On a new mathematical application concerning the discrete and the analytic functions. Mathematical connections with some sectors of Number Theory and String Theory.

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Abstract

In this work we have described a new mathematical application concerning the discrete and the analytic functions: the Volonterio's Transform and the Volonterio's Polynomial. The Volonterio's Transform (**V Transform**), indeed, work from the world of discrete functions to the world of analytic functions. We have described various mathematical applications and properties of them. Furthermore, we have showed also various examples and the possible mathematical connections with some sectors of Number Theory and String Theory.

Definition 1 (transform V)

The transformed V of a discrete function , $y(k)$ is an analytic function of a real variable (or complex) through which it is possible to pass from the world of discrete or finite mathematics to the world of differential mathematics.

Definition 2 (inverse transform V)

The inverse transform V of an analytic function $V(t)$ of a real variable t continues in the zero and infinite times differentiable (in other words a function $V(t)$ developable in Maclaurin series) is a discrete function $y(k)$ defined through

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$\mathbb{N}_0$, which it is possible the transition from the differential world to the world of discrete or finite mathematics.

DEFINITIONS NECESSARY TO UNDERSTAND THE TRANSFORM V:

We define with $\delta^{\Delta}(k)$ the following Kronecker's function:

$$\delta^{\Delta}(k):=\begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases} \quad k\in\mathbb{Z} \tag{1}$$

We define with $u^{\Delta}(k)$ the following discrete function in step (Heaviside):

$$u^{\Delta}(k):=\begin{cases} 1 & k\geqslant 0 \\ 0 & k<0 \end{cases} \quad k\in\mathbb{Z} \tag{2}$$

DEFINITION OF TRANSFORM V

Let $y(k)$ a discrete function, then we can define the transformation $V(t)$ as follows:

$$V(t)=T(y(k),t):=\sum_{k=0}^{+\infty} y(k)\frac{t^k}{k!} \quad t\in\mathbb{C} \tag{3}$$

CONDITIONS OF EXISTENCE AND UNIQUENESS OF THE TRANSFORM V

To ensure the condition of existence of the transform must be guaranteed the following relationship:

$$\lim_{k\to+\infty}\frac{|y(k)|^{1/k}}{k}<+\infty \quad e \quad \boldsymbol{R}=\lim_{k\to+\infty}\frac{k}{|y(k)|^{1/k}\mathrm{e}} \qquad with \quad \mathrm{e}=2.718281828\ldots \tag{4}$$

where $\boldsymbol{R}$ is the radius of convergence, while “e” is the Eulero-Nepero constant.

The relation (4) is a necessary condition that we have demonstrated exploiting the condition of the root of Cauchy-Hadamard while the condition of uniqueness can be relegated to the properties of series of powers where “**e**” is the Euler-Nepero constant.

DEFINITION OF INVERSE TRANSFORMATION V

We define with inverse transform of $V(t)$ the discrete function $y(k)$ obtained by the following definition:

$$y(k) = T^{-1}(V(t),k) := \frac{d^k}{dt^k} V(t)\Big|_{t=0} \tag{5}$$

or by the following alternative formula to the (5):

$$y(k) = T^{-1}(V(t),k) := \frac{2}{\pi}\Gamma(k+1)\left(\int_0^{\pi} \Re(V(e^{i\xi}))\cos(k\xi)\,d\xi\right) \quad k \in \mathbb{N}_0$$

or the more exact formula:

$$y(k) = T^{-1}(V(t),k) := \begin{cases} \frac{1}{\pi}\int_0^{\pi} \mathcal{R}(V(e^{i\xi}))d\xi \text{ ....} k = 0 \\ \frac{2}{\pi}\Gamma(k+1)\int_0^{\pi} \mathcal{R}(V(e^{i\xi}))\cos(k\xi)\,d\xi \text{ ....} k \in N \text{.....} k \neq 0 \end{cases} \tag{6}$$

where necessary and sufficient condition because (6) to be valid, is that the condition $R > \pi$ is satisfied, where R is the radius of convergence (4).
We note that in this equation, there is the Euler gamma function Γ and the constant π, fundamentals in various equations concerning the string theory.

For example, we consider the discrete functions:

$$y_1(k) = k-1 \quad \text{e} \quad y_2(k) = |k|-1 \tag{7}$$

Thence, we have that:

$$V_1(t) = (t-1)e^t \quad e \quad V_2(t) = (t-1)e^t \rightarrow V_1(k) = V_2(k) \tag{8}$$

where the inverse transform of $V(t)$ is:

$$y(k) = T^{-1}(V(t),k) = y_1(k) = k-1 \tag{9}$$

thence, it is possible conclude that :

$$y_1(k)=y_2(k) \quad \forall k \in \mathbb{N}_0 \tag{10}$$

DEFINITIONS

To be able to read and interpret the tables in complete sense, is needed clarifications on the functions and abbreviations that we have introduced and also will be fundamentals of the examples that follow after the tables. In any case, before proceeding to the list of transformations is useful to consider the following reports, definitions and functions.

Fundamental relations (see definitions (1) and (2))

$$u^{\Delta}(k)=\sum_{n=0}^{+\infty} \delta^{\Delta}(k-n) \tag{11}$$

$$u^{\Delta}(k)=[u^{\Delta}(k)]^{n} \tag{12}$$

$$u^{\Delta}(k)=u^{\Delta}(k)\,u^{\Delta}(k+n) \tag{13}$$

$$u^{\Delta}(k+n)=[u^{\Delta}(k+n)]^{m} \tag{14}$$

$$u^{\Delta}(k-n)=[u^{\Delta}(k-n)]^{m} \tag{15}$$

$$u^{\Delta}(k-n)=u^{\Delta}(k)\,u^{\Delta}(k-n) \tag{16}$$

$$u^{\Delta}(k-n)=u^{\Delta}(k+m)\,u^{\Delta}(k-n) \tag{17}$$

$$[u^{\Delta}(k)]^{0}=u^{\Delta}(-k)-\delta^{\Delta}(0)+u^{\Delta}(k) \tag{18}$$

Definition of the operator Φ_t

With the symbol Φ_t we define the following operator:

$$\Phi_t := t\frac{d}{dt} \tag{19}$$

where its application iterated n times of a certain function $V(t)$ will express by the operator of application in the following way $\Phi_t^n \circ V(t)$.

For example, we consider $V(t)=\sin(t)$, thence:

$$\Phi_t^3 \circ V(t) = t\frac{d}{dt}\left(t\frac{d}{dt}\left(t\frac{d}{dt}\sin(t)\right)\right) = t\cos(t) - 3t^2\sin(t) - t^3\cos(t)$$

Definition of Volonterio's Polynomial:

$$V_n(t)=\mathrm{e}^{-t}\Phi_t^n\circ\mathrm{e}^t \quad with \quad V_0(t):=1 \tag{20}$$

Other formulas for obtain the polynomial $V_n(t)$ (vedi TF N° 4 e 20) are the following:

$$V_{n+1}(t)=t\left(\frac{d}{dt}V_n(t)+V_n(t)\right) \tag{21}$$

or:

$$V_n(t)=\mathrm{e}^{-t}\sum_{k=0}^{+\infty}k^n\frac{t^k}{k!} \tag{22}$$

or:

$$V_n(t)=\frac{d^n}{d\xi^n}\mathrm{e}^{t(\mathrm{e}^\xi-1)}\bigg|_{\xi=0} \tag{23}$$

Below the proof of the eqs. (20) and (23), where, in this case, the polynomial is denoted with P_n . (see also the references PF N. 7, PF N. 17, TF N. 4, TF N. 20)

Observations on the Volonterio’s Polynomial

$$T(K^m,t)=\ ? \qquad V(t)=T(y(k),t)$$

A) We observe that

$$T(ky(k),t)=t\frac{d}{dt}T(y(k),t) \qquad (*)$$

Let

$$y(k)=K^m \Rightarrow T(K^{m+1},t)=t\frac{d}{dt}T(K^m,t);$$

$$T(k^0,t)=e^t=P_0(t)e^t;$$
$$T(k,t)=te^t=P_1(t)e^t;$$
$$T(k^2,t)=t(1+t)e^t=(t^2+t)e^t=P_2(t)e^t;$$
$$T(k^3,t)=t(1+3t+t^2)e^t=(t+3t^2+t^3)e^t=P_3(t)e^t;$$

$T(k^n,t)=P_n(t)e^t$ $\qquad$ $P_n(t)=$ Volonterio’s Polynomial

B) We can deduce, from the (*), that:

$$e^t P_{n+1}(t) = t\frac{d}{dt}\left[e^t P_n(t)\right];$$

$$e^t P_{n+2}(t) = t\frac{d}{dt}\left[t\frac{d}{dt}\left(e^t P_n(t)\right)\right];$$

.

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$$e^t P_{n+m}(t) = \left(t\frac{d}{dt}\right)^m \left(e^t P_n(t)\right).$$

Now if we set $\phi_t \equiv t\frac{d}{dt}$,

C) we can write generalizing:

$$P_{n+m}(t) = e^{-t}\phi_t^{\,m}\left(P_n(t)e^t\right),$$

but $P_0(t) = 1$, thence we obtain:

$$P_n(t) = e^{-t}\phi_t^{\,n} e^t$$

[N.B. From this we deduce $T\left(k^n u(k), t\right) = \phi_t^{\,n} e^t = e^t P_n(t)$]

D) Based on the above points, we can write the following mixed equation difference-differential:

$$P_{n+1}(t) = t\left(\dot{P}_n(t) + P_n(t)\right),$$

where $\dot{P}_n(t) = \frac{d}{dt}P_n(t)$.

E)
$$T\left(k^n u(k), t\right) = e^t P_n(t)$$

means that:

$$e^t P_n(t) = \sum_{k=0}^{\infty} k^n \frac{t^k}{k!},$$

namely

$$P_n(t) = e^{-t}\sum_{k=0}^{\infty} k^n \frac{t^k}{k!}.$$

F) at this point we consider $P_n(t)$ in such a way to apply the transformation V in n, i.e. we put $n = k$ and $t = \xi$ in order to avoid any confusion.

$$T(P_k(\xi),t) = ?$$

where

$$P_k(\xi) = e^{-\xi}\sum_{m=0}^{\infty} m^k \frac{\xi^m}{m!} .$$

Therefore:

$$T(P_k(\xi),t) = e^{-\xi}\sum_{m=0}^{\infty} \frac{\xi^m}{m!} T(m^k,t)$$

$$T(P_k(\xi),t) = e^{-\xi}\sum_{m=0}^{\infty} \frac{\xi^m}{m!} e^{mt} = e^{-\xi}\sum_{m=0}^{\infty} \frac{(\xi e^t)^m}{m!} = e^{-\xi} e^{\xi e^t} = e^{\xi(e^t-1)}$$

If $\xi = x$ for ease of writing, we have:

$$T(P_k(x),t) = e^{x(e^t-1)}$$

namely

$$T^{-1}\left(e^{x(e^t-1)},k\right) = P_k(x) ,$$

in other terms:

$$P_n(x) = \frac{d^n}{dt^n} e^{x(e^t-1)}\Big|_{t=0} .$$

We can express that the solution of the mixed equation

$$P_{n+1}(t) = t\left(\dot{P}_n(t) + P_n(t)\right),$$

is:

$$P_n(t) = \frac{d^n}{d\xi^n} e^{t(e^{\xi}-1)}\Big|_{\xi=0} ,$$

or

$$P_n(t) = e^{-t}\phi_t^{\,n} e^t$$

Definition of Bernoulli's Polynomial:

The Bernoulli's Polynomail (see TF N° 21) is:

$$\boldsymbol{B_k}(\alpha)=\sum_{n=0}^{k}\frac{1}{1+n}\sum_{\tau=0}^{n}(-1)^k\binom{n}{k}(\alpha+k)^k \tag{24}$$

The generating function of the Bernoulli's Polynomial is:

$$\frac{t\,\mathrm{e}^{t\alpha}}{\mathrm{e}^t-1}=\sum_{k=0}^{+\infty}\boldsymbol{B_k}(\alpha)\frac{t^k}{k!} \tag{25}$$

Definition of Euler's Polynomial:

Euler's Polynomial (see TF N° 22):

$$\boldsymbol{E_k}(\alpha)=\sum_{n=0}^{k}\frac{1}{2^n}\sum_{\tau=0}^{n}(-1)^\tau\binom{n}{\tau}(\alpha+\tau)^\tau \tag{26}$$

The generating function of the Euler's Polynomial is:

$$\frac{2\,\mathrm{e}^{t\alpha}}{\mathrm{e}^t+1}=\sum_{k=0}^{+\infty}\boldsymbol{E_k}(\alpha)\frac{t^k}{k!} \tag{27}$$

Definition of Laguerre's Polynomial:

Laguerre's Polynomial (see TF N° 15 and N° 23)

$$\boldsymbol{L_n}(t)=\frac{\mathrm{e}^t}{n!}\frac{d^n}{dt^n}\left(\mathrm{e}^{-t}t^n\right) \tag{28}$$

Definition of Bessel's Polynomial of the first kind:

$$\boldsymbol{J_n}(t)=\left(\frac{t}{2}\right)^n\sum_{k=0}^{\infty}\frac{(-1)^k(t/2)^{2k}}{\Gamma(k+n+1)} \tag{29}$$

Transform V of a discrete periodic function:

