot be a rational number and consequently the above the de Sitter minisuperspace model in D = 3 space-time dimensions is not adelic one. However D = 4 case is adelic, because $\lambda = 4\cdot 3\cdot l$ is a rational number when $l \in Z \subset Z_p$.

5. p-Adic and Adelic wave functions of the Universe. [6]

In the Vladimirov-Volovich formulation, p-adic quantum mechanics is a triple

$$\left(L_2(Q_p), W_p(z), U_p(t)\right), \quad (5.1)$$

where $W_p(z)$ corresponds to $W_p\left(\alpha\hat{x}, \beta\hat{k}\right)$ defined in the following equation

$$W_v\left(\alpha\hat{x}, \beta\hat{k}\right) = \chi_v\left(\frac{1}{2}\alpha\beta\right)\chi_v\left(-\beta\hat{k}\right)\chi_v\left(-\alpha\hat{x}\right). \quad (5.1a)$$

Adelic quantum mechanics is a natural generalization of the above formulation of ordinary and p-adic quantum mechanics: $\left(L_2(A), W_A(z), U_A(t)\right)$. In complex-valued adelic analysis it is worth mentioning an additive character

$$\chi_A(x) = \chi_\infty(x_\infty)\prod_p \chi_p(x_p), \quad (5.2)$$

a multiplicative character

$$|x|_A^s = |x_\infty|_\infty^s \prod_p |x_p|_p^s, \quad s \in C, \quad (5.3)$$

and elementary functions of the form

$$\varphi_{\mathcal{P}}(x) = \varphi_\infty(x_\infty)\prod_{p\in\mathcal{P}} \varphi_p(x_p)\prod_{p\notin\mathcal{P}} \Omega\left(|x_p|_p\right), \quad (5.4)$$

where $\varphi_\infty(x_\infty)$ is an infinitely differentiable function on R and $|x_\infty|_\infty^n \varphi_\infty(x_\infty) \to 0$ as $|x_\infty|_\infty \to \infty$ for any $n \in \{0,1,2,...\}$, $\varphi_p(x_p)$ are some locally constant functions with compact support, and

$$\Omega\left(|x_p|_p\right) = 1, \quad |x_p|_p \le 1, \qquad \Omega\left(|x_p|_p\right) = 0, \quad |x_p|_p > 1. \quad (5.5)$$

All finite linear combinations of elementary functions (5.4) make the set $\mathcal{L}(A)$ of the Schwartz-Bruhat adelic functions. The Fourier transform of $\varphi(x) \in \mathcal{L}(A)$, which maps $\mathcal{L}(A)$ onto $\mathcal{L}(A)$, is

$$\tilde{\varphi}(y) = \int_A \varphi(x)\chi_A(xy)dx, \quad (5.6)$$

where $\chi_A(xy)$ is defined by (5.2) and $dx = dx_\infty dx_2 dx_3 ...$ is the Haar measure on A. A basis of $L_2(A(\mathcal{P}))$ may be given by the corresponding orthonormal eigenfunctions in a spectral problem of the evolution operator $U_A(t)$, where $t \in A$. Such eigenfunctions have the form

$$\psi_{\mathcal{P}}(x,t) = \psi_\infty(x_\infty, t_\infty)\prod_{p\in\mathcal{P}} \psi_p(x_p, t_p)\prod_{p\notin\mathcal{P}} \Omega\left(|x_p|_p\right), \quad (5.7)$$

where $\psi_\infty \in L_2(R)$ and $\psi_p \in L_2(Q_p)$ are eigenfunctions in ordinary and p-adic cases, respectively. $\Omega\left(|x_p|_p\right)$ is an element of $L_2(Q_p)$, defined by (5.5), which is invariant under transformation of an evolution operator $U_p(t_p)$ and provides convergence of the infinite product (5.7).

p-Adic and adelic minisuperspace quantum cosmology is an application of p-adic and adelic quantum mechanics to the cosmological models, respectively. In the path integral approach to standard quantum cosmology, the starting point is Feynman's path integral method. The amplitude to go from one state with intrinsic metric h'_{ij} and matter configuration ϕ' on an initial hypersurface Σ' to another state with metric h''_{ij} and matter configuration ϕ'' on a final hypersurface Σ'' is given by the path integral

$$\mathcal{K}_\infty\left(h''_{ij},\phi'',\Sigma'';h'_{ij},\phi',\Sigma'\right)=\int \chi_\infty\left(-S_\infty\left[g_{\mu\nu},\Phi\right]\right)\mathcal{D}_\infty g_{\mu\nu}\mathcal{D}_\infty\Phi \quad (5.8)$$

over all four-geometries $g_{\mu\nu}$ and matter configurations Φ, which interpolate between the initial and final configurations. In (5.8) $S_\infty\left[g_{\mu\nu},\Phi\right]$ is an Einstein-Hilbert action for the gravitational and matter fields. To perform p-adic and adelic generalization we make first p-adic counterpart of the action using form-invariance under change of real to the p-adic number fields. Then we generalize (5.8) and introduce p-adic complex-valued cosmological amplitude

$$\mathcal{K}_p\left(h''_{ij},\phi'',\Sigma'';h'_{ij},\phi',\Sigma'\right)=\int \chi_p\left(-S_p\left[g_{\mu\nu},\Phi\right]\right)\mathcal{D}_p g_{\mu\nu}\mathcal{D}_p\Phi\,. \quad (5.9)$$

The standard minisuperspace ground-state wave function in the Hartle-Hawking (no-boundary) proposal is defined by functional integration in the Euclidean version of

$$\psi_\infty\left[h_{ij}\right]=\int \chi_\infty\left(-S_\infty\left[g_{\mu\nu},\Phi\right]\right)\mathcal{D}_\infty g_{\mu\nu}\mathcal{D}_\infty\Phi\,, \quad (5.10)$$

over all compact four-geometries $g_{\mu\nu}$ which induce h_{ij} at the compact three-manifold. This three-manifold is the only boundary of the all four-manifolds. Extending Hartle-Hawking proposal to the p-adic minisuperspace, an adelic Hartle-Hawking wave function is the infinite product

$$\psi_A(q)=\prod_v \int \chi_v\left(-S_v\left[g_{\mu\nu},\Phi\right]\right)\mathcal{D}_v g_{\mu\nu}\mathcal{D}_v\Phi\,, \quad (5.11)$$

where path integration must be performed over both, Archimedean and non-Archimedean geometries. If an evaluation of the corresponding functional integrals for a minisuperspace model yields $\psi(q_\alpha)$ in the form (5.7), the such cosmological model is a Hartle-Hawking adelic one.
Now we consider the approach consists in the following p-adic proposal for the Hartle-Hawking type of the wave function:

$$\psi_\infty(q)=\sum_{a.m.}\prod_p \int \chi_p\left(-S_p\left[g_{\mu\nu},\Phi\right]\right)\mathcal{D}_p g_{\mu\nu}\mathcal{D}_p\Phi\,, \quad (5.12)$$

where summation is over algebraic manifolds.
The de Sitter minisuperspace model in D = 4 space-time dimensions is the Hartle-Hawking adelic one. Namely, according to the Hartle-Hawking proposal one has

$$\psi_v(q)=\int \mathcal{K}_v(q,T;0,0)\,dT\,, \qquad v=\infty,2,3,\ldots,p,\ldots, \quad (5.13)$$

where

$$\mathcal{K}_v(q'',T;q',0)=\lambda_v(-8T)\left|4T\right|_v^{-\frac{1}{2}}\chi_v\left[-\frac{\lambda^2T^3}{24}+(\lambda q-2)\frac{T}{4}+\frac{q^2}{8T}\right] \quad (5.14)$$

is the kernel of the v-adic evolution operator. The functions $\lambda_v(a)$ have the properties

$$|\lambda_v(a)|_v = 1, \;\; \lambda_v(b^2 a) = \lambda_v(a), \;\; \lambda_v(a)\lambda_v(b) = \lambda_v(a+b)\lambda_v(ab(a+b)). \quad (5.15)$$

