ON SOME APPLICATIONS OF THE VOLONTERIO'S TRANSFORM: SERIES DEVELOPMENT OF TYPE Nk+M AND MATHEMATICAL CONNECTIONS WITH SOME SECTORS OF THE STRING THEORY

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Abstract

In this work we have described a new mathematical application concerning the discrete and the analytic functions: the Volonterio's Transform (**V Transform**) and the Volonterio's Polynomial. We have descrive various mathematical applications and properties of them, precisely the series development of the type Nk+M. Furthermore, we have showed also various examples and the possible mathematical connections with some sectors of Number Theory and String Theory.

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VOLONTERIO'S TRANFORM GENERALIZED AND SERIES DEVELOPMENT OF TYPE Nk+M

Definition 1 (transformed generalized V)

The transform V of a discrete function $y(k)$ is an analytic function of a real variable (or complex) through which it is possible to pass from the world of discrete or finite mathematics in the world of differential mathematics.
The transformed V provides an overview higher than it can provide a generating function.
The transformed canonical is distinguished from generalized because its existence is based on continuous functions $V(t)$ and infinitely differentiable for t = 0 while the generalized is based on a continuous function and infinitely differentiable at $t = x$ (where for $x = 0$ we obtain obviously the transform canonical).
The properties of transformation and anti-transformation of the transform V are independent from the fact that we consider the transformed canonical or generalized.

Definition 2 (inverse generalized transform V)

The inverse transform V of an analytic function $V(t)$ of a real variable t continues in the zero and infinite times differentiable at t = 0 (canonical) or at t = x (generalized) (in other words a function $V(t)$ developable in MacLaurin or Taylor series) is a discrete function $y(k)$ defined in $\mathbb{N}_0$, through which it is possible the transition from the differential world to the world of discrete or finite mathematics.

DEFINITION OF TRANSFORM V

Let $y(k)$ a discrete function, then we can define the transformation $V(t)$ as follows:

$$V(t) = T(y(k), t) := \sum_{k=0}^{+\infty} y(k) \frac{t^k}{k!} \; t \in \mathbb{C} \tag{1.1}$$

CONDITION OF EXISTENCE AND UNIQUENESS OF THE TRANSFORM V

To ensure the condition of existence of the transform must be ensured the following relation:

$$\lim_{k\to+\infty} \frac{|y(k)|^{1/k}}{k} < +\infty \; e \; R = \lim_{k\to+\infty} \frac{k}{|y(k)|^{1/k} \cdot e} \quad con \quad e = 2.718281828 \ldots \tag{a}$$

where R is the radius of the convergence while *e* is the Eulero-Nepero constant.

The relation (a) is a necessary condition that has been demonstrated exploiting the condition of the root of Cauchy-Hadamard while the condition of uniqueness can be attributed to the properties of series of powers where ***e*** is the Euler-Nepero constant.

Definition 1 of inverse transform V

We define with inverse transform of $\mathrm{V(t)}$ the discrete function $\mathrm{y(k)}$ obtained by the following definition:

$$y(k) = T^{-1}(V(t), k) := \frac{d^k}{dt^k} V(t) \Big|_{t=0} \tag{b}$$

Definition 2 of inverse transform V

Or by the following formula alternative to the (b):

$$y(k) = T^{-1}(V(t), k) := \begin{cases} \dfrac{1}{\pi} \displaystyle\int_0^{\pi} \Re\left(V\left(e^{i\xi}\right)\right) d\xi & for \quad k = 0 \\ \dfrac{2}{\pi} \Gamma(k+1) \displaystyle\int_0^{\pi} \Re\left(V\left(e^{i\xi}\right)\right) \cos(k\xi)\, d\xi & for \quad k \in \mathbb{N} \; k \neq 0 \end{cases} \tag{c}$$

where necessary and sufficient condition because (c) is valid is that is satisfied the condition $R > \pi$ where R is the radius of convergence (a).