Let $y_N(k)$ be a particular discrete function in which the following relation applies:

$$y_N(k)=y_N(k+N) \quad \forall k\in\mathbb{N}_0 \;\; e \;\; N\in\mathbb{N} \tag{30}$$

thanks to the Fourier series in the discrete domain:

$$y_N(k)=\frac{1}{N}\sum_{n=0}^{N-1} C_n \mathrm{e}^{i\frac{2\pi nk}{N}} \quad with \quad C_n=\sum_{k=0}^{N-1} y_N(k)\mathrm{e}^{-i\frac{2\pi nk}{N}} \tag{31}$$

we can apply the transformation V to the (31) as in the (32):

$$Tv(y_N(k),t)=\frac{1}{N}\sum_{n=0}^{N-1} C_n Tv(\exp(i2\pi nk/N),t) \tag{32}$$

thanks to the transformation tables we obtain:

$$Y_N(t)=\frac{1}{N}\sum_{n=0}^{N-1} C_n \mathrm{e}^{\left(t\exp\left(i\frac{2\pi n}{N}\right)\right)} \tag{33}$$

$$Y_N(t)=\frac{1}{N}\sum_{n=0}^{N-1} C_n \exp\left(t\cos\left(\frac{2\pi n}{N}\right)+i\,t\sin\left(\frac{2\pi n}{N}\right)\right) \tag{34}$$

$$Y_N(t)=\sum_{n=0}^{N-1} C_N \mathrm{e}^{t\cos\left(\frac{2\pi n}{N}\right)}\left(\cos\left(t\sin\left(\frac{2\pi n}{N}\right)\right)+i\sin\left(t\sin\left(\frac{2\pi n}{N}\right)\right)\right) \tag{35}$$

Now substituting C_n of the (31) in the (35) we have the eq. (36):

$$Y_N(t)=\frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{k=0}^{N-1} y_N(k)\mathrm{e}^{-i2\pi nk/N}\right)\mathrm{e}^{t\cos(2\pi n/N)}\left(\cos\left(t\sin\left(\frac{2\pi n}{N}\right)\right)+i\sin\left(t\sin\left(\frac{2\pi n}{N}\right)\right)\right) \tag{36}$$

$$Y_N(t)=\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k)\mathrm{e}^{t\cos(2\pi n/N)}\,\mathrm{e}^{-i2\pi nk/N}\left(\cos\left(t\sin\left(\frac{2\pi n}{N}\right)\right)+i\sin\left(t\sin\left(\frac{2\pi n}{N}\right)\right)\right) \tag{37}$$

So expanding the term $\mathrm{e}^{-i2\pi nk/N}$ we obtain:

$$Y_N(t) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k) e^{t\cos\left(\frac{2\pi n}{N}\right)}\left(\cos\left(\frac{2\pi nk}{N}\right) - i\sin\left(\frac{2\pi nk}{N}\right)\right)\left(\cos\left(t\sin\left(\frac{2\pi n}{N}\right)\right) + \right.$$
$$\left. + i\sin\left(t\sin\left(\frac{2\pi n}{N}\right)\right)\right) \qquad (38)$$

Now, thanks to the same definition of transform V , we have:

$$Y_N(t) = \sum_{k=0}^{+\infty} y_N(k)\frac{t^k}{k!} \qquad (39)$$

i.e. for $t \in \mathbb{R} \Rightarrow Y_N(t) \in \mathbb{R}$ thence $\Re(Y_N(t)) = Y_N(t)$ e $\Im(Y_N(t)) = 0$ that is ultimately:

$$\Re Y_N(t) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k) e^{t\cos\left(\frac{2\pi n}{N}\right)}\cos\left(\frac{2\pi n}{N} - t\sin\left(\frac{2\pi n}{N}\right)\right) \qquad (40)$$

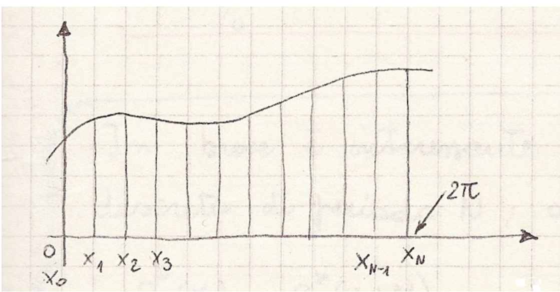
$$\Im Y_N(t) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k) e^{t\cos\left(\frac{2\pi n}{N}\right)}\sin\left(\frac{2\pi n}{N} - t\sin\left(\frac{2\pi n}{N}\right)\right) \equiv 0 \qquad (41)$$

from which we deduce the following equations:

$$Y_N(t) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k) e^{t\cos\left(\frac{2\pi n}{N}\right)}\cos\left(\frac{2\pi n}{N} - t\sin\left(\frac{2\pi n}{N}\right)\right) \qquad (42)$$

$$\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k) e^{t\cos\left(\frac{2\pi n}{N}\right)}\sin\left(\frac{2\pi n k}{N} - t\sin\left(\frac{2\pi n}{N}\right)\right) \equiv 0 \qquad (43)$$

In the particular case where the period N of the periodic discrete function is very large or even tending to infinity, we proceed in the following way:



$x_n = \frac{2\pi n}{N}$ where the step is $h = \frac{2\pi}{N}$ i.e. $h = \Delta x_n$ then rewrite the (36) as follows:

$$Y_N(t) = \sum_{n=0}^{N-1} y_N(k) \frac{1}{2\pi} \left(\frac{2\pi}{N} \sum_{k=0}^{N-1} e^{t\cos\left(\frac{2\pi n}{N}\right)} \cos\left(\frac{2\pi n}{N} - t\sin\left(\frac{2\pi n}{N}\right)\right)\right) \quad (44)$$

and for $N \to +\infty$ we have $\lim_{N\to+\infty} y_N(k) = y(k)$ thence we can write the (44) as follows:

$$Y_N(t) = \lim_{N\to\infty} \left(\sum_{n=0}^{N-1} y_N(k) \frac{1}{2\pi} \left(\sum_{k=0}^{N-1} h e^{t\cos(x_n)} \cos(k x_n - t\sin(x_n))\right)\right) \quad (45)$$

i.e.:

$$Y(t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} y(k) \int_0^{2\pi} e^{t\cos(x)} \cos(kx - t\sin(x)) dx \quad (46)$$

Since $V(t)$ is the transform V of $y(k)$ we deduce the equality:

$$\frac{1}{2\pi} \sum_{n=0}^{+\infty} y(k) \int_0^{2\pi} e^{t\cos(x)} \cos(kx - t\sin(x)) dx = \sum_{k=0}^{+\infty} y(k) \frac{t^k}{k!} \quad (47)$$

from which follows:

$$\frac{1}{2\pi} \int_0^{2\pi} e^{t\cos(x)} \cos(kx - t\sin(x)) dx = \frac{t^k}{k!} \quad \forall t \in \mathbb{C},\ \forall k \in \mathbb{N} \quad (48)$$

$$\sum_{k=0}^{+\infty} y_N(k) \frac{t^k}{k!} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} y_N(k) e^{t\cos\left(\frac{2\pi n}{N}\right)} \cos\left(\frac{2\pi n}{N} - t\sin\left(\frac{2\pi n}{N}\right)\right)$$

only and only if $y_N(k+N) = y_N(k)$ (49)

The eq. (49) shows how a development in Taylor's series can be reduced to a double summation (N^2 elements) when is valid the condition (30). This is a formula that represents the generalized solution of the differential equation $\frac{d^N V(t)}{d^N t} = V(t)$ where $T(y(k),t) = V(t)$

PROPERTIES OF THE TRANSFORM V		
PF N.	***Function*** $y(k) \quad k \in \mathbb{N}_0$	***Definition*** $V(t) := T(y(k), t) \quad t \in \mathbb{R}$
1	$\alpha\, y_1(k) + \beta\, y_2(k)$	$\alpha\, V_1(t) + \beta\, V_2(t)$
2	$y(k)\big\|_{k=0}$	$V(t)\big\|_{t=0}$
3	$y(k)\big\|_{k=n}$	$\left.\frac{d^n}{dt^n} V(t)\right\|_{t=0}$
4	$y(k)\,\delta^{\Delta}(k-n)$	$\frac{t^n}{n!}\, y(n)$
5	$y(k)\, u^{\Delta}(k-n)$	$V(t) - \sum_{k=0}^{n-1} y(k) \frac{t^k}{k!}$
6	$y(k)\, k$	$t \frac{d}{dt} V(t)$
7	$y(k)\, k^n$	$\Phi^n \circ V(t)$ *con* $\Phi := t \frac{d}{dt}$
8	$y(k)\, \alpha^k$	$V(\alpha k)$
9	$y(k)\, \mathrm{e}^{\alpha k}$	$V(t\, \mathrm{e}^{\alpha})$
10	$y(k+n)$	$\frac{d^n}{dt^n} V(t)$
11	$y(k-1)$	$y_\tau(-1) + \int_0^t V(\xi)\, d\xi$
12	$y(k) \sin(\alpha k)$	$\Re\left(V(t\, \mathrm{e}^{i\alpha})\right)$
13	$y(k) \cos(\alpha k)$	$\Im\left(V(t\, \mathrm{e}^{i\alpha})\right)$
14	$\binom{k}{n} y(k-n+m)$	$\frac{t^n}{n!} \frac{d^m}{dt^m} V(t)$
15	$\sum_{\tau=0}^{+\infty} \binom{k}{\tau} y(\tau)$	$V(t)\, \mathrm{e}^t$ where $V(t) := T\left(y(\tau)\big\|_{\tau=k}, t\right)$
16	$\frac{y(1+k)}{1+k}$	$\frac{V(t) - V(0)}{t}$
17	$y_1(k) * y_2(k)$	$V_1(k)\, V_2(k)$
18	$T(T^{-1}(\mathrm{e}^t\, T(y(k)\, y(\xi-k), t), k)\big\|_{\xi=k}, t)$	$V(t)^2$

TRANSFORMATION V OF SOME NOTE FUNCTIONS		
T F N.	***Function*** $y(k) \quad k \in \mathbb{N}_0$	***Definition*** $V(t) := T(y(k), t) \quad t \in \mathbb{R}$
1	$\delta^{\Delta}(k-n)$	$\frac{t^n}{n!}$
2	$u^{\Delta}(k)$	e^t
3	k	$t\,e^t$
4	k^n	$\Phi^n \circ e^t = \boldsymbol{V}_n(t) e^t \quad ove \quad \Phi := t \frac{d}{dt}$
5	α^k	$e^{\alpha t}$
6	$e^{\alpha k}$	$\exp(t\,e^{\alpha})$
7	$\sin(\alpha k)$	$e^{t\cos(\alpha)} \sin(t \sin(\alpha))$
8	$\cos(\alpha k)$	$e^{t\cos(\alpha)} \cos(t \sin(\alpha))$
9	$\alpha^k \sin(k\pi/2)$	$\sin(\alpha t)$
10	$\alpha^k \cos(k\pi/2)$	$\cos(\alpha t)$
11	$\Im((\beta + i\alpha)^k)$	$\sin(\alpha t) e^{\beta t}$
12	$\Re((\beta + i\alpha)^k)$	$\cos(\alpha t) e^{\beta t}$
13	$k!$	$\frac{t}{1-t}$
14	$(-1)^k k!$	$\frac{t}{1+t}$
15	$\frac{(k+n)!}{k!}$	$\boldsymbol{L}_n(-t) e^t \quad con \quad \boldsymbol{L}_n(-t) = e^{-t} (-1)^n \frac{d^n}{dt^n}(e^t t^n)$
16	$\binom{k}{n}$	$\frac{t^n e^t}{n!}$
17	$\frac{t}{1+k}$	$\frac{e^t - 1}{t}$
18	$\frac{1}{k!}$	$\boldsymbol{J}_0(2i\sqrt{t}) \quad con \quad \boldsymbol{J}_n(t) = \sum_{k=0}^{+\infty} \frac{(-1)^k (t/2)^{n+2k}}{k!\,\Gamma(k+1-n)}$
19	$\ln(1+k)$	$\int_0^1 \frac{e^{\xi t} - e^t}{\ln(t)} d\xi$
20	$y_{\alpha}(k) := \boldsymbol{V}_k(\alpha)$	$\exp(\alpha e^t - 1)$
21	$y_{\alpha}(k) := \boldsymbol{B}_k(\alpha)$	$\frac{t\,e^{\alpha t}}{e^t - 1}$

TRANSFORMATION V OF SOME NOTE FUNCTIONS		
T F N.	***Function*** $y(k) \quad k \in \mathbb{N}_0$	***Definition*** $V(t):=T(y(k),t) \quad t \in \mathbb{R}$
22	$y_\alpha(k):=\boldsymbol{E}_k(\alpha)$	$\frac{2e^{\alpha t}}{e^t+1}$
23	$y_\alpha(k):=\boldsymbol{L}_k(\alpha)$	$\frac{1}{1-t}\exp\left(\frac{\alpha t}{t-1}\right)$
24	$y_\alpha(k):=\frac{1}{k!}\boldsymbol{L}_k(\alpha)$	$\boldsymbol{J}_0(2\sqrt{\alpha t})e^t$
25	$\frac{a+bk}{c+k}$	$(-1)^t(a\boldsymbol{\Gamma}(c)-b\boldsymbol{\Gamma}(1+c)-a\boldsymbol{\Gamma}(c,-t)+b\boldsymbol{\Gamma}(c+1,-t))$
26	$\zeta(k+2)\Gamma(k+1)$	$-\frac{\Psi^{(0)}(1-t)}{t}-\frac{\gamma}{t}$ *ove* $\gamma=0.577215664901\ldots$
27	$\boldsymbol{\Gamma}(k+\alpha)$	$\frac{\boldsymbol{\Gamma}(\alpha)}{(1-t)^\alpha}$
28	$y(k):=f_N(k)$ ove $f_N(k)=f_N(k+N)$	$\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}y(k)e^{2\pi n/N}\cos\left(\frac{2\pi n k}{N}-t\sin\left(\frac{2\pi n}{N}\right)\right)$
29	$\frac{2^{2k}(2^{2k}-1)\boldsymbol{B}_{k+1}}{k+1}\cdot\sin\left(\frac{\pi\cdot k}{2}\right)$	$\tan(k)$