Employing the p-adic Gauss integral

$$\int_{Q_p} \chi_p(\alpha x^2 + \beta x)dx = \lambda_p(\alpha)|2\alpha|_p^{-\frac{1}{2}} \chi_p\left(-\frac{\beta^2}{4\alpha}\right), \quad \alpha \neq 0, \quad (5.16)$$

one can rewrite p-adic version of (5.13) in the form

$$\psi_p(q) = \int_{Q_p} dx \chi_p(qx) \int DT \chi_p\left[-\frac{\lambda^2 T^3}{24} + \left(\frac{\lambda q}{4} - \frac{1}{2} - 2x^2\right)T\right]. \quad (5.17)$$

Taking the region of integration to be $|T|_p \leq 1$ one obtains

$$\psi_p(q) = \int_{Q_p} dx \chi_p(qx) \Omega\left(\left|\frac{\lambda q}{4} - \frac{1}{2} - 2x^2\right|_p\right), \quad \left|\frac{\lambda^2}{24}\right|_p \leq 1. \quad (5.18)$$

An evaluation of the integral (5.18) yields

$$\psi_p(q) = \exp\left(i\pi \delta^1_{|q|_2} \delta^2_p\right)\Omega\left(|q|_p\right), \quad \left|\frac{\lambda^2}{24}\right|_p \leq 1, \quad (5.19)$$

where δ^b_a is the Kronecker symbol. With regard $\psi_\infty(q_\infty)$, the result depends on the contour of integration and has an exact solution

$$\psi_\infty(q_\infty) = \exp\left(\frac{1}{3\lambda}\right) Ai\left(\frac{1-\lambda q_\infty}{(2\lambda)^{\frac{2}{3}}}\right), \quad (5.20)$$

that can be rewritten also

$$\frac{(2\lambda)^{2/3}}{Ai(1-\lambda q_\infty)} \ln[\psi_\infty(q_\infty)] = \frac{1}{3\lambda}, \quad (5.20b)$$

where $Ai(x)$ is the Airy function. Thence, we obtain an adelic wave function for the de Sitter cosmological model in the form

$$\psi_A(q) = \psi_\infty(q_\infty) \prod_p \exp\left(i\pi \delta^1_{|q|_2} \delta^2_p\right)\Omega\left(|q_p|_p\right), \quad \left|\frac{\lambda^2}{24}\right|_p \leq 1. \quad (5.21)$$

The necessary condition that a system can be regarded as the adelic one is the existence of p-adic ground state $\Omega(|q_\alpha|_p)$ $(\alpha = 1,2,...,n)$ in the way

$$\int_{|q'_\alpha|_p \leq 1} \mathcal{K}_p(q''_\alpha, T; q'_\alpha, 0) dq'_\alpha = \Omega(|q''_\alpha|_p) \quad (5.22)$$

for all p but a finite set $\mathcal{P}$. For the case of de Sitter model one obtains

$$\psi_p(q) = \Omega(|q|_p), \quad |T|_p \leq 1, \quad \left|\frac{\lambda^2}{24}\right|_p \leq 1, \quad p \neq 2,$$

$$\psi_p(q) = \Omega(|q|_2), \quad |T|_2 \leq \frac{1}{2}, \quad \left|\frac{\lambda^2}{24}\right|_2 \leq 1, \quad p = 2, \quad (5.23)$$

what is in a good agreement with the result (5.21) obtained by the Hartle-Hawking proposal.

6. Number Theory: On Some Equations Concerning the Riemann Zeta Function.

A. The Goldston-Montgomery Theorem [7]

In the chapter "Goldbach's numbers in short intervals" of Languasco's paper "The Goldbach's conjecture", is described the Goldston-Montgomery theorem.

THEOREM 1

Assume the Riemann hypothesis. We have the following implications: (1) If $0 < B_1 \leq B_2 \leq 1$ and $F(X,T) \approx \frac{1}{2\pi} T \log T$ uniformly for $\frac{X^{B_1}}{\log^3 X} \leq T \leq X^{B_2} \log^3 X$, then

$$\int_1^X (\psi(1+\delta)x) - \psi(x) - \delta(x)^2 dx \approx \frac{1}{2} \delta X^2 \log \frac{1}{\delta}, \quad (6.1)$$

uniformly for $\frac{1}{X^{B_2}} \leq \delta \leq \frac{1}{X^{B_1}}$.

(2) If $1 < A_1 \leq A_2 < \infty$ and $\int_1^X (\psi((1+\delta)x) - \psi(x) - \delta x)^2 dx \approx \frac{1}{2} \delta X^2 \log \frac{1}{\delta}$ uniformly for $\frac{1}{X^{1/A_1} \log^3 X} \leq T \leq \frac{1}{X^{1/A_2}} \log^3 X$, then $F(X,T) \approx \frac{1}{2\pi} T \log T$ uniformly for

$$T^{A_1} \le X \le T^{A_2}.$$

Now, for show this theorem, we must to obtain some preliminary results .

Preliminaries Lemma. (Goldston-Montgomery)

Lemma 1.

We have $f(y) \ge 0 \quad \forall y \in R$ and let $I(Y) = \int_{-\infty}^{+\infty} e^{-2|y|} f(Y+y)dy = 1 + \varepsilon(Y)$. If R(y) is a Riemann-integrable function, we have:

$$\int_a^b R(y) f(Y+y)dy = \left(\int_a^b R(y)dy \right)(1 + \varepsilon'(y)).$$

Furthermore, fixed R, $|\varepsilon'(Y)|$ is little if $|\varepsilon(y)|$ is uniformly small for $Y + a - 1 \le y \le Y + b + 1$.

Lemma 2.

Let $f(t) \ge 0$ a continuous function defined on $[0, +\infty)$ such that $f(t) << \log^2(t+2)$.
If

$$J(T) = \int_0^T f(t)dt = (1 + \varepsilon(T))T \log T,$$

then

$$\int_0^\infty \left(\frac{\sin ku}{u} \right)^2 f(u)du = \left(\frac{\pi}{2} + \varepsilon'(k) \right) k \log \frac{1}{k},$$

with $|\varepsilon'(k)|$ small for $k \to 0^+$ if $|\varepsilon(T)|$ is uniformly small for

$$\frac{1}{k \log^2 k} \le T \le \frac{1}{k} \log^2 k.$$

Lemma 3.