FUNDAMENTAL PROPERTIES OF THE INVERSE TRANSFORM V		
P A N.	***Function*** $V(t) \quad t \in \mathbb{R}$	***Definition*** $y^\Delta(k):=T^{-1}(V(t),k) \quad k \in \mathbb{N}_0$
1	$\alpha V_1(t)+\beta V_2(t)$	$\alpha y_1(k)+\beta y_2(k)$
2	$V(t)\vert_{t=0}$	$V(t):=T(y(k),t) \quad t \in \mathbb{R} \quad y(k)\vert_{k=0}$
3	$V(\alpha t)$	$\alpha^k y(k)$
4	$t V(t)$	$k\, y(k-1)$
5	$t^n V(t)$	$\frac{k!}{(k-n)!}y(k-n)$
6	$\alpha^t V(t)$	$\sum_{\tau=0}^{+\infty}(\ln\alpha)^\tau\binom{k}{\tau}y(k-\tau)$
7	$e^{\alpha t}V(t)$	$\sum_{\tau=0}^{+\infty}\alpha^\tau\binom{k}{\tau}y(k-\tau)$

FUNDAMENTAL PROPERTIES OF THE INVERSE TRANSFORM V		
P A N.	***Function*** $V(t) \quad t \in \mathbb{R}$	***Definition*** $y^{\Delta}(k) := T^{-1}(V(t), k) \quad k \in \mathbb{N}_0$
8	$\sin(\alpha t) V(t)$	$\sum_{\tau=0}^{+\infty} (-1)^{\tau} \alpha^{2\tau+1} \binom{k}{2\tau+1} y(k-2\tau-1)$
9	$\cos(\alpha t) V(t)$	$\sum_{\tau=0}^{+\infty} (-1)^{\tau} \alpha^{2\tau} \binom{k}{2\tau} y(k-2\tau)$
10	$\frac{d^n}{dt^n} V(t)$	$y(k+n)$
11	$\frac{d^n}{dt^n} V(t) \Big\|_{t=0}$	$y(k)\|_{k=0}$
12	$\int_0^t V(\xi) d\xi$	$y(k-1) - y(-1)$
13	$V_1(t) V_2(t)$	$y_1(t) * y_2(t)$
14	$V^2(t)$	$T(T^{-1}(T(y(k) y(\xi-k), t), k)\|_{\xi=k}, t)$
15	$\ln(t)$	$-\gamma \delta^{\Delta}(k-1) - \frac{1}{2\pi i} \oint \ln(z) z^{k-2} dz$
16	$\ln(t+1)$	$-\frac{1}{2\pi i} \oint e^z \Gamma(0,z) z^{k-1} dz \quad con \quad \Gamma(0,z) := \int_z^{+\infty} \xi^{a-1} e^{-\xi} d\xi$

INVERSE TRANSFORMATION OF THE NOTE FUNCTIONS		
AF N.	***Function*** $V(t) \quad t \in \mathbb{R}$	***Definition*** $y^{\Delta}(k) := T^{-1}(V(t), k) \quad k \in \mathbb{N}_0$
1	1	$\delta^{\Delta}(k)$
2	t^n	$n! \delta^{\Delta}(k-n)$
3	α^t	$(\ln(k))^{\alpha}$
4	$e^{\alpha t}$	α^k
5	$t^n e^t$	$\frac{k!}{(k-n)!}$
6	$\sin(t)$	$\sin(k\pi/2)$
7	$\cos(t)$	$\cos(k\pi/2)$
8	$\sin(\alpha t)$	$\alpha^k \sin(k\pi/2)$
9	$\cos(\alpha t)$	$\alpha^k \cos(k\pi/2)$

INVERSE TRANSFORMATION OF THE NOTE FUNCTIONS		
AF N.	***Function*** $V(t) \quad t \in \mathbb{R}$	***Definition*** $y^{\Delta}(k) := T^{-1}(V(t), k) \quad k \in \mathbb{N}_0$
10	$\sin(\alpha t) e^{\beta t}$	$\Im((\beta + i\alpha)^k)$
11	$\cos(\alpha t) e^{\beta t}$	$\Re((\beta + i\alpha)^k)$
12	$\frac{1}{1+t}$	$(-1)^k k!$
13	$\frac{1}{1-t}$	$k!$
14	$\frac{(n+t)!}{t!}$	$\frac{d^k}{d\xi^k} \Gamma(n+1+\xi)\Big\vert_{\xi=0}$
15	e^{-t^2}	$H_k(0)$

EXAMPLE 1
PROBLEM
Solve the following equation to the finite difference of the 2° order.

$$Y_{k+2} - 3Y_{k+1} + 2Y_k = 0 \quad Y_0 = 2 \quad Y_1 = 3 \tag{50}$$

Now, to solve such a simple equation to the finite difference of the second order homogeneous with constant coefficients, may be used various methods, including the method of the generating function and the method through the transform V that we have realized

* * * * *

SOLUTION

a) **METHOD OF THE GENERATING FUNCTION**
We consider the following generating function:

$$G(t) := \sum_{k=0}^{+\infty} Y_k \cdot t^k \tag{51}$$

$$\sum_{k=0}^{+\infty} Y_{k+2} \cdot t^k - 3 \sum_{k=0}^{+\infty} Y_{k+1} \cdot t^k + 2 \sum_{k=0}^{+\infty} Y_k \cdot t^k = 0 \tag{52}$$

$$(Y_2 + Y_3 t + Y_4 t^2 + \ldots) - 3(Y_1 + Y_2 t + Y_3 t^2 + \ldots) + 2G(t) = 0 \tag{53}$$

$$\frac{G(t)-Y_0-Y_1 t}{t^2}-3\frac{G(t)-Y_0}{t}+2\mathrm{G}(t)=0 \tag{54}$$

$$G(t)=\frac{2-3\mathrm{t}}{1-3\mathrm{t}+2\mathrm{t}^2}=\frac{2-3\mathrm{t}}{(1-t)(1-2\mathrm{t})}=\frac{1}{1-t}+\frac{1}{1-2\mathrm{t}} \tag{55}$$

Now, keep in mind the following observations:

$$\sum_{k=0}^{+\infty} t^k=\frac{1}{1-t} \quad |t|<1 \qquad \sum_{k=0}^{+\infty}(2\mathrm{t})^k=\frac{1}{1-2\mathrm{t}} \quad |2\mathrm{t}|<1 \tag{56}$$

from which we obtain the new generating function (already have been considered the initial condition):

$$G(t)=\sum_{k=0}^{+\infty}(1+2^k)t^k \tag{57}$$

b) **RISOLUTION METHOD THROUGH THE TRANSFORM V**

Calling with T such transformation from variable $k\in\mathbb{N}$ to variable $t\in\mathbb{R}$ and placing $Y_k=y(k)$ with $V(t):=T(y(k),t)$ we have the following:

$$T(y(k+2),t)-3\mathrm{T}(y(k+1),t)+2\mathrm{T}(y(k),t)=0 \tag{58}$$

$$\frac{d^2}{dt^2}V(t)-3\frac{d}{dt}V(t)+2\mathrm{V}(t)=0 \quad V(0)=2 \quad \dot{V}(t)=3 \tag{59}$$

The associated characteristic equation is:

$$r^2-3\mathrm{r}+2=0 \quad \rightarrow \quad r_1=1 \quad r_2=2 \tag{60}$$

thence the solution of the differential equation with the initial conditions is:

$$V(t)=\mathrm{e}^t+\mathrm{e}^{2\mathrm{t}} \tag{61}$$

where now using the inverse transform, we have the desired solution:

$$Y_k=y(k)=T^{-1}\left(\mathrm{e}^t+\mathrm{e}^{2\mathrm{t}},k\right)=1+2^k \tag{62}$$

EXAMPLE 2
PROBLEM
Given the following Taylor series expansion:

$$V(t)=\sum_{k=0}^{+\infty}\frac{\sin(k)t^{1+k}}{(1+k)!} \tag{63}$$

determine the function that has generated such series expansion.

*   *   *   *   *

SOLUTION

Rewrite the expression (63) as follows

$$V(t)=t\cdot\sum_{k=0}^{+\infty}\frac{t^k}{k!}\frac{\sin(k)}{(k+1)} \tag{64}$$

for the transformation tables, we can write:

$$T\left(\frac{\sin(k)}{k+1},t\right)=\frac{V(t)}{t} \quad \text{where} \quad T\left(\frac{y(k+1)}{k+1},t\right)=\frac{y(0)}{t}+\frac{T(y(k),t)}{t} \tag{65}$$

but we have also $V(t):=T(y(k),t) \qquad t\in\mathbb{R}$

$$y(1+k)=\sin(k) \quad\rightarrow\quad y(k)=\sin(k-1)=\sin(k)\cos(1)-\cos(k)\sin(1) \tag{66}$$

now reassemble:

$$T\left(\frac{\sin(k)}{k+1},t\right)=-\frac{\sin(-1)}{t}+\frac{\cos(1)T(\sin(k),t)}{t}-\frac{\sin(1)T(\cos(k),t)}{t} \tag{67}$$

$$T\left(\frac{\sin(k)}{k+1},t\right)=\frac{\sin(1)}{t}+\frac{\cos(1)e^{t\cos(1)}\sin(t\sin(1))}{t}-\frac{\sin(1)e^{t\cos(1)}\cos(t\sin(1))}{t} \tag{68}$$

and then the searched solution is

$$V(t)=\sin(1)+\left(\cos(1)\sin(t\sin(1))-\sin(1)\cos(t\sin(1))\right)e^{t\cos(1)} \tag{69}$$

EXAMPLE 3
PROBLEM
Given the following composite function, find the generalized expression of the k-th derivative.

$$f(x)=\sin(bx)e^{ax} \tag{70}$$

* * * * *

SOLUTION
In order to use the transformed V we make the following changes in the function (70)

$$V(t)=f(x+t)=\sin(b(t+x))e^{a(t+x)}=(\cos(bx)\sin(bt)+\sin(bx)\cos(bt))e^{at}e^{ax} \tag{71}$$

So, after some trivial step, we have:

$$y(k)=\cos(bx)e^{ax}T^{-1}(\sin(bt)e^{at},k)+\sin(bx)e^{ax}T^{-1}(\cos(bt)e^{at},k) \tag{72}$$

where thanks to the direct and inverse transformation tables from which we highlight two properties:

$$T^{-1}(\sin(bt)V(t),k)=\Im(T^{-1}(e^{ibt}V(t),k)) \tag{73}$$

$$T^{-1}(\cos(bt)V(t),k)=\Re(T^{-1}(e^{ibt}V(t),k)) \tag{74}$$

we obtain:

$$T^{-1}(\cos(bt)e^{at})=\Re(T^{-1}(e^{at+ibt},k))=\Re((a+ib)^k) \tag{75}$$

$$T^{-1}(\sin(bt)e^{at})=\Im(T^{-1}(e^{at+ibt},k))=\Im((a+ib)^k) \tag{76}$$

So thanks to other rules associated with this transformation we have the solution searched

$$\frac{d^k}{dx^k}V(x)=\cos(bx)e^{ax}\Im((a+ib)^k)+\sin(bx)e^{ax}\Re((a+ib)^k) \tag{77}$$

EXAMPLE 4
PROBLEM
Develop in binomial series the (sin) function, i.e.:

$$\sin(x)=\sum_{n=0}^{+\infty}\binom{x}{n}a_n \tag{78}$$

* * * * *

SOLUTION
In order to use the transform V we perform the following formal changes

$$\sin(x)\big|_{x=k}=\sin(k) \quad \textit{thence} \quad a_n=y(n) \tag{79}$$

therefore the expression posed by the problem becomes:

$$z(k)=\sin(k)=\sum_{n=0}^{+\infty}\binom{k}{n}y(k) \tag{80}$$

where thanks to the following rules for the binomial expansion

$$z(k)=T^{-1}\left(e^{t}\,T(y(n),t),k\right) \quad \rightarrow \quad y(k)=T^{-1}\left(e^{-t}\,T(z(k),t),n\right) \tag{81}$$

we have

$$T(\sin(k),t)=\sin(t\sin(1))\,e^{t\cos(1)} \tag{82}$$

thence

$$a_n=y(n)=T^{-1}\left(\sin(t\sin(1))\,e^{t\cos(1)-t},n\right) \tag{83}$$

where putting $\alpha=\sin(1)$ e $\beta=\cos(1)-1$ and taking into account the AF #10 we have:

$$a_n=\Im\left((\beta+i\alpha)^n\right)=\Im\left((\cos(1)-1+i\sin(1))^n\right)=\Im\left(\left(e^{i}-1\right)^n\right) \tag{84}$$

and therefore as required by the problem is

$$\sin(k)=\sum_{n=0}^{+\infty}\binom{k}{n}\Im\left(\left(e^{i}-1\right)^n\right) \tag{85}$$

where extending the variable k in the domain of the real, we have for each x the following relation:

$$\sin(x)=\sum_{n=0}^{+\infty}\frac{\Gamma(x+1)}{n!\,\Gamma(x-n+1)}\Im\left(\left(e^{i}-1\right)^n\right) \tag{86}$$