Let $f(t) \ge 0$ a continuous function defined on $[0, +\infty)$ such that $f(t) << \log^2(t+2)$. If

$$I(k) = \int_0^\infty \left(\frac{\sin ku}{u} \right)^2 f(u)du = \left(\frac{\pi}{2} + \varepsilon'(k) \right) k \log \frac{1}{k}, \quad (6.2) \qquad \text{then}$$

$$J(T) = \int_0^T f(t)dt = (1 + \varepsilon')T \log T, \quad (6.3)$$

with $|\varepsilon'|$ small if $|\varepsilon(k)| \le \varepsilon$ uniformly for $\frac{1}{T \log T} \le k \le \frac{1}{T} \log^2 T$.

Lemma 4.

Let $F(X,T):=\sum_{0<\gamma,\gamma'<T}\frac{4X^{i}(\gamma-\gamma')}{4+(\gamma-\gamma')^2}$. Then (i) $F(X,T)\geq 0$; (ii) $F(X,T)=F(1/X,T)$; (iii) If

The Riemann hypothesis is preserved, then we have

$$F(X,T)=T\left(\frac{1}{X^2}\log^2 T+\log X\right)\left(\frac{1}{2\pi}+O\left(\sqrt{\frac{\log\log T}{\log T}}\right)\right)$$

uniformly for $1\leq X\leq T$.

Lemma 5.

Let $\delta\in(0,1]$ and $a(s)=\frac{(1+\delta)^s-1}{s}$. If $c(\gamma)\leq 1\ \ \forall y$ we have that

$$\int_{-\infty}^{+\infty}|a(it)|^2\left|\sum_{\gamma}\frac{c(\gamma)}{1+(t-\gamma)^2}\right|^2 dt=\int_{-\infty}^{+\infty}\left|\sum_{|\gamma|\leq Z}a(1/2+i\gamma)\frac{c(\gamma)}{1+(t-\gamma)^2}\right|^2 dt+O\left(\delta^2\log^3\frac{2}{\delta}\right)+O\left(\frac{1}{Z}\log^3 Z\right)$$

for $Z>\frac{1}{\delta}$.

For to show the Theorem 1, there are two parts. We go to prove (1).
We define

$$J(X,T)=4\int_0^T\left|\sum_{\gamma}\frac{X^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt.$$

Montgomery has proved that $J(X,T)=2\pi F(X,T)+O(\log^3 T)$ and thence the hypothesis $F(X,T)\approx\frac{1}{2\pi}T\log T$ is equal to $J(X,T)=(1+o(1))T\log T$. Putting $k=\frac{1}{2}\log(1+\delta)$, we have

$$|a(it)|^2=4\left(\frac{\sin kt}{t}\right)^2.$$

For the Lemma 2, we obtain that

$$\int_0^{\infty}|a(it)|^2\left|\sum_{\gamma}\frac{X^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt=\left(\frac{\pi}{2}+o(1)\right)k\log\frac{1}{k}=\left(\frac{\pi}{4}+o(1)\right)\delta\log\frac{1}{\delta}$$

for
$$\frac{1}{\delta\log^2\frac{1}{\delta}}\leq T\leq\frac{3}{\delta}\log^2\frac{1}{\delta}.$$

For the Lemma 5 and the parity of the integrand, we have that

$$\int_{-\infty}^{+\infty}\left|\sum_{|\gamma|\le Z} a(\rho)\frac{X^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt=\left(\frac{\pi}{2}+o(1)\right)\delta\log\frac{1}{\delta}\qquad \text{(a)}$$

if $Z\ge\frac{1}{\delta}\log^3\frac{1}{\delta}$.

From the $S(t)=\sum_{|\gamma|\le Z} a(\rho)\frac{X^{i\gamma}}{1+(t-\gamma)^2}$ we note that the Fourier's transformed verify that

$$\hat{S}(u)=\pi\sum_{|\gamma|\le Z} a(\rho)X^{i\gamma}e(-\gamma u)e^{-2\pi|u|}.$$

From the Plancherel identity, we have that

$$\int_{-\infty}^{+\infty}\left|\sum_{|\gamma|\le Z} a(\rho)X^{i\gamma}e(-\gamma u)\right|^2 e^{-4\pi|u|}du=\left(\frac{2}{\pi}+o(1)\right)\delta\log\frac{1}{\delta}.$$

For the substitution $Y=\log X$, $-2\pi u=y$ we obtain

$$\int_{-\infty}^{+\infty}\left|\sum_{|\gamma|\le Z} a(\rho)e^{i\gamma(Y+y)}\right|^2 e^{-2|y|}dy=(1+o(1))\delta\log\frac{1}{\delta}.\qquad \text{(b)}$$

Using the Lemma 1 with $R(y)=e^{2y}$ if $0\le y\le\log 2$ and $R(y)=0$ otherwise, and putting $x=e^{Y+y}$ we have that

$$\int_{X}^{2X}\left|\sum_{|\gamma|\le Z} a(\rho)x^{\rho}\right|^2 dx=\left(\frac{3}{2}+o(1)\right)\delta X^2\log\frac{1}{\delta}.$$

Substituting X with $X2^{-j}$, summarizing on j, $1\le j\le K$, and using the explicit formula for $\psi(x)$ with $Z=X\log^3 X$ we obtain

$$\int_{X2^{-K}}^{X}(\psi((1+\delta)x)-\psi(x)-\delta x)^2dx=\frac{1}{2}\left(1-2^{-2K}+o(1)\right)\delta X^2\log\frac{1}{\delta}.$$

Furthermore, we put $K=[\log\log X]$ and we utilize, for the interval $1\le x\le X2^{-K}$, the estimate of Lemma 4 (placing $X2^{-K}$ for X). Thus, we obtain (1).
Now, we prove (2).
We fix an real number X_1. Making an integration for parts between X_1 and $X_2=X_1\log^{2/3}X_1$ we obtain, remembering that for hypothesis we have

$$\int_{1}^{X}(\psi((1+\delta)x)-\psi(x)-\delta x)^2dx\approx\frac{1}{2}\delta X^2\log\frac{1}{\delta},$$

that $$\int_{X_1}^{X_2}(\psi((1+\delta)x)-\psi(x)-\delta x)^2 x^{-4}dx=\left(\frac{1}{2}+o(1)\right)\delta X_1^{-2}\log\frac{1}{\delta}. \quad \text{(c)}$$

Utilizing the estimate, valid under the Riemann hypothesis

$$\int_1^X(\psi((1+\delta)x)-\psi(x)-\delta x)^2 dx << \delta X^2\log^2\frac{2}{\delta},$$

we obtain analogously as before that

$$\int_{X_2}^{\infty}(\psi((1+\delta)x)-\psi(x)-\delta x)^2 x^{-4}dx << \delta X_2^{-2}\log^2\frac{1}{\delta}=o\left(\delta X_1^{-2}\log\frac{1}{\delta}\right). \quad \text{(d)}$$

Now, summarizing (c) and (d) and multiplying the sum for X_1^2 we obtain

$$\int_1^{\infty}\min\left(\frac{x^2}{X_1^2},\frac{X_1^2}{x^2}\right)(\psi((1+\delta)x)-\psi(x)-\delta x)^2 x^{-2}dx=(1+o(1))\delta\log\frac{1}{\delta}.$$

Putting $X_1 = X$, $Y = \log X$, $x = e^{Y+y}$ and using the explicit formula for $\psi(x)$ with $Z = X\log^3 X$, we obtain the equation (b).