EXAMPLE 5
Determine the sum of the following binomial expansion:

$$S(k)=\sum_{n=0}^{+\infty}\binom{k}{n}n\,\alpha^n \tag{87}$$

thanks to the rules of the binomial expansion, we get

$$S(k)=T^{-1}\left(e^{t}T(k\alpha^{k},t),k\right) \tag{88}$$

EXAMPLE 6

Determine the analytical expression of the following finite sum.

$$S(n)=\sum_{k=0}^{n}k^{3} \tag{89}$$

SOLUTION

Thanks to the following theorem of the finite sum expressed by the following formula:

$$S(n)=T^{-1}\left(\mathrm{e}^{t}\int_{0}^{t}e^{-\xi}\frac{d}{d\,\xi}T(y(k),\xi)\,d\,\xi,n\right)+y(0)\delta(n)\quad with\quad \delta(n)=\begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases} \tag{90}$$

we proceed as follows:

$$y(k)=k^{3} \tag{91}$$

where for the TF 4 we have that:

$$T(k^{3},t)=\boldsymbol{V}_{3}(t)\,\mathrm{e}^{t}=(t+3t^{2}+t^{3})\,\mathrm{e}^{t} \tag{92}$$

where $\boldsymbol{V}_{3}(t)=t+3t^{2}+t^{3}$ is the Volonterio's polynomial of the third order. Thence:

$$S(n)=T^{-1}\left(\mathrm{e}^{t}\int_{0}^{t}\mathrm{e}^{-\xi}\frac{d}{d\xi}\left(\mathrm{e}^{\xi}(\xi+3\xi^{2}+\xi^{3})\right)d\xi,n\right) \tag{93}$$

i.e.:

$$S(n)=T^{-1}\left(\mathrm{e}^{t}\int_{0}^{t}(1+7\xi+6\xi^{2}+\xi^{3})\,d\xi,n\right) \tag{94}$$

After some calculations, we obtain:

$$S(n)=T^{-1}\left(\mathrm{e}^{t}\left(t+\frac{7}{2}t^{2}+2t^{3}+\frac{t^{4}}{4}\right),n\right) \tag{95}$$

i.e.:

$$S(n)=T^{-1}\left(\mathrm{e}^{t}\left(t+\frac{7}{2}t^{2}+2t^{3}+\frac{t^{4}}{4}\right),n\right)=\frac{k!}{(k-1)!}+\frac{7}{2}\frac{k!}{(k-2)!}+2\frac{k!}{(k-3)!}+\frac{1}{4}\frac{k!}{(k-4)!} \tag{96}$$

$$S(n)=k+\frac{7}{2}k(k-1)+2k(k-1)(k-2)+\frac{1}{4}k(k-1)(k-2)(k-3) \tag{97}$$

In conclusion, we have:

$$S(n)=\frac{k^4}{4}+\frac{k^3}{2}+\frac{k^2}{4}=\left(\frac{k(k+1)}{2}\right)^2 \tag{98}$$

Multiplying the eq.(98) for 1/6, we obtain an equivalent formula that can be connected to the Ramanujan's modular function, linked to the modes corresponding to the physical vibrations of the bosonic strings.

We observe that the sum of the cubes of the numbers of the succession of natural numbers is the square of the sum of the numbers of the succession of natural numbers, namely:

$$\sum_{k=0}^{n} k^3=\left(\sum_{k=0}^{n} k\right)^2 \quad \textit{where} \quad \sum_{k=0}^{n} k=\frac{k(k+1)}{2} \tag{99}$$

EXAMPLE 7

PROBLEM

Determine the generalized term of the Maclaurin's series expansion of the following composite analytic function:

$$y(t)=\mathrm{e}^{\alpha t}\sin(\beta t) \tag{100}$$

SOLUTION

For the same definition of transform and inverse transform V and for the AF N° 10:

$$c_k:=y(k)=T^{-1}\left(\mathrm{e}^{\alpha t}\sin(\beta t),k\right)=\Im\left((\alpha+i\beta)^k\right) \tag{101}$$

EXAMPLE 8

PROBLEM

Determine the generalized expression of the Fibonacci’s numbers.

SOLUTION

The law governing the Fibonacci's numbers is the following:

$$Y_k+Y_{k+1}=Y_{k+2} \qquad Y_0=0 \qquad Y_1=1 \tag{102}$$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Y_k	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	967

In order to apply the transformation V we must make, for formalities, the following changes:

$$Y_k = y(k) \qquad V(t) = T(y(k), t) \tag{103}$$

So rewriting everything using the transform V we obtain:

$$T(y(k), t) = T(y(k+1), t) + T(y(k+2), t) \tag{104}$$

$$V(t) + \frac{d}{dt}V(t) = \frac{d^2}{dt^2}V(t) \qquad V(0) = 0 \qquad \frac{d}{dt}V(0) = 1 \tag{105}$$

where from this differential equation we obtain the following associated characteristic:

$$r^2 - r - 1 = 0 \quad with \quad r_{1,2} = \frac{1 \pm \sqrt{5}}{2} \tag{106}$$

and thence

$$V(t) = a\,e^{r_1 t} + b\,e^{r_2 t} \quad with \quad V(0) = 0 \quad e \quad \dot{V}(0) = 1 \tag{107}$$

In conclusion, the solution of the differential equation is:

$$V(t) = \frac{1}{\sqrt{5}} e^{\frac{1+\sqrt{5}}{2}t} - \frac{1}{\sqrt{5}} e^{\frac{1-\sqrt{5}}{2}t} \tag{108}$$

Now inv-transform obtaining the searched solution:

$$Y_k = y(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right) \tag{109}$$

This expression can be connected with the Rogers-Ramanujan identities, that are connected to the aurea ratio, thence to the physical vibrations of the supersrings that are connected to the number 8, that is a Fibonacci's number (we remember that the ratio between a Fibonacci's number and the subsequent tend to the aurea ratio (1,61803398...)

We have the following expression:

$$Y_k = y(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right),$$

that is the eq. (109).

Now, it is well-known that the series of Fibonacci's numbers exhibits a fractal character, where the forms repeat their similarity starting from the reduction factor $1/\phi = 0{,}618033 = \frac{\sqrt{5}-1}{2}$ (Peitgen et al. 1986). Such a factor appears also in the famous fractal Ramanujan identity (Hardy 1927):

$$0{,}618033 = 1/\phi = \frac{\sqrt{5}-1}{2} = R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)},$$

and

$$\pi = 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right],$$

where

$$\Phi = \frac{\sqrt{5}+1}{2}.$$

We have also that

$$\pi = \frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]. \quad (109a)$$

From (109a), we have that

$$24 = \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (109b)$$

Furthermore, with regard the number 24, this is related to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \qquad (109c)$$

Furthermore, eqs. (109b) and (109c) are related. Indeed, we can write also the following expression:

$$24 = \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Thence, after some calculations, we can rewrite the expression (109) also as follows:

$$Y_k = y(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - (-0,61803398...)^k\right) =$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(-R(q) - \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t)^{1/5}}\frac{dt}{t^{4/5}}\right)}\right)\right], \qquad (109d)$$

from which we can obtain the following mathematical connections with the eq. (109c):

$$Y_k = y(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - (-0,61803398...)^k\right) =$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(-R(q) - \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t)^{1/5}}\frac{dt}{t^{4/5}}\right)}\right)\right] \Rightarrow$$

$$\Rightarrow\ 24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (109e)$$

EXAMPLE 9

Determine the function (analytic) generated by the following infinite sum:

$$f(x)=\sum_{k=0}^{k=+\infty}\frac{\sin(x)}{(k-1)!}\cdot(x-3)^k \quad (110)$$

We proceed in such a way as to make this equation conforms with the expression of the definition of transform V.

we observe that $(k+1)!=(k+1)\cdot k!$ and furthermore we set $x-3=t$ obtaining:

$$f(t+3)=\sum_{k=0}^{k=+\infty}\frac{\sin(k)}{k+1}\cdot\frac{t^k}{k!} \quad (111)$$

of course we can see now clearly that (111) can be rewritten as follows:

$$V(t)=\sum_{k=0}^{k=+\infty}y(k)\cdot\frac{t^k}{k!} \quad (112)$$

by placing respectively, the following substitutions:

$$V(t)=f(t+3) \text{ and } y(k)=\frac{\sin(k)}{k+1} \quad (113)$$

Now we need to consider if is possible to perform the transformation (this is equivalent to show that the eq. (110) converges), thence:

$$\lim_{k\to+\infty}\left|\frac{\sin(k)}{k+1}\right|^{\frac{1}{k}}\leqslant\lim_{k\to+\infty}\left|\frac{1}{k+1}\right|^{\frac{1}{k}}=0\leq+\infty \text{ c.v.v.} \quad (114)$$

Equation (114) is to verify that the condition of existence of the transform of $y(k)$ is true and then we proceed to the transformation using the attached tables in this paper:

$$V(t)=T\left(\frac{y(k+1)}{k+1!},t\right)=\frac{V_1(t)-V_1(0)}{t} \quad \text{with} \quad y(k+1)=\sin(k)\cdot u(k) \tag{115}$$

where for the initial value Theorem:

$$u(k)=\begin{cases}1 & k\geqslant 0\\ 0 & k<0\end{cases} \quad k\in\mathbb{N}$$

For the PF 16 we observe that:

$$V(t):=T\left(\frac{y(1+k)}{1+k},t\right)=\frac{V(t)-V(0)}{t}$$

where

$$y(1+k)=\sin(k) \quad \textit{where} \quad y(k)=\sin(k-1)=\sin(k)\cos(1)-\sin(1)\cos(k) \tag{116}$$

thence

$$T\left(\frac{y(1+k)}{1+k},t\right)=\frac{V(t)-V(0)}{t} \quad \textit{where} \quad V(t)=T(y(k),t) \tag{117}$$

$$T(\sin(k)\cos(1)-\cos(k)\sin(1),t) = e^{t\cos(1)}\sin(t\sin(1))\cos(1) - \\ - e^{t\cos(1)}\cos(t\sin(1))\sin(1) \tag{118}$$

$$V(t)=\mathrm{e}^{t\cos(1)}\sin(t\sin(1))\cos(1)-\mathrm{e}^{t\cos(1)}\cos(t\sin(1))\sin(1)=\mathrm{e}^{t\cos(1)}\sin(t\sin(1)-1) \tag{119}$$

$$V(0)=\mathrm{e}^{t\cos(1)}\sin(t\sin(1)-1)=-\sin(1) \tag{120}$$

thence the eq. (117) becomes:

$$f(t)=\frac{\mathrm{e}^{(t-3)\cos(1)}\sin(t\sin(1)-1)+\sin(1)}{(t-3)} \tag{121}$$

and finally the searched result of (110) is:

$$\sum_{k=0}^{k=+\infty} \frac{\sin(x)}{(k-1)!} \cdot (x-3)^k = \frac{e^{(x-3)\cos(1)} \sin((x-3)\sin(1)-1)+\sin(1)}{(x-3)} \qquad (122)$$

EXAMPLE 10

PROBLEM

Represent the Hermite's numbers using an integral formula.

SOLUTION

Consider the following famous relationships concerning the polynomials and the Hermite's numbers:

$$H_k(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2} \qquad (123)$$

$$e^{-t^2} = \sum_{k=0}^{+\infty} H_k(0) \frac{t^k}{k!} \qquad t \in \mathrm{R} \qquad (124)$$

The formula (124) satisfies the necessary and sufficient condition because the (6) is valid, as it is convergent for each $t \in$ R, and then the radius of convergence is $+\infty > \pi$, then in such case, the relation (6) is valid. Below there are some Hermite's polynomials :

$$H_0(t) = 1; \quad H_1(t) = 2t; \quad H_2(t) = 4t^2 - 2; \quad H_3(t) = 8t^3 - 12t; \quad H_4(t) = 16t^4 - 48t^2 + 12, \qquad (125)$$

where if we place $t = 0$, we obtain the Hermite's numbers:

$$H_0(0) = 1; \quad H_1(0) = 0; \quad H_2(0) = -2; \quad H_3(0) = 0; \quad H_4(0) = 12; \quad H_5(0) = 0; \quad H_6(0) = -120. \qquad (126)$$

So, by combining the formula (6) with the (123), we have that:

$$V(t) := e^{-t^2}\Big|_{t=e^{i\xi}} = e^{-\cos(2\xi)} \cdot (\cos(\sin(2\xi) - i\sin(\sin(2\xi)))), \qquad (127)$$

thence, the real part of (127) is:

$$e^{-\cos(2\xi)} \cdot (\cos(\sin(2\xi))) \qquad (128)$$

which, substituted into (6), becomes a new formula of the Hermite's numbers:

$$H_k(0) = (-1)^n = 1 \qquad k = 0;$$

$$H_k(0) = (-1)^n = \frac{2}{\pi}\Gamma(k+1)\int_0^{\pi} e^{-\cos(2\xi)} \cdot (\cos(\sin(2\xi)))\cos(k\xi)\,d\xi \qquad k \neq 0. \qquad (129)$$

We note that for $H_4(0) = 12$ and $H_6(0) = -120$, and know that 12 = 24/2 and 120 = 24× 5, we have a possible relationship with the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log\frac{\int_0^{\infty}\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Thence, we obtain the following mathematical connection with the second equation (129):

$$H_k(0) = (-1)^n = \frac{2}{\pi}\Gamma(k+1)\int_0^{\pi} e^{-\cos(2\xi)} \cdot (\cos(\sin(2\xi)))\cos(k\xi)\,d\xi \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^{\infty}\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

EXAMPLE 11

PROBLEM

Calculate $(-1)^k$ by an integral expression using the (6).