B. On the study of the behaviour of the argument of the Riemann function $\zeta(s)$ with the condition that s lies on the critical line $s=\frac{1}{2}+it$, where t is real. [8]

We introduce the known functions $S(t)$, $S_1(t)$.

Definition 1. For real t, not equal to the imaginary part of a zero of $\zeta(s)$,

$$S(t)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+it\right),$$

where $\arg\zeta\left(\frac{1}{2}+it\right)$ is obtained by continuous extension of $\arg\zeta(s)$ along the polygonal line starting at the point $s=2$ $(\arg\zeta(2)=0)$, going to the point $s=2+it$ and then to the point $s=\frac{1}{2}+it$. If t is the imaginary part of a zero of $\zeta(s)$ then

$$S(t)=\lim_{\delta\to 0}\frac{1}{2}\{S(t+\delta)+S(t-\delta)\}.$$

Definition 2. For positive t the function $S_1(t)$ is defined by

$$S_1(t)=\int_0^t S(u)du.$$

Definition 3. The following function is known as *Selberg's function*:

$$\eta_\sigma(s) = \sum_{\nu<x} \lambda_\nu \nu^{-s} .$$

Furthermore, we have the following four theorems.

Theorem A. *With* $H = T^{27/82+\alpha}$, $0 < \alpha < 0.001$, $T \ge T_1(\alpha) > 0$, *and* $1/2 \le \sigma \le 1$ *we have*

$$N(\sigma, T+H) - N(\sigma, T-H) = O\left(HT^{-0.1\alpha(2\sigma-1)} \log T\right),$$

where $N(\sigma, T)$ *is the number of zeros of the Riemann zeta function in the rectangle* $\operatorname{Re} s \ge \sigma$, $0 < \operatorname{Im} s \le T$, *and the constant in the symbol* O *depends only on* α.

Theorem B. *Suppose that* $H = T^{27/82+\alpha}$, $0 < \alpha < 0.001$, $T \ge T_1(\alpha) > 0$, *and* k *is a natural number. Then the following asymptotic formula holds:*

$$\int_T^{T+H} |S(t)|^{2k} dt = \frac{(2k)!}{k!(2\pi)^{2k}} H(\log\log T)^k + O\left(H(\log\log T)^{k-1/2}\right),$$

where the constant in the symbol O *depends only on* α *and* k.

Theorem C. *Suppose that* $H = T^{27/82+\alpha}$, $0 < \alpha < 0.001$, $T \ge T_1(\alpha) > 0$, *and* k *is a natural number. Then the following asymptotic formula holds:*

$$\int_T^{T+H} |S_1(t)|^{2k} dt = c_k H + O\left(H \log^{-1} T\right),$$

where c_k *is a positive constant depending only on* k, *and the constant in the symbol* O *depends only on* α *and* k.

Theorem D. *Let* $H = T^{27/82+\alpha}$, $0 < \alpha < 0.001$. *Then there are positive numbers* $T_1 = T_1(\alpha)$ *and* $A = A(\alpha)$ *such that for* $T \ge T_1$ *the function* $S(t)$ *changes sign in the interval* $(T, T+H)$ *no fewer than* K *times,*

$$K = \left\lfloor H(\log T)^{1/3} \exp\left(-A\sqrt{\log\log T}\right)\right\rfloor.$$

Now suppose that $t \ge 2$, $2 \le x \le t^2$ and $\rho = \beta + i\gamma$ run through the values of zeros of the Riemann zeta function with the condition

$$|t-\gamma| \le \frac{x^{3|\beta-0.5|}}{\log x} .$$

Definition 4. We write

$$\sigma_{x,t} = \frac{1}{2} + 2\max\left(\left|\beta - \frac{1}{2}\right|, \frac{2}{\log x}\right).$$

Theorem 1. *If* $t \ge 2$ *and* $2 \le x \le t^2$ *we have*

$$S(t) = -\frac{1}{\pi}\sum_{n<x^3}\frac{\Lambda_x(n)}{n^{\sigma_{x,t}}}\cdot\frac{\sin(t\log n)}{\log n} + O\left(\left(\sigma_{x,t}-\frac{1}{2}\right)\left|\sum_{n<x^3}\frac{\Lambda_x(n)}{n^{\sigma_{x,t}+it}}\right|\right) + O\left(\left(\sigma_{x,t}-\frac{1}{2}\right)\log t\right),$$

where

$$\Lambda_x(n) = \Lambda(n), \quad 1 \le n \le x,$$

$$\Lambda_x(n) = \Lambda(n)\frac{\log^2\frac{x^3}{n} - 2\log^2\frac{x^2}{n}}{2\log^2 x}, \quad x \le n \le x^2,$$

$$\Lambda_x(n) = \Lambda(n)\frac{\log^2\frac{x^3}{n}}{2\log^2 x}, \quad x^2 \le n \le x^3.$$

Theorem 2. *If* $t \ge 2$ *and* $2 \le x \le t^2$ *we have*

$$S_1(t) = -\frac{1}{\pi}\int_{0.5}^{\infty}\log|\zeta(\sigma)|d\sigma + \frac{1}{\pi}\sum_{n<x^3}\frac{\Lambda_x(n)}{n^{\sigma_{x,t}}\log^2 n}\left(1+\left(\sigma_{x,t}-\frac{1}{2}\right)\log n\right)\cos(t\log n)$$
$$+ O\left(\left(\sigma_{x,t}-\frac{1}{2}\right)^2\left|\sum_{n<x^3}\frac{\Lambda_x(n)}{n^{\sigma_{x,t}+it}}\right|\right) + O\left(\left(\sigma_{x,t}-\frac{1}{2}\right)^2\log t\right).$$

Now we describe the estimates of the mean deviations of $S(t)$ and $S_1(t)$ from the corresponding original segments of the Dirichlet series.
To prove the following Theorems 3 and 4, on which the proofs of Theorem B and C are based, we require the following Lemma 1 and 2.

Lemma 1. *Suppose that* $H = T^{27/82+\alpha}$, $0 < \alpha < 0.001$, $T \ge T_1(\alpha) > 0$, $x \ge 2$, k *is a positive integer,* $1 \le y \le x^{8k}$, $x^3y^2 \le T^{0.1\alpha}$. *Then for* $0 \le \nu \le 8k$ *we have:*

$$I(\nu) = \int_T^{T+H}\left(\sigma_{x,t}-\frac{1}{2}\right)^{\nu} y^{\sigma_{x,t}-\frac{1}{2}}dt = O\left(H(\log x)^{-\nu}\right),$$

where the constant within the symbol O *depends only on* α *and* k.