SOLUTION

For the (4), follow that the radius of convergence is $+\infty > \pi$, and then we can use the (6). From TF N.5, placing $\alpha = -1$, we have:

$$V(t) = T^{-1}\left((-1)^k, t\right) = e^{-t} \qquad (130)$$

therefore in the (6), we have:

$$y(k) = T^{-1}(V(t),k) := \frac{1}{\pi}\int_0^{\pi} \mathcal{R}\left(V\left(e^{i\xi}\right)\right)d\xi \qquad k = 0;$$

$$y(k) = T^{-1}(V(t),k) := \frac{2}{\pi}\Gamma(k+1)\int_0^{\pi} \mathcal{R}\left(V\left(e^{i\xi}\right)\right)\cos(k\xi)d\xi \qquad k \in \mathrm{N} \quad k \neq 0 \quad (131)$$

where:

$$\mathcal{R}\left(V\left(e^{i\xi}\right)\right) = \mathcal{R}\left(e^{-e^{i\xi}}\right) = e^{-\cos(\xi)}\cos(\sin(\xi)), \quad (132)$$

therefore:

$$(-1)^k = \frac{1}{\pi}\int_0^{\pi} e^{-\cos(\xi)}\cos(\sin(\xi))d\xi \qquad k = 0;$$

$$(-1)^k = \frac{2}{\pi}\Gamma(k+1)\int_0^{\pi} e^{-\cos(\xi)}\cos(\sin(\xi))\cos(k\xi)d\xi \qquad k \in \mathrm{N} \quad k \neq 0. \quad (133)$$

EXAMPLE 12

PROBLEM

Express the relation that exists between the Volonterio's polynomials and the inverse transform (6).

SOLUTION

From the formula (23) of the Volonterio's polynomials we have:

$$V_n(t)e^t = \sum_{k=0}^{+\infty} k^n \frac{t^k}{k!}. \quad (134)$$

Since the radius of convergence is $+\infty > \pi$, it is possible use the (6) and thence:

$$k^n = \frac{1}{\pi}\int_0^{\pi} \mathcal{R}\left(V\left(e^{i\xi}\right)\right)d\xi \qquad k = 0;$$

$$k^n = \frac{2}{\pi}\Gamma(k+1)\int_0^{\pi} \mathcal{R}\left(V\left(e^{i\xi}\right)\right)\cos(k\xi)d\xi \qquad k \in \mathrm{N}, \quad k \neq 0. \quad (135)$$

namely:

$$k^n = 0 \qquad k = 0;$$

$$k^n = \frac{2}{\pi}\Gamma(k+1)\int_0^{\pi} \mathcal{R}\left(V_n\left(e^{i\xi}\right)e^{\cos(\xi)+i\sin(\xi)}\right)\cos(k\xi)d\xi \qquad k \in \mathrm{N}, \quad k \neq 0. \quad (136)$$

where:

$$V_0(t) = 1; \quad V_1(t) = t; \quad V_2(t) = t + t^2; \quad V_3(t) = t + 3t^2 + t^3; \quad V_4(t) = t + 7t^2 + 6t^3 + t^4.$$

Now, putting $t = 2$ in the expression $t + 7t^2 + 6t^3 + t^4$, we have the following result:

$$2 + 7\cdot 2^2 + 6\cdot 2^3 + 2^4 = 2 + 7\cdot 4 + 6\cdot 8 + 16 = 2 + 28 + 48 + 16 = 94;$$

We note that $16 = (48/3);(8\times 2)$ and $48 = 24\times 2$ are related with the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Thence, also for the second equation (136), we can write the following mathematical connection:

$$k^n = \frac{2}{\pi}\Gamma(k+1)\int_0^\pi \mathcal{R}\left(V_n\left(e^{i\xi}\right)e^{\cos(\xi)+i\sin(\xi)}\right)\cos(k\xi)\,d\xi \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Appendix 1

***On some mathematical connections concerning some sectors of string theory*.**

Now we analyze the possible mathematical connections between some equations regarding some sectors of string theory and some expressions already described previously in this paper (see eqs. (6) and (86)).

We define as inverse transform of $V(t)$ the discrete function $y(k)$ obtained by the following definition: $y(k) = T(V(t),k) := \frac{d^k}{dt^k} V(t)|_{t=0}$ or by the following alternative formula for $k \in \mathrm{N}_0$:

$$y(k) = T^{-1}(V(t),k) := \frac{2}{\pi} \Gamma(k+1) \left(\int_0^{\pi} \mathcal{R}\left(V\left(e^{i\xi}\right)\right) \cos(k\xi) d\xi \right), \quad \textbf{(a)}$$

where Γ is the **Euler Gamma function** and e is the Eulero-Nepero's constant ($e = 2.718281828....$).

If we develop in binomial series the following function: $\sin(x) = \sum_{n=0}^{+\infty} \binom{x}{n} a_n$, applying and using the transformed V, we obtain the following relationship:

$$\sin(x) = \sum_{n=0}^{+\infty} \frac{\Gamma(x+1)}{n!\Gamma(x-n+1)} \mathcal{J}\left(\left(e^i - 1\right)^n\right),$$

from which, integrating, we can to obtain:

$$\int \sin(x) = \int \sum_{n=0}^{+\infty} \frac{\Gamma(x+1)}{n!\Gamma(x-n+1)} \mathcal{J}\left(\left(e^i - 1\right)^n\right).$$

We know that $\int \sin(x) = -\cos(x)$, thence, we obtain the following relationship:

$$\int \sum_{n=0}^{+\infty} \frac{\Gamma(x+1)}{n!\Gamma(x-n+1)} \mathcal{J}\left(\left(e^i - 1\right)^n\right) = -\cos(x). \quad \textbf{(b)}$$

a)4-point tachyon amplitude

In 1968 Veneziano proposed the following heuristic answer

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-(\alpha(s)+\alpha(t)))} \quad (1.1)$$

with $\alpha(s) = \alpha(0) + \alpha' s$.

Euler Gamma function has poles in the negative real axis at integer values $\alpha(s) = n$ with residue

$$\Gamma(-\alpha(s)) = \frac{\Gamma(-\alpha(s)+1)}{-\alpha(s)} = \frac{\Gamma(-\alpha(s)+n+1)}{-\alpha(s)(-\alpha(s)+1)(-\alpha(s)+2)\ldots(-\alpha(s)+(n-1))(-\alpha(s)+n)}$$
$$\stackrel{\alpha(s)\to n}{\to} \frac{(-1)^n}{n!}\frac{1}{-\alpha(s)+n}. \quad (1.2)$$

Hence, at fixed t, the amplitude has infinitely many poles at $s \in (0, \infty)$ for $\alpha(s) = \alpha(0) + s\alpha' = n$ or

$$s = \frac{n - \alpha(0)}{\alpha'} = M_n^2 \quad (1.3)$$

with residue

$$A^{(4)}(s,t) \stackrel{\alpha(s)\to n}{=} \frac{(-1)^n}{n!}\frac{\Gamma(-\alpha(t))}{\Gamma(-n-\alpha(t))}\frac{1}{\alpha(s)-n} = \frac{(\alpha(t)+1)(\alpha(t)+2)\ldots(\alpha(t)+n)}{n!}\frac{1}{\alpha(s)-n}. \quad (1.4)$$

In the bosonic string the simplest vertex operator is the one for the tachyon state $N = 0$ hence $M^2 = -4/\alpha'$. We have:

$$\mathcal{V}(0;p) = g_s \int d^2 z e^{ip\cdot X} = g_s \int d^2 z V(z,\bar{z};p). \quad (1.5)$$

With regard the 4-point tachyon amplitude, we have the following equation:

$$\mathrm{A}^{(m)}(\Lambda_i, p_i) = \delta\left(\sum_i p_i\right)\frac{g_s^{m-2}}{Vol(SL(2,C))}\int \prod_{i=1}^{m} d^2 z_i \prod_{j<l} |z_j - z_l|^{\alpha' p_j \cdot p_l}. \quad (1.6)$$

Setting $m = 4$ we end up with

$$\mathrm{A}^{(4)}(\Lambda_i, p_i) = \delta\left(\sum_i p_i\right)\frac{g_s^{2}}{Vol(SL(2,C))}\int \prod_{i=1}^{4} d^2 z_i \prod_{j<l} |z_j - z_l|^{\alpha' p_j \cdot p_l}. \quad (1.7)$$

After fixing the $SL(2,C)$ invariance by putting the insertion points at $0,1,z$ and $z_4 \to \infty$ we end up with

$$\mathrm{A}^{(4)} \approx g_s^2 \delta\left(\sum_i p_i\right)\int d^2 z |z|^{\alpha' p_1 \cdot p_3} |1-z|^{\alpha' p_2 \cdot p_3} \quad (1.8)$$

using **Gamma function identities** this expression can be given a nice form. One must use the integral representation

$$\int d^2z|z|^{2a-2}|1-z|^{2b-2} = \frac{2\pi\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)} \quad (1.9)$$

where $a+b+c=1$. With this, (1.8) can be shown to be equal to

$$A^{(4)} \approx g_s^2\delta\left(\sum_i p_i\right)\frac{\Gamma(-1-\alpha's/4)\Gamma(-1-\alpha't/4)\Gamma(-1-\alpha'u/4)}{\Gamma(2+\alpha's/4)\Gamma(2+\alpha't/4)\Gamma(2+\alpha'u/4)} \quad (1.10)$$

in terms of the Mandelstam variables

$$s=-(p_1+p_2)^2; \quad t=-(p_2+p_3)^2; \quad u=-(p_1+p_4)^2 \quad (1.11)$$

which satisfy on shell (i.e. use the tachyon mass $-p_i^2 = M^2 = -4/\alpha'$)

$$s+t+u = -\sum_{i=1}^4 p_i^2 = \sum M_i^2 = -\frac{16}{\alpha'}. \quad (1.12)$$

We can write also the following mathematical connection:

$$A^{(4)} \approx g_s^2\delta\left(\sum_i p_i\right)\int d^2z|z|^{\alpha' p_1\cdot p_3}|1-z|^{\alpha' p_2\cdot p_3} \approx$$

$$\approx g_s^2\delta\left(\sum_i p_i\right)\frac{\Gamma(-1-\alpha's/4)\Gamma(-1-\alpha't/4)\Gamma(-1-\alpha'u/4)}{\Gamma(2+\alpha's/4)\Gamma(2+\alpha't/4)\Gamma(2+\alpha'u/4)} \Rightarrow$$

$$\Rightarrow -\sum_{i=1}^4 p_i^2 = \sum M_i^2 = -\frac{16}{\alpha'}. \quad (1.13)$$

This expression can be related with the following Ramanujan's modular equation linked with the "modes" (i.e. 8 that is also a Fibonacci's number) that correspond to the physical vibrations of the superstrings:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.14)$$

Thence, we have the following relationship:

$$A^{(4)} \approx g_s^2 \delta\left(\sum_i p_i\right) \int d^2 z |z|^{\alpha' p_1 \cdot p_3} |1 - z|^{\alpha' p_2 \cdot p_3} \approx$$

$$\approx g_s^2 \delta\left(\sum_i p_i\right) \frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' t/4)\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\Gamma(2+\alpha' t/4)\Gamma(2+\alpha' u/4)} \Rightarrow$$

$$\Rightarrow -\sum_{i=1}^{4} p_i^2 = \sum M_i^2 = -\frac{16}{\alpha'} \Rightarrow$$

$$\Rightarrow -2 \cdot \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\alpha' \log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)} \right]} . \quad (1.15)$$

We note that this relationship ca be related also with the eq. (**a**), i.e. the inverse transform of $V(t)$, thence we obtain this further mathematical connection:

$$T^{-1}(V(t),k) := \frac{2}{\pi}\Gamma(k+1)\left(\int_0^\pi \mathcal{R}\left(V\left(e^{i\xi}\right)\right) \cos(k\xi) d\xi \right) \Rightarrow$$

$$\Rightarrow g_s^2 \delta\left(\sum_i p_i\right) \int d^2 z |z|^{\alpha' p_1 \cdot p_3} |1 - z|^{\alpha' p_2 \cdot p_3} \approx$$

$$\approx g_s^2 \delta\left(\sum_i p_i\right) \frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' t/4)\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\Gamma(2+\alpha' t/4)\Gamma(2+\alpha' u/4)} \Rightarrow$$

$$\Rightarrow -\sum_{i=1}^{4} p_i^2 = \sum M_i^2 = -\frac{16}{\alpha'} \Rightarrow$$

$$\Rightarrow -2 \cdot \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\alpha' \log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)} \right]} . \quad (1.15b)$$

b)the open string scattering

With regard the open string scattering, the amplitude is computed with operator insertions along the boundary of the disk which maps onto the real axis of the complex plane. The equation of the amplitude is:

$$A^{(4)} = \frac{g_s}{Vol(SL(2,R))} \int \prod_{i=1}^{4} dx_i \left\langle e^{ip_1 \hat{X}(x_1)} ... e^{ip_1 \hat{X}(x_4)} \right\rangle \approx \frac{g_s}{Vol(SL(2,R))} \delta^{26}\left(\delta \sum_i p_i \right) \int \prod_{i=1}^{4} dx_i \prod_{j \le l} \left| x_i - x_j \right|^{2\alpha' p_i \cdot p_j} .$$

(1.16)