Proof. Using the definition of $\sigma_{x,t}$ we obtain

$$I(\nu) << H(\log x)^{-\nu} + R(\nu), \quad (6.4)$$

where

$R(\nu) = \sum_{\substack{T-H<\gamma\le T+2H \\ \beta>0.5}}\left(\beta-\frac{1}{2}\right)^{\nu}\frac{(x^3y^2)^{\beta-\frac{1}{2}}}{\log x}$, thence

$$I(\nu) << H(\log x)^{-\nu} + \sum_{\substack{T-H<\gamma\le T+2H \\ \beta>0.5}} \left(\beta-\frac{1}{2}\right)^{\nu} \frac{\left(x^3y^2\right)^{\beta-\frac{1}{2}}}{\log x},$$

and summation in the last sum is taken over zeros of $\zeta(s)$ of the form $\rho=\beta+i\gamma$. We present the terms in $R(\nu)$ in a somewhat different form. Since

$$\int_{1/2}^{\beta} \frac{\nu}{\log x}\left(u-\frac{1}{2}\right)^{\nu-1}\left(x^3y^2\right)^{u-\frac{1}{2}}du = \frac{\left(\beta-\frac{1}{2}\right)^{\nu}\left(x^3y^2\right)^{\beta-\frac{1}{2}}}{\log x} - \int_{1/2}^{\beta} \frac{\log x^3y^2}{\log x}\left(u-\frac{1}{2}\right)^{\nu}\left(x^3y^2\right)^{u-\frac{1}{2}}du,$$

it follows that, by increasing somewhat the right-hand side of $R(\nu)$, we obtain

$$R(\nu) \le \sum_{T-2H<\gamma\le T+2H} \int_{1/2}^{\beta} \left\{ \frac{\log x^3y^2}{\log x}\left(u-\frac{1}{2}\right)^{\nu}\left(x^3y^2\right)^{u-\frac{1}{2}} + \frac{\nu}{\log x}\left(u-\frac{1}{2}\right)^{\nu-1}\left(x^3y^2\right)^{u-\frac{1}{2}} \right\} du =$$

$$= \int_{1/2}^{1} \left\{ \frac{\log x^3y^2}{\log x}\left(u-\frac{1}{2}\right)^{\nu} + \frac{\nu}{\log x}\left(u-\frac{1}{2}\right)^{\nu-1} \right\} \times \left(x^3y^2\right)^{u-\frac{1}{2}} \left(\sum_{T-2H<\gamma\le T+2H} g(\gamma;u) \right) du,$$

where

$$g(\gamma;u)=1, \quad 1/2<u<\beta,$$
$$g(\gamma;u)=0, \quad u\ge\beta.$$

Further, we have

$$\sum_{T-2H<\gamma\le T+2H} g(\gamma;u) = \sum_{\substack{T-2H<\gamma\le T+2H \\ \beta>u}} 1 = N(u,T+2H) - N(u,T-2H).$$

By Theorem A

$$N(u,T+2H) - N(u,T-2H) = O\left(HT^{-0.2\alpha\left(u-\frac{1}{2}\right)} \log T \right);$$

and by the hypothesis of the lemma,

$$\left(x^3y^2\right)^{u-\frac{1}{2}} \le T^{0.1\alpha\left(u-\frac{1}{2}\right)}, \quad \log x^3y^2 \le \log T.$$

Hence we arrive at an estimate for $R(\nu)$ given by

$$R(\nu) << \int_{1/2}^{1} \left\{ \frac{\log T}{\log x}\left(u-\frac{1}{2}\right)^{\nu} + \frac{\nu}{\log x}\left(u-\frac{1}{2}\right)^{\nu-1} \right\} HT^{-0.1\alpha\left(u-\frac{1}{2}\right)} (\log T) du. \quad (6.5)$$

It is easy to estimate the last integral. In fact, on changing the variable of integration to $v = 0.1\alpha\left(u - \frac{1}{2}\right)\log T$, we obtain

$$\int_{1/2}^{1}\left(u-\frac{1}{2}\right)^{\nu} T^{-0.1\alpha\left(u-\frac{1}{2}\right)}du = \int_{0}^{0.05\alpha\log T}\frac{(10v)^{\nu+1}e^{-v}}{(\alpha\log T)^{\nu+1}}dv \le 10^{\nu+1}(\alpha\log T)^{-(\nu+1)}\int_{0}^{\infty}v^{\nu+1}e^{-v}dv =$$
$$= (\nu+1)!10^{\nu+1}(\alpha\log T)^{-(\nu+1)}. \quad (6.6)$$

From (6.5) and (6.6) we obtain

$$R(\nu) << \frac{\log^2 T}{\log x}H(\log T)^{-(\nu+1)} + \frac{\log T}{\log x}H(\log T)^{-\nu} << H(\log x)^{-\nu},$$

thence

$$R(\nu) << \int_{1/2}^{1}\left\{\frac{\log T}{\log x}\left(u-\frac{1}{2}\right)^{\nu} + \frac{\nu}{\log x}\left(u-\frac{1}{2}\right)^{\nu-1}\right\}HT^{-0.1\alpha\left(u-\frac{1}{2}\right)}(\log T)du <<$$
$$<< \frac{\log^2 T}{\log x}H(\log T)^{-(\nu+1)} + \frac{\log T}{\log x}H(\log T)^{-\nu} << H(\log x)^{-\nu}. \quad (6.6b)$$

The lemma follows from (6.4) and the last inequality.

Lemma 2. *Suppose that* $H_0 > 1$, $1 < y \le H_0^{1/k}$, k *is a natural number and, for prime* $p \le y$,

$$|\alpha_p| \le c_1\frac{\log p}{\log y}, \qquad |\alpha_p^{'}| \le c_2.$$

Then

$$\int_{0}^{H_0}\left|\sum_{p<y}\alpha_p p^{-\frac{1}{2}-it}\right|^{2k}dt = O(H_0); \quad \int_{0}^{H_0}\left|\sum_{p<y}\alpha_p^{'}p^{-1-i2t}\right|^{2k} = O(H_0).$$

Theorem 3. *Suppose that* $H = T^{27/82+\alpha}$, $0 < \alpha < 0.001$, $T \ge T_1(\alpha) > 0$, k *is a natural number,* $T^{\alpha/k} < x < H^{1/k}$. *Then*

$$\int_{T}^{T+H}\left|S(t) + \frac{1}{\pi}\sum_{p<x}\frac{\sin(t\log p)}{\sqrt{p}}\right|^{2k}dt = O(H),$$

where the constant in the symbol O *depends only on* α *and* k.

Theorem 4. *Suppose that* $H = T^{27/82+\alpha}$, $0 < \alpha < 0.001$, $T \ge T_1(\alpha) > 0$, k *is a natural number,* $T^{\alpha/k} < x < H^{1/k}$.
Then

$$\int_{T}^{T+H}\left|S_1(t) + \frac{1}{\pi}\int_{0.5}^{\infty}\log|\zeta(\sigma)|d\sigma - \frac{1}{\pi}\sum_{n<x}\frac{\Lambda(n)}{\sqrt{n}\log^2 n}\cos(t\log n)\right|^{2k}dt = O(HL^{-2k}),$$

where the constant in the symbol O depends only on α and k.

We base the proofs of Theorems 3 and 4 on Theorems 1 and 2 and Lemmas 1 and 2. We give only the proof of Theorem 3. In Theorem 1 we put $x = T^{\alpha/60k}$ and add to both sides of the equation in this theorem the sum

$$\frac{1}{\pi}\sum_{p<x^3}\frac{\sin(t\log p)}{\sqrt{p}}.$$

On proceeding to estimates on the right-hand side we then have

$$S(t)+\frac{1}{\pi}\sum_{p<x^3}\frac{\sin(t\log p)}{\sqrt{p}} = O\left(\sum_{j=1}^{7} R_j\right), \quad (6.7)$$

where

$$R_1 = \left|\sum_{p<x^3}\frac{\Lambda(p)-\Lambda_x(p)}{\sqrt{p}\log p}p^{-it}\right|, \quad R_2 = \left|\sum_{p<x^3}\frac{\Lambda_x(p)}{\sqrt{p}\log p}\left(1-p^{0.5-\sigma_{x,t}}\right)p^{-it}\right|, \quad R_3 = \left(\sigma_{x,t}-\frac{1}{2}\right)\left|\sum_{p<x^3}\frac{\Lambda_x(p)}{p^{\sigma_{x,t}+it}}\right|,$$

$$R_4 = \left|\sum_{p<x^{1.5}}\frac{\Lambda_x(p^2)}{p\log p}p^{-i2t}\right|, \quad R_5 = \left|\sum_{p<x^{1.5}}\frac{\Lambda_x(p^2)}{p\log p}\left(1-p^{1-2\sigma_{x,t}}\right)p^{-i2t}\right|, \quad R_6 = \left(\sigma_{x,t}-\frac{1}{2}\right)\left|\sum_{p<x^{1.5}}\frac{\Lambda_x(p^2)}{p^{2\sigma_{x,t}+i2t}}\right|,$$

$$R_7 = \sum_{r>2}\sum_{p^r<x^3}\frac{\log p}{p^{r/2}}.$$

We transform each of the R_j, $j = 2,5,6,7$, to a uniform form (similar to the form of R_3). First of all we have, obviously,

$$R_7 << 1 << \left(\sigma_{x,t}-\frac{1}{2}\right)\log T, \quad R_6 << \left(\sigma_{x,t}-\frac{1}{2}\right)\sum_{p<x^{1.5}}\frac{\log p}{p} << \left(\sigma_{x,t}-\frac{1}{2}\right)\log T,$$

$$R_5 << \sum_{p<x^{1.5}}\frac{1}{p}\left(1-p^{1-2\sigma_{x,t}}\right) << \left(\sigma_{x,t}-\frac{1}{2}\right)\sum_{p<x^{1.5}}\frac{\log p}{p} << \left(\sigma_{x,t}-\frac{1}{2}\right)\log T.$$

We now consider R_2. It is easy to see that

$$R_2 = \left|\int_{0.5}^{\sigma_{x,t}}\sum_{p<x^3}\frac{\Lambda_x(p)}{p^{u+it}}du\right| \le \int_{0.5}^{\sigma_{x,t}}\left|\sum_{p<x^3}\frac{\Lambda_x(p)}{p^{u+it}}\right|du.$$

Further, for $1/2 \le u \le \sigma_{x,t}$ we have

$$\left|\sum_{p<x^3}\frac{\Lambda_x(p)}{p^{u+it}}\right| = \left|x^{u-\frac{1}{2}}\int_u^\infty x^{\frac{1}{2}-v}\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{v+it}}dv\right| \le x^{\sigma_{x,t}-\frac{1}{2}}\int_{0.5}^\infty x^{\frac{1}{2}-v}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{v+it}}\right|dv.$$

Therefore

$$R_2 << \left(\sigma_{x,t}-\frac{1}{2}\right)x^{\sigma_{x,t}-\frac{1}{2}}\int_{0.5}^\infty x^{\frac{1}{2}-v}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{v+it}}\right|dv.$$

From (6.7) and the estimates already obtained we obtain

$$\int_T^{T+H} \left| S(t) + \frac{1}{\pi} \sum_{p<x^3} \frac{\sin(t \log p)}{\sqrt{p}} \right|^{2k} dt = O\left(\sum_{j=1}^{4} K_j \right), \quad (6.8)$$

where

$$K_1 = \int_T^{T+H} \left| \sum_{p<x^3} \frac{\Lambda(p) - \Lambda_x(p)}{\sqrt{p} \log p} p^{-it} \right|^{2k} dt, \quad K_2 = \int_T^{T+H} \left| \sum_{p<x^{1.5}} \frac{\Lambda_x(p^2)}{p \log p} p^{-i2t} \right|^{2k} dt,$$

$$K_3 = (\log T)^{2k} \int_T^{T+H} \left(\sigma_{x,t} - \frac{1}{2} \right)^{2k} dt,$$

$$K_4 = \int_T^{T+H} \left(\sigma_{x,t} - \frac{1}{2} \right)^{2k} x^{2k\left(\sigma_{x,t} - \frac{1}{2}\right)} \times \left\{ \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p) \log(xp)}{p^{u+it}} \right| du \right\}^{2k} dt.$$

From Lemma 2 we have

$$K_1 = O(H), \quad K_2 = O(H);$$

from Lemma 1 follows that

$$K_3 = O\left(H \left(\frac{\log T}{\log x} \right)^{2k} \right) = O(H).$$

Applying Cauchy's inequality to K_4, we obtain

$$K_4 \ll \left\{ \int_T^{T+H} \left(\sigma_{x,t} - \frac{1}{2} \right)^{4k} x^{4k\left(\sigma_{x,t} - \frac{1}{2}\right)} dt \right\}^{1/2} \times \left\{ \int_T^{T+H} \left(\int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p) \log(xp)}{p^{u+it}} \right| du \right)^{4k} dt \right\}^{1/2}. \quad (6.9)$$

By Lemma 1 the first integral in (6.9) is estimated by

$$I_1 \ll H (\log x)^{-4k}. \quad (6.10)$$

Further, using the Holder inequality, we obtain

$$\left(\int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p) \log(xp)}{p^{u+it}} \right| du \right)^{4k} \le \left(\int_{0.5}^{\infty} x^{\frac{1}{2}-u} du \right)^{4k-1} \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p) \log(xp)}{p^{u+it}} \right| du =$$

$$= (\log x)^{-4k+1} \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p) \log(xp)}{p^{u+it}} \right|^{4k} du.$$

The second integral in (6.9) is therefore estimated by

$$I_2 << (\log x)^{-4k+1} \int_T^{T+H} \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it}} \right|^{4k} dtdu =$$

$$= (\log x)^{4k+1} \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left(\int_T^{T+H} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it}\log^2 x} \right| dt \right) du << H(\log x)^{4k} \quad (6.11)$$

(we have again used Lemma 2, estimating the last integral with respect to t).

Therefore from (6.9)-(6.11) we obtain

$$K_4 << H.$$

From (6.8) and the estimates $K_j << H$, $j = 1,2,3,4$, it follows that

$$\int_T^{T+H} \left| S(t) + \frac{1}{\pi} \sum_{p<y} \frac{\sin(t\log p)}{\sqrt{p}} \right|^{2k} dt = O(H),$$

where $y = T^{\alpha/20k}$. Now let $T^{\alpha/k} < x < H^{1/k}$. Going over to inequalities, using the relations already proved and Lemma 2, we have

$$\int_T^{T+H} \left| S(t) + \frac{1}{\pi} \sum_{p<x} \frac{\sin(t\log p)}{\sqrt{p}} \right|^{2k} dt << \int_T^{T+H} \left| S(t) + \frac{1}{\pi} \sum_{p<y} \frac{\sin(t\log p)}{\sqrt{p}} \right|^{2k} dt + \int_T^{T+H} \left| \sum_{y\le p<x} \frac{\sin(t\log p)}{\sqrt{p}} \right|^{2k} dt << H$$

which is was required to prove.