For a given ordering, the residual symmetry can be used to fix 3 points to $x_1 = 0, x_2 = 0, x_3 = x$ and $x_4 = \infty$. The resulting expression contains a single integration for $0 \le x \le 1$

$$A^{(4)} \approx g_s \int_0^1 dx \left| x \right|^{2\alpha' p_1 \cdot p_2} \left| 1 - x \right|^{2\alpha' p_2 \cdot p_3} . \quad (1.17)$$

This integral is related to the Euler Beta function (thence with the Euler Gamma function)

$$B(a,b) = \int_0^1 dx x^{a-1} (1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} . \quad (1.18)$$

Whence, using now the tachyon mass $M^2 = -1/\alpha'$ one recovers the <u>Veneziano amplitude</u>

$$A^{(4)} \approx g_s \left(\frac{\Gamma(-1-\alpha' s)\Gamma(-1-\alpha' t)}{\Gamma(-1-\alpha'(s+t))} \right) . \quad (1.19)$$

Thence, we have the following possible mathematical relationship between (1.16), (1.17) and (1.19):

$$A^{(4)} = \frac{g_s}{Vol(SL(2,R))} \int \prod_{i=1}^{4} dx_i \left\langle e^{ip_1 \hat{X}(x_1)} ... e^{ip_1 \hat{X}(x_4)} \right\rangle \approx \frac{g_s}{Vol(SL(2,R))} \delta^{26}\left(\delta \sum_i p_i \right) \int \prod_{i=1}^{4} dx_i \prod_{j \le l} \left| x_i - x_j \right|^{2\alpha' p_i \cdot p_j}$$
$$\Rightarrow g_s \int_0^1 dx \left| x \right|^{2\alpha' p_1 \cdot p_2} \left| 1 - x \right|^{2\alpha' p_2 \cdot p_3} \Rightarrow g_s \left(\frac{\Gamma(-1-\alpha' s)\Gamma(-1-\alpha' t)}{\Gamma(-1-\alpha'(s+t))} \right) . \quad (1.20)$$

Also this relationship can be related with eq. (**a**), thence we obtain this further mathematical connection:

$$T^{-1}(V(t),k) := \frac{2}{\pi} \Gamma(k+1) \left(\int_0^{\pi} \mathcal{R}\left(V\left(e^{i\xi}\right)\right) \cos(k\xi)\, d\xi \right) \Rightarrow$$
$$\Rightarrow \frac{g_s}{Vol(SL(2,R))} \int \prod_{i=1}^{4} dx_i \left\langle e^{ip_1 \hat{X}(x_1)} ... e^{ip_1 \hat{X}(x_4)} \right\rangle \approx \frac{g_s}{Vol(SL(2,R))} \delta^{26}\left(\delta \sum_i p_i \right) \int \prod_{i=1}^{4} dx_i \prod_{j \le l} \left| x_i - x_j \right|^{2\alpha' p_i \cdot p_j}$$
$$\Rightarrow g_s \int_0^1 dx \left| x \right|^{2\alpha' p_1 \cdot p_2} \left| 1 - x \right|^{2\alpha' p_2 \cdot p_3} \Rightarrow g_s \left(\frac{\Gamma(-1-\alpha' s)\Gamma(-1-\alpha' t)}{\Gamma(-1-\alpha'(s+t))} \right) . \quad (1.20b)$$

c) Four point amplitude for the tachyons from CFT

The ground state tachyons in the twisted sector corresponds to:

$$M^2 = -\left(1 - \frac{k}{N}\right) \quad (1.21)$$

For the near marginal tachyons, in the large N limit, which are in the $(N-k)th$ sector, the vertex operator in the $(-1,-1)$ representation is,

$$V^{+}_{(-1,-1)}(z,\bar{z}) = e^{-\phi} e^{-\tilde{\phi}} e^{i\left(1-\frac{k}{N}\right)H(z)} e^{-i\left(1-\frac{k}{N}\right)\tilde{H}(\bar{z})} \sigma_{+} e^{ik.x}(z,\bar{z}). \quad (1.22)$$

The four point amplitude for these lowest lying tachyons can now be computed by taking two vertices in the $(0,0)$ representation and two in the $(-1,-1)$ representation.

$$C\int_C d^2z \left\langle V^{-}_{(-1,-1)}(z_\infty, \bar{z}_\infty) e^{\phi} T_F e^{\tilde{\phi}} \tilde{T}_F V^{+}_{(-1,-1)}(1) V^{-}_{(-1,-1)}(z,\bar{z}) e^{\phi} T_F e^{\tilde{\phi}} \tilde{T}_F V^{+}_{(-1,-1)}(0) \right\rangle. \quad (1.23)$$

The constant $C = g_c^4 C_s^2$, where C_s^2 is related to g_c by

$$C_s^2 = \frac{4\pi}{g_c^2}. \quad (1.24)$$

This amplitude can now be computed and is given by,

$$I = C(k_1.k_3)^2 \int_C d^2z \frac{|z|^{-2-s}|1-z|^{-2-t}}{|F(z)|^2}, \quad (1.25)$$

where $F(z)$ is the hypergeometric function,

$$F(z) \equiv F\left(\frac{k}{N}, 1-\frac{k}{N}; 1; z\right) = \frac{1}{\pi}\int_0^1 dy y^{-\frac{k}{N}} (1-y)^{-\left(1-\frac{k}{N}\right)} (1-yz)^{-\frac{k}{N}}, \quad (1.26)$$

and $s = -(k_1+k_2)^2$, $t = -(k_2+k_3)^2$, $s = -(k_3+k_1)^2$.
In the large N approximation,

$$F(z) \approx 1 + \frac{k}{N}\left(z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \ldots\right) + O\left((k/N)^2\right). \quad (1.27)$$

Note that the terms proportional to k/N in (1.27) shift the s-channel pole. There is an additional factor of $(k_1.k_2)^2$, due to which the contact term from any of the terms of (1.27) apart from 1, would at least be of $O\left((k/N)^2\right)$. With this observation, the integral can now be performed for $F(z) \to 1$.

$$I = C2\pi\,(k_1.k_3)^2 \frac{\Gamma\left(-\frac{s}{2}\right)\Gamma\left(-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}+\frac{t}{2}\right)}{\Gamma\left(-\frac{s}{2}-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1+\frac{t}{2}\right)} =$$

$$= -(4\pi)^2 g_c^2 \times \frac{1}{4}\left(u-2m^2\right)^2\left(\frac{1}{s}+\frac{1}{t}\right)\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}+\frac{t}{2}\right)}{\Gamma\left(1-\frac{s}{2}-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1+\frac{t}{2}\right)}. \quad (1.28)$$

Now using $s+t+u=4m^2$,

$$I = -4\pi^2 g_c^2\left[\frac{\left(t-2m^2\right)^2}{s}+\frac{\left(s-2m^2\right)^2}{t}+3(s+t)-8m^2\right]\times\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}+\frac{t}{2}\right)}{\Gamma\left(1-\frac{s}{2}-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1+\frac{t}{2}\right)}. \quad (1.29)$$

where we have to expand the gamma functions.
Also here we can write the following relationship between (1.25) and (1.29):

$$I = C(k_1.k_3)^2\int_C d^2z \frac{|z|^{-2-s}|1-z|^{-2-t}}{|F(z)|^2} \Rightarrow$$

$$\Rightarrow -4\pi^2 g_c^2\left[\frac{\left(t-2m^2\right)^2}{s}+\frac{\left(s-2m^2\right)^2}{t}+3(s+t)-8m^2\right]\times\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}+\frac{t}{2}\right)}{\Gamma\left(1-\frac{s}{2}-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1+\frac{t}{2}\right)}. \quad (1.30)$$

Also this expression can be related with the eq. (**a**) and with the Ramanujan's modular equation concerning the number 8 and thence, we obtain this further mathematical connection:

$$T^{-1}(V(t),k) := \frac{2}{\pi}\Gamma(k+1)\left(\int_0^\pi \mathcal{R}\left(V\left(e^{i\xi}\right)\right)\cos(k\xi)\,d\xi\right) \Rightarrow$$

$$\Rightarrow C(k_1.k_3)^2\int_C d^2z \frac{|z|^{-2-s}|1-z|^{-2-t}}{|F(z)|^2} \Rightarrow$$

$$\Rightarrow -4\pi^2 g_c^2\left[\frac{\left(t-2m^2\right)^2}{s}+\frac{\left(s-2m^2\right)^2}{t}+3(s+t)-8m^2\right]\times\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}+\frac{t}{2}\right)}{\Gamma\left(1-\frac{s}{2}-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1+\frac{t}{2}\right)}\Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.30b)$$

d) expressions concerning the four tachyon amplitude in CSFT

With regard a closed analytical expression for the off-shell four tachyon amplitude in CSFT, Giddings gave an explicit conformal map that takes the Riemann surfaces defined by the Witten diagrams to the standard disc with four tachyon vertex operators on the boundary. This conformal map is defined in terms of four parameters $\alpha,\beta,\gamma,\delta$.
The four parameters are not independent variables. They satisfy the relations

$$\alpha\beta=1,\quad \gamma\delta=1 \quad (1.31)$$

and

$$\frac{1}{2}=\Lambda_0(\theta_1,k)-\Lambda_0(\theta_2,k), \quad (1.32)$$

where $\Lambda_0(\theta,k)$ is defined by

$$\Lambda_0(\theta,k)=\frac{2}{\pi}\left(E(k)F(\theta,k')+K(k)E(\theta,k')-K(k)F(\theta,k')\right). \quad (1.33)$$

In (1.33) $K(k)$ and $E(k)$ are complete elliptic functions of the first and second kinds, $F(\theta,k)$ is the incomplete elliptic integral of the first kind. The parameters θ_1,θ_2,k and k' satisfy

$$k^2=\frac{\gamma^2}{\delta^2},\qquad k'^2=1-k^2, \qquad (1.34)$$

$$\sin^2\theta_1=\frac{\beta^2}{\beta^2+\gamma^2},\qquad \sin^2\theta_2=\frac{\alpha^2}{\alpha^2+\gamma^2}. \quad (1.35)$$

By using the integral representations of the elliptic functions it is possible to write the equation (1.32) in a useful form

$$E(\gamma^2)\int_{\alpha\gamma}^{\gamma/\alpha} dt \frac{1}{\sqrt{t^2+\gamma^4}\sqrt{1+t^2}} - (1-\gamma^4)K(\gamma^2)\int_{\alpha\gamma}^{\gamma/\alpha} dt \frac{1}{\sqrt{t^2+\gamma^4}\left(\sqrt{1+t^2}\right)^3} = \frac{\pi}{4}. \quad (1.36)$$

To expand (1.36) for small γ and α we have to divide the integration region into three intervals in such a way that the square roots in the denominators of (1.36) can be consistently expanded and the integrals in t performed. For example consider the integral in the first term of (1.36), it can be rewritten as

$$\int_{\alpha\gamma}^{\gamma/\alpha} dt \frac{1}{\sqrt{t^2+\gamma^4}\sqrt{1+t^2}} = \int_{\alpha\gamma}^{\gamma^2} dt \frac{1}{\gamma^2\sqrt{1+\frac{t^2}{\gamma^4}}\sqrt{1+t^2}} + \int_{\gamma^2}^{1} dt \frac{1}{t\sqrt{1+\frac{\gamma^4}{t^2}}\sqrt{1+t^2}} + \int_{1}^{\gamma/\alpha} dt \frac{1}{t^2\sqrt{1+\frac{\gamma^4}{t^2}}\sqrt{1+\frac{1}{t^2}}}$$

(1.37)

In each integral of the rhs the integration domain is contained in the convergence radius of the Taylor expansions of the square roots containing γ, so that they can be safely expanded and the integrals in t performed. With this procedure one gets the following equation equivalent to (1.36):

$$E(\gamma^2)\sum_{n,k=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}-k\right)n!k!}\left\{\frac{2}{2n+2k+1}\left[\gamma^{4k}-\left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right]+(1-\delta_{kn})\right.$$

$$\left.\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n}-\delta_{kn}\gamma^{4n}\ln\gamma^2\right\} - (1-\gamma^4)K(\gamma^2)\sum_{n,k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(-\frac{1}{2}-k\right)n!k!}\left\{\frac{1}{2n+2k+1}\left[\gamma^{4k}-\left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right]+\right.$$

$$\left.+(1-\delta_{kn})\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n}-\delta_{kn}\gamma^{4n}\ln\gamma^2+\frac{1}{2n+2k+3}\left[\gamma^{4n}-\left(\frac{\alpha}{\gamma}\right)^{2k+3}(\alpha\gamma)^{2n}\right]\right\} = \frac{\pi}{4}. \quad (1.38)$$

Thence, from (1.36) and (1.38) we can write the following mathematical relationship:

$$E(\gamma^2)\int_{\alpha\gamma}^{\gamma/\alpha} dt \frac{1}{\sqrt{t^2+\gamma^4}\sqrt{1+t^2}} - (1-\gamma^4)K(\gamma^2)\int_{\alpha\gamma}^{\gamma/\alpha} dt \frac{1}{\sqrt{t^2+\gamma^4}\left(\sqrt{1+t^2}\right)^3} =$$

$$= E(\gamma^2)\sum_{n,k=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}-k\right)n!k!}\left\{\frac{2}{2n+2k+1}\left[\gamma^{4k}-\left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right]+(1-\delta_{kn})\right.$$

$$\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n}-\delta_{kn}\gamma^{4n}\ln\gamma^2\Bigg\}-\left(1-\gamma^4\right)K\left(\gamma^2\right)\sum_{n,k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(-\frac{1}{2}-k\right)n!k!}\Bigg\{\frac{1}{2n+2k+1}\left[\gamma^{4k}-\left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right]+$$

$$+\left(1-\delta_{kn}\right)\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n}-\delta_{kn}\gamma^{4n}\ln\gamma^2+\frac{1}{2n+2k+3}\left[\gamma^{4n}-\left(\frac{\alpha}{\gamma}\right)^{2k+3}(\alpha\gamma)^{2n}\right]\Bigg\}=\frac{\pi}{4}. \quad (1.38b)$$