C. The P-N Model (Palumbo-Nardelli model) and the Ramanujan identities. [9]

Palumbo (2001) ha proposed a simple model of the birth and of the evolution of the Universe. Nardelli (2005) has compared this model with the theory of the strings, and translated it in terms of the latter obtaining:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu}G_{\rho\sigma}) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] =$$

$$= \int_0^{\infty} \frac{1}{2\kappa_{10}^2} \int d^{10}x(-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu\left(\left|F_2\right|^2\right) \right], \quad (6.12)$$

A general relationship that links bosonic and fermionic strings acting in all natural systems.
It is well-known that the series of Fibonacci's numbers exhibits a fractal character, where the forms repeat their similarity starting from the reduction factor $1/\phi$ = 0,618033 = $\frac{\sqrt{5}-1}{2}$ (Peitgen et al. 1986). Such a factor appears also in the famous fractal Ramanujan identity (Hardy 1927):

$$0,618033 = 1/\phi = \frac{\sqrt{5}-1}{2} = R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}, \qquad (6.13)$$

and $$\pi = 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right], \qquad (6.14)$$

where $$\Phi = \frac{\sqrt{5}+1}{2}.$$

Furthermore, we remember that π arises also from the following identity (Ramanujan's modular equations and approximations to π):

$$\pi = \frac{12}{\sqrt{130}}\log\left[\frac{\left(2+\sqrt{5}\right)\left(3+\sqrt{13}\right)}{\sqrt{2}}\right], \quad (6.14a) \quad \text{and} \quad \pi = \frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right].$$
(6.14b)

From (6.14b), we have that

$$24 = \frac{\pi\sqrt{142}}{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (6.14c)$$

We remember that the "Ramanujan function" (an elliptic modular function that satisfies the "conformal symmetry") has 24 "modes" that correspond to the physical vibrations of a bosonic string.
The introduction of (6.13) and (6.14) in (6.12) provides:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16G}\cdot\frac{1}{2\Phi-\frac{3}{20}\left(R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f\left(-t^{1/5}\right)}\frac{dt}{t^{4/5}}\right)}\right)}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)+\right.$$

$$-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \;\;] = \int_0^\infty \frac{R}{\kappa_{11}^2}\cdot 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right].$$

$$\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{11}^2}{2\Phi - \frac{3}{20}\left(R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right)2Rg_{10}^2} Tr_v\left(\left|F_2\right|^2\right)\right], \qquad (6.15)$$

which is the translation of (6.12) in the terms of the Theory of the Numbers, specifically the possible connection between the Ramanujan identity and the relationship concerning the Palumbo-Nardelli model.

7. On some possible mathematical connections.

In this section we describe some possible mathematical connections between some equations of arguments above discussed and some equations concerning the Riemann zeta-function, the Ramanujan's modular equations and the Palumbo-Nardelli model.

First of all, now we describe the following possible mathematical connections.
If we take the eq. (1.52) of **section 1** and the eq. (2.51) of **section 2**, we note that

$$-\frac{1}{16\pi G_4}\int d^4x\sqrt{\hat{\gamma}}\left(12R_4^2 - \frac{2}{R_4^2}h^{ij}h_{ij} + \frac{1}{4R_4^2}h^{ij}\hat{\nabla}^2 h_{ij}\right) \Rightarrow$$
$$\Rightarrow \frac{1}{16\pi G_4}\int d^4x\sqrt{g}\left[R + \frac{l^2}{2}C_{\mu\nu\alpha\beta}k_1(\Box)C^{\mu\nu\alpha\beta} - 6l^2\left(\Box\psi + \frac{R}{6}\right)k_2(\Box)\left(\Box\psi + \frac{R}{6}\right)\right], \quad (7.1)$$

hence the possible mathematical connection.
Furthermore, we note that the eqs. (3.70), (3.72) of **section 3** and (5.10) of **section 5**, can be related, and we obtain the following connections

$$\int d\chi e^{\frac{i}{\pi}S(X,\chi)}\psi_{top}(\chi) \Rightarrow \left|\det \operatorname{Im}\tau\right|^{\frac{1}{2}}\int d\chi e^{iS(x,\chi;Z,\bar{Z})}\psi_{top}(\chi) \Rightarrow$$
$$\Rightarrow \int \chi_\infty\left(-S_\infty\left[g_{\mu\nu},\Phi\right]\right)\mathcal{D}_\infty g_{\mu\nu}\mathcal{D}_\infty\Phi. \quad (7.2)$$

Also the eqs. (3.70), (3.72), of **section 3**, (4.17), of **section 4**, and (5.17) of **section 5**, can be related, and we obtain the following interesting connections

$$\int d\chi e^{\frac{i}{\pi}S(X,\chi)}\psi_{top}(\chi)\Rightarrow |\det \operatorname{Im}\tau|^{\frac{1}{2}}\int d\chi e^{iS(x,\chi;Z,\bar{Z})}\psi_{top}(\chi)\Rightarrow$$

$$\Rightarrow \int_{|T|_p\leq 1} dT\,\frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2T^3}{24}+(\lambda q-2)\frac{T}{4}+\frac{q^2}{8T}\right)\Rightarrow$$

$$\Rightarrow \int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left[-\frac{\lambda^2T^3}{24}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right]. \quad (7.3)$$

With regard the possible mathematical connections concerning the Ramanujan's modular equations, we note that the eqs. (4.17) of **section 4** and (5.17) of **section 5**, can be related with the eq. (6.14c) of **section 6**, obtaining the following connection

$$\int_{|T|_p\leq 1} dT\,\frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2T^3}{24}+(\lambda q-2)\frac{T}{4}+\frac{q^2}{8T}\right)\Rightarrow$$

$$\Rightarrow \int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left[-\frac{\lambda^2T^3}{24}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right]\Rightarrow$$

$$\Rightarrow \int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left\{-\lambda^2T^3\,\frac{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}{\pi\sqrt{142}}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right\}. \quad (7.4)$$

But, remember that the 24 "modes" correspond to the physical vibrations of a bosonic string, it is possible to obtain the following interesting connection concerning the Palumbo-Nardelli model in the terms of Number Theory

$$\int_{|T|_p\leq 1} dT\,\frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2T^3}{24}+(\lambda q-2)\frac{T}{4}+\frac{q^2}{8T}\right)\Rightarrow$$

$$\Rightarrow \int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left[-\frac{\lambda^2T^3}{24}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right]\Rightarrow$$

$$\Rightarrow \int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left\{-\lambda^2T^3\,\frac{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}{\pi\sqrt{142}}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right\}\Rightarrow$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16G}\cdot\frac{1}{2\Phi-\frac{3}{20}\left(R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f\left(-t^{1/5}\right)}\frac{dt}{t^{4/5}}\right)}\right)}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)+\right.$$

$$\left.-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\;\right]=\int_0^\infty\frac{R}{\kappa_{11}^2}\cdot 2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\cdot$$

$$\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{11}^2}{2\Phi-\frac{3}{20}\left(R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right)2Rg_{10}^2}Tr_\nu\right.$$

$$\left.\left(\left|F_2\right|^2\right)\;\right] \quad (7.5)$$

Now we describe the possible mathematical connections with some equations concerning the Goldston-Montgomery theorem and the Riemann zeta-function.

We take the eqs. (1.29) of **section 1**, (6.3), (6.9) and (6.11) of **section 6**, then we obtain the following connections

$$\frac{3N^2\Omega_4}{8\pi^2}\log\frac{R}{\bar{l}}=W_{CFT}+\frac{\bar{l}^3}{16\pi G}\int d^4x\sqrt{\hat{\gamma}}\left(-\frac{1}{4\bar{l}^4}h^{ij}\partial_y h_{ij}+\frac{1}{\bar{l}^4}h^{ij}h_{ij}\left(\frac{3}{2}-\sqrt{1+\frac{\bar{l}^2}{R^2}}\right)+\frac{1}{\bar{l}^2R^2}h^{ij}h_{ij}-\frac{1}{8\bar{l}^2R^2}h^{ij}\hat{\nabla}^2h_{ij}\right)$$

$$\Rightarrow(1+\varepsilon')T\log T=\int_0^T f(t)dt\Rightarrow\left\{\int_T^{T+H}\left(\int_{0.5}^\infty x^{\frac{1}{2}-u}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}}\right|du\right)^{4k}dt\right\}^{1/2} \ll$$

$$\ll(\log x)^{-4k+1}\int_T^{T+H}\int_{0.5}^\infty x^{\frac{1}{2}-u}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}}\right|^{4k}dtdu=$$

$$=(\log x)^{4k+1}\int_{0.5}^\infty x^{\frac{1}{2}-u}\left(\int_T^{T+H}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}\log^2 x}\right|dt\right)du\ll H(\log x)^{4k}\,. \quad (7.6)$$

With the eqs. (1.50) of **section 1**, (6.3), (6.9) and (6.11) of **section 6**, we obtain the following connections

$$-\frac{1}{16\pi G_4}\int d^4x\sqrt{\gamma}R+W_{CFT}=-\frac{3\Omega_4 R_4^2}{4\pi G_4}+\frac{3N^2\Omega_4}{8\pi^2}\log\frac{R_4}{\rho}\Rightarrow$$

$$\Rightarrow\int_0^T f(t)dt=(1+\varepsilon')T\log T\Rightarrow\left\{\int_T^{T+H}\left(\int_{0.5}^{\infty}x^{\frac{1}{2}-u}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}}\right|du\right)^{4k}dt\right\}^{1/2}<<$$

$$<<(\log x)^{-4k+1}\int_T^{T+H}\int_{0.5}^{\infty}x^{\frac{1}{2}-u}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}}\right|^{4k}dtdu=$$

$$=(\log x)^{4k+1}\int_{0.5}^{\infty}x^{\frac{1}{2}-u}\left(\int_T^{T+H}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}\log^2 x}\right|dt\right)du<<H(\log x)^{4k}. \quad (7.7)$$

With the eqs. (1.53) of **section 1**, (6.3), (6.9) and (6.11) of **section 6**, we obtain the following connections

$$-\frac{3N^2\Omega_4}{16\pi^2}+\frac{3N^2\Omega_4}{8\pi^2}\log\frac{R_4}{\rho}=S-\sum_p\left(\int d^4x'\sqrt{\hat{\gamma}}h^{kl}(x')H_{kl}^{(p)}(x')\right)^2\left[\frac{1}{64\pi G_4 R_4^2}\left(p^2+3p+6\right)\right.$$
$$\left.+\frac{N^2}{256\pi^2 R_4^4}\left(2p(p+1)(p+2)(p+3)\log(\rho/R_4)+\Psi(p)\right)\right]\Rightarrow$$

$$\Rightarrow(1+\varepsilon')T\log T=\int_0^T f(t)dt\Rightarrow\left\{\int_T^{T+H}\left(\int_{0.5}^{\infty}x^{\frac{1}{2}-u}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}}\right|du\right)^{4k}dt\right\}^{1/2}<<$$

$$<<(\log x)^{-4k+1}\int_T^{T+H}\int_{0.5}^{\infty}x^{\frac{1}{2}-u}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}}\right|^{4k}dtdu=$$

$$=(\log x)^{4k+1}\int_{0.5}^{\infty}x^{\frac{1}{2}-u}\left(\int_T^{T+H}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}\log^2 x}\right|dt\right)du<<H(\log x)^{4k}. \quad (7.8)$$

With the eqs. (2.51), (2.54) of **section 2**, (6.3), (6.9) and (6.11) of **section 6**, we obtain the following connections

$$\frac{1}{16\pi G_4}\int d^4x\sqrt{g}\left[R+\frac{l^2}{2}C_{\mu\nu\alpha\beta}k_1(\Box)C^{\mu\nu\alpha\beta}-6l^2\left(\Box\psi+\frac{R}{6}\right)k_2(\Box)\left(\Box\psi+\frac{R}{6}\right)\right]\Rightarrow$$

$$\Rightarrow k(\Box)+\frac{1}{4}\mathbf{C}\equiv\frac{1}{4}\ln\frac{4}{l^2(-\Box)}\Rightarrow$$

$$\Rightarrow\int_0^T f(t)dt=(1+\varepsilon')T\log T\Rightarrow\left\{\int_T^{T+H}\left(\int_{0.5}^{\infty}x^{\frac{1}{2}-u}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}}\right|du\right)^{4k}dt\right\}^{1/2}<<$$

$$<<(\log x)^{-4k+1}\int_T^{T+H}\int_{0.5}^{\infty}x^{\frac{1}{2}-u}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}}\right|^{4k}dtdu=$$

$$=(\log x)^{4k+1}\int_{0.5}^{\infty}x^{\frac{1}{2}-u}\left(\int_T^{T+H}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}\log^2 x}\right|dt\right)du<<H(\log x)^{4k}. \quad (7.9)$$

Furthermore, with the eqs. (5.17), (5.20b) of **section 5**, (6.3), (6.9) and (6.11) of **section 6**, we obtain the following connections

$$\int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left[-\frac{\lambda^2T^3}{24}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right]\Rightarrow$$
$$\Rightarrow\frac{(2\lambda)^{2/3}}{Ai(1-\lambda q_\infty)}\ln[\psi_\infty(q_\infty)]=\frac{1}{3\lambda}\Rightarrow$$
$$\Rightarrow(1+\varepsilon')T\log T=\int_0^T f(t)dt\Rightarrow\left\{\int_T^{T+H}\left(\int_{0.5}^{\infty}x^{\frac{1}{2}-u}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}}\right|du\right)^{4k}dt\right\}^{1/2}<<$$
$$<<(\log x)^{-4k+1}\int_T^{T+H}\int_{0.5}^{\infty}x^{\frac{1}{2}-u}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}}\right|^{4k}dtdu=$$
$$=(\log x)^{4k+1}\int_{0.5}^{\infty}x^{\frac{1}{2}-u}\left(\int_T^{T+H}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}\log^2 x}\right|dt\right)du<<H(\log x)^{4k}\ . \quad (7.10)$$

Conclusion

Hence, in conclusion, also for some mathematical sectors concerning the Hartle-Hawking no boundary proposal concerning the Randall-Sundrum cosmological scenario, Hartle-Hawking wave-function in the mini-superspace sector of physical superstring theory and p-adic Hartle-Hawking wave function, can be obtained interesting and new possible connections between them, String Theory and some sectors of Number Theory, principally the Ramanujan's modular equations and some formulae related to the Riemann zeta function.
Furthermore, also the fundamental relationship concerning the Palumbo-Nardelli model, can be related with some equations (see eq. (7.5)) regarding p-adic models in the Hartle-Hawking proposal and p-adic and adelic wave functions of the Universe.

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