Also this expression can be related with the eq. (**a**), and thence we obtain this further mathematical connection:

$$T^{-1}(V(t),k):=\frac{2}{\pi}\Gamma(k+1)\left(\int_0^{\pi}\mathcal{R}\left(V\left(e^{i\xi}\right)\right)\cos(k\xi)d\xi\right)\Rightarrow$$

$$\Rightarrow E\left(\gamma^2\right)\int_{\alpha\gamma}^{\gamma/\alpha}dt\frac{1}{\sqrt{t^2+\gamma^4}\sqrt{1+t^2}}-\left(1-\gamma^4\right)K\left(\gamma^2\right)\int_{\alpha\gamma}^{\gamma/\alpha}dt\frac{1}{\sqrt{t^2+\gamma^4}\left(\sqrt{1+t^2}\right)^3}=$$

$$=E\left(\gamma^2\right)\sum_{n,k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}-k\right)n!k!}\Bigg\{\frac{2}{2n+2k+1}\left[\gamma^{4k}-\left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right]+\left(1-\delta_{kn}\right)$$

$$\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n}-\delta_{kn}\gamma^{4n}\ln\gamma^2\Bigg\}-\left(1-\gamma^4\right)K\left(\gamma^2\right)\sum_{n,k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(-\frac{1}{2}-k\right)n!k!}\Bigg\{\frac{1}{2n+2k+1}\left[\gamma^{4k}-\left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right]+$$

$$+\left(1-\delta_{kn}\right)\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n}-\delta_{kn}\gamma^{4n}\ln\gamma^2+\frac{1}{2n+2k+3}\left[\gamma^{4n}-\left(\frac{\alpha}{\gamma}\right)^{2k+3}(\alpha\gamma)^{2n}\right]\Bigg\}=\frac{\pi}{4}. \quad (1.38c)$$

Around $x=1/2$, i.e $\alpha=\sqrt{2}-1$ and $\gamma=1$, it is possible to obtain only x (or α) as a function of γ and not the opposite. Such an expansion can be obtained by first expanding eq.(1.36) around $\gamma=1$ and then looking for an expansion of α in terms of powers of $1-\gamma$ and $\ln(1-\gamma)$

$$\alpha=\sqrt{2}-1+a_1(1-\gamma)+a_2(1-\gamma)^2+\ldots+b_1(1-\gamma)\ln(1-\gamma)+b_2(1-\gamma)^2\ln(1-\gamma)+\ldots+$$
$$+c_1(1-\gamma)(\ln(1-\gamma))^2+c_2(1-\gamma)^2(\ln(1-\gamma))^2+\ldots \quad (1.39)$$

The coefficients in (1.39) are determined by requiring that (1.36) is satisfied. We provide here directly the expansion of x as a function of $1-\gamma$ up to the ninth order

$$x=\frac{1}{2}+\frac{1}{8}(1-\gamma)^2\left[1-2\log\left(\frac{1-\gamma}{4}\right)\right]-\frac{1}{4}(1-\gamma)^3\log\left(\frac{1-\gamma}{4}\right)-\frac{1}{16}(1-\gamma)^4\left[1+3\log\left(\frac{1-\gamma}{4}\right)\right]+$$

$$-\frac{1}{96}(1-\gamma)^5\left[7+12\log\left(\frac{1-\gamma}{4}\right)\right]+\frac{1}{1536}(1-\gamma)^6\left[-97-108\log\left(\frac{1-\gamma}{4}\right)-24\log^2\left(\frac{1-\gamma}{4}\right)+64\log^3\left(\frac{1-\gamma}{4}\right)\right]+$$

$$-\frac{1}{2560}(1-\gamma)^7\left[119+100\log\left(\frac{1-\gamma}{4}\right)-40\log^2\left(\frac{1-\gamma}{4}\right)-320\log^3\left(\frac{1-\gamma}{4}\right)\right]+\frac{1}{10240}(1-\gamma)^8[-321+$$
$$-60\log\left(\frac{1-\gamma}{4}\right)+1240\log^2\left(\frac{1-\gamma}{4}\right)+2240\log^3\left(\frac{1-\gamma}{4}\right)\Bigg]+\frac{1}{107520}(1-\gamma)^9\left[-1871+5740\log\left(\frac{1-\gamma}{4}\right)+\right.$$
$$\left.+29120\log^2\left(\frac{1-\gamma}{4}\right)+31360\log^3\left(\frac{1-\gamma}{4}\right)\right]+\ldots \quad (1.40)$$

e) physical interpretation of the nontrivial zeta zeros in terms of tachyonic string poles

The four-point dual string amplitude obtained by Veneziano was

$$A_4 = A(s,t) + A(t,s) + A(u,s) = \int_R dx|x|^{\alpha-1}|1-x|^{\beta-1} = B(\alpha,\beta), \quad (1.41)$$

where the Regge trajectories in the respective s,t,u channels are:

$$-\alpha(s) = 1+\frac{1}{2}s, \qquad -\beta(t) = 1+\frac{1}{2}t, \qquad -\gamma(u) = 1+\frac{1}{2}u. \quad (1.42)$$

The conservation of the energy-momentum yields:

$$k_1 + k_2 = k_3 + k_4 \Rightarrow k_1 + k_2 - k_3 - k_4 = 0. \quad (1.43)$$

We have also that the sum

$$s + t + u = 2(k_1^2 + k_2^2 + k_3^2) + 2(k_1\cdot k_2 - k_2\cdot k_3 - k_1\cdot k_3) = -8 \quad (1.44)$$

in mass units of $m_{\text{Planck}} = 1$, when all the four particles are tachyons and one has the on-shell condition:

$$k_1^2 = k_2^2 = k_3^2 = m^2 = -2m^2_{\text{Planck}} = -2 \quad (1.45)$$

in the natural units $\hbar = c = G = 1 \Rightarrow L_{\text{Planck}} = 1$ such that the string slope parameter in those units is given by $\alpha' = (1/2)L^2_{\text{Planck}} = 1/2$ and the string mass spectrum is quantized in multiplies of the Planck mass $m_{\text{Planck}} = 1$.
From the conservation of energy-momentum (1.43) and the tachyon on-shell condition eq. (1.45), one can deduce that:

$$(k_1 + k_2)^2 = (k_3 + k_4)^2 \Rightarrow k_1\cdot k_2 = k_3\cdot k_4. \quad (1.46)$$

Therefore, from eqs. (1.44) – (1.46) it is straightforward to show:

$$s + t + u = 2(-2-2-2) + 2(k_1\cdot k_2 - k_3\cdot(k_1+k_2)) = -12 + 2(k_1\cdot k_2 - k_3\cdot(k_3+k_4)) =$$
$$= -12 + 2(k_1\cdot k_2 - k_3\cdot k_4 - k_3\cdot k_3) = -12 - 2k_3\cdot k_3 = -12 + 4 = -8. \quad (1.47)$$

This relationship among $s+t+u=4m^2=-8$ will be crucial in what follows next. From eqs. (1.42), (1.44), and (1.47) we learn that:

$$\alpha+\beta+\gamma=1. \qquad (1.48)$$

There exists a well-known relation among the Γ functions (Euler Gamma function) in terms of ζ functions (Riemann zeta function) appearing in the expression for $A(s,t,u)$ when α,β fall inside the critical strip. In this case, the integration region in the real line that defines $A(s,t,u)$ in eq. (1.41) can be divided into three parts and leads to the very important identity

$$A(s,t,u)=B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}+\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}+\frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}=\frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1-\beta)}{\zeta(\beta)}\frac{\zeta(1-\gamma)}{\zeta(\gamma)} \qquad (1.49)$$

where $\alpha+\beta+\gamma=1$ and α,β are confined to the interior of the critical strip.
The derivation behind eq. (1.49) relies on the condition $\alpha+\beta+\gamma=1$ eq. (1.48) and the identities

$$\sin\pi(\alpha+\beta)+\sin\pi(\alpha+\gamma)+\sin\pi(\beta+\gamma)=4\cos\frac{\pi\alpha}{2}\cos\frac{\pi\beta}{2}\cos\frac{\pi\gamma}{2}, \qquad (1.50)$$

$$\Gamma(\gamma)=\Gamma(1-\alpha-\beta)=\frac{1}{\Gamma(\alpha+\beta)}\frac{\pi}{\sin\pi(\alpha+\beta)}, \qquad (1.51)$$

plus the remaining cyclic permutations from which one can infer

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\frac{\sin\pi(\alpha+\beta)}{\pi}, \qquad (1.52)$$

$$\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}=\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\frac{\sin\pi(\alpha+\gamma)}{\pi}, \qquad (1.53)$$

$$\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}=\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\frac{\sin\pi(\beta+\gamma)}{\pi}. \qquad (1.54)$$

Therefore, eqs. (1.50) – (1.54) allow us to recast the left hand side of (1.49) as

$$A(s,t,u)=B(\alpha,\beta)=\frac{4}{\pi}\cos\frac{\pi\alpha}{2}\cos\frac{\pi\beta}{2}\cos\frac{\pi\gamma}{2}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma). \qquad (1.55)$$

And, finally, the known functional relation

$$(2\pi)^z\zeta(1-z)=2\cos\frac{\pi z}{2}\Gamma(z)\zeta(z), \qquad (1.56)$$

in conjunction with the condition $\alpha+\beta+\gamma=1$ such that $(2\pi)^{\alpha+\beta+\gamma}=2\pi$ is what

establishes **the important identity (1.49) expressing explicitly the string amplitude $A(s,t,u)$ either in terms of zeta functions or in terms of Γ functions**.
In conclusion, we have the following interesting relationship between the eqs. (1.41), (1.49) and (1.55):

$$A_4 = A(s,t) + A(t,s) + A(u,s) = \int_R dx|x|^{\alpha-1}|1-x|^{\beta-1} = B(\alpha,\beta) \Rightarrow$$

$$\Rightarrow A(s,t,u) = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} + \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)} = \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1-\beta)}{\zeta(\beta)}\frac{\zeta(1-\gamma)}{\zeta(\gamma)} =$$

$$= \frac{4}{\pi}\cos\frac{\pi\alpha}{2}\cos\frac{\pi\beta}{2}\cos\frac{\pi\gamma}{2}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma), \quad (1.57)$$

from which we can to obtain the following equivalent expression:

$$\frac{1}{4}A_4 = \frac{1}{4}\int_R dx|x|^{\alpha-1}|1-x|^{\beta-1} = \frac{1}{4}B(\alpha,\beta) = \frac{1}{\pi}\cos\frac{\pi\alpha}{8}\cos\frac{\pi\beta}{8}\cos\frac{\pi\gamma}{8}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma). \quad (1.58)$$

In this expression there are both π and 8, i.e. the number that is connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.59)$$

Thence the final mathematical connection:

$$\frac{1}{4}A_4 = \frac{1}{4}\int_R dx|x|^{\alpha-1}|1-x|^{\beta-1} = \frac{1}{4}B(\alpha,\beta) = \frac{1}{\pi}\cos\frac{\pi\alpha}{8}\cos\frac{\pi\beta}{8}\cos\frac{\pi\gamma}{8}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma) \Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.60)$$

Also here we can to obtain an interesting mathematical connection between the eq. (1.57) and the eq. (**b**). Thence, we have the following relationship:

$$\int \sum_{n=0}^{+\infty} \frac{\Gamma(x+1)}{n!\Gamma(x-n+1)} \mathcal{J}\left(\left(e^{i}-1\right)^{n}\right) = -\cos(x) \Rightarrow$$

$$\Rightarrow A(s,t) + A(t,s) + A(u,s) = \int_R dx |x|^{\alpha-1} |1-x|^{\beta-1} = B(\alpha,\beta) \Rightarrow$$

$$\Rightarrow A(s,t,u) = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} + \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)} = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \frac{\zeta(1-\beta)}{\zeta(\beta)} \frac{\zeta(1-\gamma)}{\zeta(\gamma)} =$$

$$= \frac{4}{\pi} \cos\frac{\pi\alpha}{2} \cos\frac{\pi\beta}{2} \cos\frac{\pi\gamma}{2} \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma). \quad (1.60b)$$

Mathematical connections between Gamma function and Riemann's zeta function

Riemann's zeta function ζ is defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad (1.61)$$

for $\operatorname{Re} z > 1$. In that region the series converges uniformly on compact sets and represents a holomorphic function. If we consider the expression for the Gamma function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \quad (1.62)$$

for $\operatorname{Re} z > 0$, and change the variable t into $t = ns$ for $n \in N$, we get

$$\Gamma(z) = n^z \int_0^{\infty} s^{z-1} e^{-ns} ds \quad (1.63)$$

i.e., we have

$$\frac{\Gamma(z)}{n^z} = \int_0^{\infty} t^{z-1} e^{-nt} dt \quad (1.64)$$

for any $n \in N$ and $\operatorname{Re} z > 0$. This implies that for $\operatorname{Re} z > 1$, we have

$$\Gamma(z)\zeta(z) = \sum_{n=1}^{\infty} \int_0^{\infty} t^{z-1} e^{-nt} dt = \int_0^{\infty} t^{z-1} \left(\sum_{n=1}^{\infty} e^{-nt} \right) dt = \int_0^{\infty} t^{z-1} \frac{e^{-t}}{1-e^{-t}} dt = \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt. \quad (1.65)$$

This establishes the following integral representation for the zeta function.
For $\operatorname{Re} z > 1$, we have

$$\Gamma(z)\zeta(z) = \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt. \quad (1.66)$$

For any $z \in$ C, we have

$$\zeta(z) = \pi^{z-1} 2^z \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z) \zeta(1-z). \quad (1.67)$$

Let C be a path. We cut the complex plane along positive real axis and consider the integral

$$\int_C \frac{w^{z-1}}{e^w - 1} dw. \quad (1.68)$$

If the radius ε of the arc C_ε is less than 2π, it follows immediately from the Cauchy theorem that this integral doesn't depend on ε and the distance of the horizontal lines from the positive real axis. We can estimate the integral over C_ε as

$$\left| \int_{C_\varepsilon} \frac{w^{z-1}}{e^w - 1} dw \right| \le M\varepsilon^{\operatorname{Re} z} \int_0^{2\pi} \frac{1}{\left| e^{\varepsilon e^{i\phi}} - 1 \right|} d\phi . \quad (1.69)$$

Since $w \to e^w - 1$ has a simple zero at the origin, $e^w - 1 = wg(w)$ where g is holomorphic near the origin and $g(0) = 1$. It follows that

$$\left| e^w - 1 \right| \ge \frac{1}{2} |w| \quad (1.70)$$

for small w. Hence, we have

$$\left| \int_{C_\varepsilon} \frac{w^{z-1}}{e^w - 1} dw \right| \le 4\pi M \varepsilon^{\operatorname{Re} z - 1} \quad (1.71)$$

for small ε.
From (1.71), we obtain the following interesting inverse formula:

$$\left| \int_{C_\varepsilon} \frac{w^{z-1}}{e^w - 1} dw \right| \frac{1}{4M\varepsilon^{\operatorname{Re} z - 1}} \le \pi \quad (1.71b)$$

In particular, if $\operatorname{Re} z > 1$, we see that the integral over C_ε tends to 0 as $\varepsilon \to 0$. Hence, by taking the limit as ε goes to zero, and the horizontal lines tend to the real axis, we get

$$\int_C \frac{w^{z-1}}{e^w - 1} dw = \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt - e^{2\pi (z-1) i} \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt = \left(1 - e^{2\pi i z}\right) \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt = -2ie^{i\pi z} \sin(\pi z) \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt =$$
$$= -2ie^{i\pi z} \sin(\pi z) \Gamma(z) \zeta(z) \quad (1.72)$$

by (1.66). This implies that

$$- 2ie^{i\pi z}\sin(\pi z)\Gamma(z)\zeta(z) = \int_C \frac{w^{z-1}}{e^w - 1}dw \quad (1.73)$$

for any path C we considered above and $\operatorname{Re} z > 1$.
Furthermore, we have also the following mathematical connection:

$$- 2ie^{i\pi z}\sin(\pi z)\Gamma(z)\zeta(z) = \int_C \frac{w^{z-1}}{e^w - 1}dw \Rightarrow \left|\int_{C_\varepsilon} \frac{w^{z-1}}{e^w - 1}dw\right| \frac{1}{4M\varepsilon^{\operatorname{Re} z - 1}} \leq \pi \ . \quad (1.74)$$

We remember that

$$\pi = 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right], \quad (1.75)$$

where

$$\Phi = \frac{\sqrt{5}+1}{2}.$$

Furthermore, we remember that π arises also from the following Ramanujan's identities:

$$\pi = \frac{12}{\sqrt{130}}\log\left[\frac{(2+\sqrt{5})(3+\sqrt{13})}{\sqrt{2}}\right], \quad (1.75a)$$

and

$$\pi = \frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]. \quad (1.75b)$$

From (1.75b), we have that

$$24 = \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.75c).$$

Thence, with the eq. (1.74), we can obtain the following mathematical connections with π and Φ:

$$-2ie^{i\pi z}\sin(\pi z)\Gamma(z)\zeta(z) = \int_C \frac{w^{z-1}}{e^w - 1}dw \Rightarrow \left|\int_{C_\varepsilon} \frac{w^{z-1}}{e^w - 1}dw\right| \frac{1}{4M\varepsilon^{\operatorname{Re} z - 1}} \leq \pi \Rightarrow$$

$$\Rightarrow \pi = 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right], \quad (1.76)$$

and

$$-2ie^{i\pi z}\sin(\pi z)\Gamma(z)\zeta(z) = \int_C \frac{w^{z-1}}{e^w - 1}dw \Rightarrow \left|\int_{C_\varepsilon} \frac{w^{z-1}}{e^w - 1}dw\right| \frac{1}{4M\varepsilon^{\operatorname{Re} z - 1}} \leq \pi \Rightarrow$$

$$\Rightarrow \pi = \frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]. \quad (1.77)$$

Thence, also mathematical connection with 24, i.e. the number concerning the "modes" that correspond to the physical vibrations of the bosonic strings.

In conclusion, we have also a mathematical connection between the eq. (1.77) and the eq. (6)

$$y(k) = T^{-1}(V(t),k) := \frac{2}{\pi}\Gamma(k+1)\int_0^\pi \mathcal{R}\left(V\left(e^{i\xi}\right)\right)\cos(k\xi)d\xi, \qquad k \in \mathbb{N}, \quad k \neq 0.$$

We have indeed:

$$y(k) = T^{-1}(V(t),k) := \frac{2}{\pi}\Gamma(k+1)\int_0^\pi \mathcal{R}\left(V\left(e^{i\xi}\right)\right)\cos(k\xi)d\xi \Rightarrow$$

$$\Rightarrow -2ie^{i\pi z}\sin(\pi z)\Gamma(z)\zeta(z) = \int_C \frac{w^{z-1}}{e^w - 1}dw \Rightarrow \left|\int_{C_\varepsilon} \frac{w^{z-1}}{e^w - 1}dw\right| \frac{1}{4M\varepsilon^{\operatorname{Re} z - 1}} \leq \pi \Rightarrow$$

$$\Rightarrow \pi = \frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]. \quad (1.78)$$

Appendix 2.

Now we take the eq. (1.40):

$$x = \frac{1}{2} + \frac{1}{8}(1-\gamma)^2\left[1 - 2\log\left(\frac{1-\gamma}{4}\right)\right] - \frac{1}{4}(1-\gamma)^3\log\left(\frac{1-\gamma}{4}\right) - \frac{1}{16}(1-\gamma)^4\left[1 + 3\log\left(\frac{1-\gamma}{4}\right)\right] +$$

$$-\frac{1}{96}(1-\gamma)^5\left[7 + 12\log\left(\frac{1-\gamma}{4}\right)\right] + \frac{1}{1536}(1-\gamma)^6\left[-97 - 108\log\left(\frac{1-\gamma}{4}\right) - 24\log^2\left(\frac{1-\gamma}{4}\right) + 64\log^3\left(\frac{1-\gamma}{4}\right)\right] +$$

$$-\frac{1}{2560}(1-\gamma)^7\left[119 + 100\log\left(\frac{1-\gamma}{4}\right) - 40\log^2\left(\frac{1-\gamma}{4}\right) - 320\log^3\left(\frac{1-\gamma}{4}\right)\right] + \frac{1}{10240}(1-\gamma)^8[-321 +$$

$$-60\log\left(\frac{1-\gamma}{4}\right) + 1240\log^2\left(\frac{1-\gamma}{4}\right) + 2240\log^3\left(\frac{1-\gamma}{4}\right)\Bigg] + \frac{1}{107520}(1-\gamma)^9\left[-1871 + 5740\log\left(\frac{1-\gamma}{4}\right) +\right.$$

$$\left. + 29120\log^2\left(\frac{1-\gamma}{4}\right) + 31360\log^3\left(\frac{1-\gamma}{4}\right)\right] + \ldots$$

This is an expression concerning the four tachyon amplitude in CSFT. From this equation, we take the following numbers:

1, 2, 3, 4, 7, 8, 12, 16, 24, 40, 60, 64, 96, 97, 100, 108, 119, 320, 321

1240, 1536, 1871, 2240, 2560, 5740, 10240, 29120, 31360, 107520

First series

1, 2, 3, 4, 7, 8, 12, 16, 24, 40, 60, 64, 96, 97, 100, 108, 119, 320,

321,

Numbers	Triangular equals or near	Fibonacci's equals or near	Partitions Equals or near	**squares**	**Subsequent ratios**
1	**1**	**1**	**1**	**1**	
2		**2**	**2**		2/1=2
3	**3**	**3**	**3**		3/2= **1,50**
4	≈ 3	≈ 5		**4**	4/3 =1,33 ≈√√*3,14*
7	≈ 6		**7**		7/4=1,75

					≈√*3,14*
8	≈ 6	**8**			8/7 =1,14
12	≈10	≈ 13	≈11		…**1,50**
16	≈15		≈15	**16**	…1.33 ≈√√*3,14*
24	≈21	≈21	≈22		…**1,50**
40	40,5≈mean between 36 and 45	≈ 34	≈42		…1,66≈ ***1,618***
60	≈55	≈ 55 = 50 mean between 60 and 40			…**1,50**
64	≈66			**64**	…1,06 ≈ ≈ $\sqrt[8]{1{,}618}$
96	≈91	≈89			…**1,50**
97	≈100				…1,01 ≈ ≈ $\sqrt[64]{2{,}718}$
100	≈105		≈101	**100**	…1,03 ≈ ≈ $\sqrt[32]{2{,}718}$
108	≈105				…1,08 ≈ ≈ $\sqrt[16]{\pi}$
119	≈120	≈116,5 mean between 89 and 144			…1,10 ≈ ≈ $\sqrt[4]{\Phi}$
320	≈325	≈ 305 mean between 233 and 377			…2,68≈ ***2,718***
321	≈325	≈ 305 mean between 233 and 377	≈341 mean between 297 and 395		1,003 ≈ ≈ $\sqrt[128]{\Phi}$

Observations.

The numbers of the series are equal or near, initially, to the triangular numbers, Fibonacci's numbers, or partitions, and after there are the arithmetic means. We observe, however, the presence of five squares, average distributed:
one for every two or three numbers.
About the subsequent ratios (a number divided by the previous), always interesting, we can notice that include square roots, cube and so on of π =3,14 , Φ = 1,618, e = 2,718 . the subsequent ratio more frequent 1,50, could be related to the

approximate mean between 1,34 and 1,67 that is (1,34 + 1,67)/2 = 3,01 /2 = 1,505 ≈ 1,50; or even better between 1,335 and 1,665, that is (1.335 + 1,665)/2 = 3/2 = 1,50 the exact value of the subsequent ratio, that is present five times in the last column. That are look good connections. Even the mean between $\sqrt{2{,}718}$ = 1,6486 and $\sqrt{1{,}61803398}$ = 1,2720 is equal to (1,6486 + 1,2720)/2 = 2,9206/2 = 1,4603. Furthermore $\pi/2$ = 1,5708 and the mean between 1,6486; 1,2720 and 1,5708 is 1,497 ≈ 1,50 thence very near to the value of the ratio. Therefore mean between the value of the square root of e , the value of the square root of Φ and of the value of $\pi/2$

Triangular

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, 1225, 1275, 1326, 1378, 1431, 1485, 1540, 1596, 1653, 1711, 1770, 1830, 1891, 1953, 2016, 2080, 2145, 2211, 2278, 2346, 2415, 2485, 2556, 2628, 2701, 2775, 2850, 2926, 3003, 3081, 3160, 3240 ecc.

Partitions

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, 6842, 8349, 10143, 12310, 14883, 17977, 21637, 26015, 31185, 37338, 44583, 53174, 63261, 75175, 89134, 105558,

Now we see the second series, with larger numbers:

Second series:

1240, 1536, 1871, 2240, 2560, 5740, 10240, 29120, 31360, 107520

Numbers	Triangular equals or near	Fibonacci's	Partitions	Squares no	Subsequent ratios
1240	**≈ Mean 1250 between 1225 and 1275**	**987 1597 Mean 1292**	**1255**		
1536	≈1540	**≈1597**	**≈1575**		**1,23** **8** ***√3,14***
1871	≈1830 ≈1891 **Mean 1860,5**		 ≈1575 ≈1958 **Mean 1766**		**1,21** **8** ***√3,14***
2240	**≈2211 ≈2278 Mean 2244,5**	**≈1597 2≈584 mean 2090,5**	**1958 2436 Mean 2197**		**1,19** **8** ***√3,14***
2560	**≈ 2556**	**≈ 2584**	**≈2436**		**1,14≈ 1,15=** **8** ***√3,14***
5740	**-**	**≈ 4181 ≈ 6765 Mean 5473**	**≈ 5604**		
10240	**-**	**≈ 10946**	**≈10143**		**1,78 ≈1,77 =*√3,14***
29120	**-**	**≈ 28657**	≈26015, ≈31185 **Mean 28600**		**2,84** ***≈ 2,718***
31360	**-**	≈28657	≈ 31185		**1,076** **8** **≈ √1,618 =1,0619**
107520	**-**	≈ 75025	≈ 105558		**3,42**

		≈121 393 **Mean** **98 209**			***≈ 3,14***

Again, the numbers of this second series are near to triangular numbers, Fibonacci's numbers or partition numbers, or to the means of two respective consecutive numbers. There are not squares, as in the first series. The subsequent ratios (also here included generally between 1 and 2) are also here linked to constant as Φ, π and e or to their square roots, cube and so on. Even here there are some possible connections with the three most famous mathematical constants Φ, π and e.

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