Definition 3 of inverse transform V

$$y(k) = T^{-1}(V(t),k) := \frac{2}{\pi}\Gamma(k+1)\int_0^{\pi} \Im\left(V\left(e^{i\xi}\right)\right)\sin(k\xi)d\xi \qquad k \in \mathbb{N} \qquad (1.2)$$

or, for $k \in N \quad k \neq 0$ (see c):

$$T^{-1}(V(t),k) := \frac{2}{\pi}\Gamma(k+1)\left(\int_0^{\pi} \mathcal{R}\left(V\left(e^{i\xi}\right)\right)\cos(k\xi)d\xi\right) \Rightarrow \qquad (d)$$

$$\Rightarrow \quad y(k) = T^{-1}(V(t),k) := \begin{cases} \frac{1}{\pi}\int_0^{\pi} \Re\left(V\left(e^{i\xi}\right)\right)d\xi & for \quad k = 0 \\ \frac{2}{\pi}\Gamma(k+1)\int_0^{\pi} \Re\left(V\left(e^{i\xi}\right)\right)\cos(k\xi)\,d\xi & for \quad k \in \mathbb{N}\ k \neq 0 \end{cases}$$

Generalized definition of transform V

As follows, we define the generalized transformed $V(t)$:

$$V(t) = T_x(y(k),t) := \sum_{k=0}^{+\infty} y(k)\,\frac{(t-x)^k}{k!} \qquad t \in \mathbb{C} \qquad 1.3$$

Generalized definition of inverse transform V

We define as generalized inverse transform of $V(t)$ the discrete function $y(k)$ obtained by the following definition:

$$y(k) = T_x^{-1}(V(t),k) := \frac{d^k}{dt^k}V(t+x)\Big|_{t=0} \qquad 1.4$$

This definition is particularly useful in all those cases where the function $V(t)$ can not exist in 0.
Another alternative definition is the following:

$$y(k) = T_x^{-1}(V(t),k) := \frac{2}{\pi}\Gamma(k+1)\int_0^{1} \Im\left(V\left(x+e^{i\pi\xi}\right)\right)\sin(k\xi)d\xi \qquad k \in \mathbb{N}_0 \qquad 1.5$$

Fundamental properties of the transform V

Among the generalized transform and the canonical transform, is useful to keep in mind the following identity:

$$T(y(k), t) = T_0(y(k), t) \qquad 1.6$$

$$T_x(y(k), t) = T(y(k), t - x) \qquad 1.7$$

$$T_x(y(k), t + x) = T(y(k), t) \qquad 1.8$$

RELATION BETWEEN THE VOLONTERIO'S TRANFORM, OF LAPLACE AND ZETA

We consider the following definitions of Gamma function, Laplace's Transform and Zeta Transform:

$$\Gamma(k) = \int_0^\infty e^{-\xi}\xi^{k-1}d\xi, \quad \mathcal{L}(\mathrm{F(x)}, s) := \int_0^\infty e^{-sx}\cdot \mathrm{F}(x)dx, \qquad \mathcal{Z}(y(k), z) := \sum_{k=0}^{\infty} y(k)z^{-k} \qquad 1.9$$

we have:

$$T(y(k)\Gamma(k+1), t) = \int_0^\infty e^{-\xi}\cdot T(y(k)\xi^k, t)d\xi = \int_0^\infty e^{-\xi}\cdot T(y(k), \xi t)d\xi = \int_0^\infty e^{-\xi}\cdot V(\xi t)d\xi \qquad 1.10$$

thence:

$$T(y(k)\Gamma(k+1), t) = \sum_{k=0}^{\infty} y(k)\Gamma(k+1)\frac{t^k}{k!} = \sum_{k=0}^{\infty} y(k)t^k = \int_0^\infty e^{-\xi}\cdot V(\xi t)d\xi \qquad 1.11$$

$$T(y(k)\Gamma(k+1), t) = \sum_{k=0}^{\infty} y(k)t^k = \int_0^\infty e^{-\xi}\cdot V(\xi t)d\xi \qquad 1.12$$

but, for the left-hand side we note that:

$$\xi = x/t, \qquad d\xi = dx/t, \qquad x_{min} = 0, \qquad x_{max} = +\infty \qquad 1.13$$

thence:

$$\int_0^\infty e^{-\xi}\cdot V(\xi t)d\xi = \frac{1}{t}\int_0^\infty e^{-x/t}\cdot V(x)dx = \frac{1}{t}\mathcal{L}\left(V(x), \frac{1}{t}\right) \qquad 1.14$$

while, for the right-hand side putting:

$$\sum_{k=0}^{\infty} y(k)t^k = \mathcal{Z}\left(y(k), \frac{1}{t}\right) = \frac{1}{t}\mathcal{L}\left(V(x), \frac{1}{t}\right) \qquad 1.15$$

putting $\frac{1}{t} = \zeta,\ x = t$ we have:

$$\mathcal{Z}(y(k), \zeta) = \zeta\mathcal{L}(V(t), \zeta) \qquad 1.16$$

$$y(k) = \mathcal{Z}^{-1}(\zeta\cdot\mathcal{L}(V(t), \zeta), k) \qquad 1.17$$

vice versa

$$V(t) = \mathcal{L}^{-1}\left(\frac{1}{\zeta}\mathcal{Z}(y(k), \zeta), t\right) \quad 1.18$$

more generally, in the case of generalized transform V, we have:

$$T_x^{-1}(V(t), k) = \mathcal{Z}^{-1}(\zeta \cdot \mathcal{L}(V(t+x), \zeta), k) \quad 1.19$$

$$T_x(y(k), t) = \mathcal{L}^{-1}\left(\frac{1}{\zeta}\mathcal{Z}(y(k), \zeta), t - x\right) \quad 1.20$$

ADDITIONAL DEFINITIONS

In order to read and interpret tables in complete sense clarifications is needed on the functions and abbreviations that have been introduced and also will be essential of the examples that follow after the tables. In any case, before proceeding to the list of transformations is useful to consider the following relations, definitions and functions.

Definition of operator Φ_t

With the symbol Φ_t we define the following operator:

$$\Phi_t := t\frac{d}{dt} \tag{1.21}$$

where its application iterated n times on a determined function $V(t)$ is expressed by the operator of the application in the following way $\Phi_t^n \circ V(t)$.

for example, we consider $V(t)=\sin(t)$,thence:

$$\Phi_t^3 \circ V(t) = t\frac{d}{dt}\left(t\frac{d}{dt}\left(t\frac{d}{dt}\sin(t)\right)\right) = t\cos(t) - 3t^2\sin(t) - t^3\cos(t)$$

Definition and properties of the Volonterio's polynomials

$$\boldsymbol{V}_n(t) = e^{-t}\sum_{k=0}^{+\infty} k^n \frac{t^k}{k!} \tag{1.22}$$

other formulas to determine the polynomial $\boldsymbol{V}_n(t)$ (see TF N° 4 and 20) are the following:

$$\boldsymbol{V}_n(t) = e^{-t}\Phi_t^n \circ e^t \;\; con \;\; \boldsymbol{V}_0(t) := 1 \tag{1.23}$$

or by the following recursive formulas:

$$\boldsymbol{V}_{n+1}(t) = t\left(\frac{d}{dt}\boldsymbol{V}_n(t) + \boldsymbol{V}_n(t)\right) \tag{1.24}$$

$$\boldsymbol{V}_n(t) = t\sum_{k=0}^{+\infty}\binom{n-1}{k}\boldsymbol{V}_k(t) \quad with \quad \boldsymbol{V}_0(t) := 1 \quad and \quad n \in \mathbb{N} \tag{1.25}$$

or:

$$\boldsymbol{V}_n(t) = \frac{d^n}{d\xi^n}e^{t(e^{\xi}-1)}\bigg|_{\xi=0} \tag{1.26}$$

From the above definition, by the Volonterio's transform:

$$e^{\alpha(e^t-1)} = \sum_{k=0}^{+\infty} \boldsymbol{V}_k(\alpha)\frac{t^k}{k!} \tag{1.27}$$

Definition of Bernoulli's Polynomial:

The Bernoulli's polynomial (see TF N° 21) is:

$$\boldsymbol{B}_k(\alpha) = \sum_{n=0}^{k} \frac{1}{1+n} \sum_{\tau=0}^{n} (-1)^{\tau} \binom{n}{\tau} (\alpha+\tau)^k \tag{1.28}$$

The generating function of the Bernoulli's polynomial is:

$$\frac{te^{t\alpha}}{e^t-1} = \sum_{k=0}^{+\infty} \boldsymbol{B}_k(\alpha)\frac{t^k}{k!} \tag{1.29}$$

Definition of Eulero's Polynomial:

Eulero's Polynomial (see TF N° 22):

$$\boldsymbol{E}_k(\alpha) = \sum_{n=0}^{k} \frac{1}{2^n} \sum_{\tau=0}^{n} (-1)^{\tau} \binom{n}{\tau} (\alpha+\tau)^{\tau} \tag{1.30}$$

The generating function of the Eulero's polynomial is:

$$\frac{2e^{t\alpha}}{e^t+1} = \sum_{k=0}^{+\infty} \boldsymbol{E}_k(\alpha)\frac{t^k}{k!} \tag{1.31}$$

Definition of Laguerre's Polynomial:

Laguerre's Polynomial (see TF N° 15 and N° 23)

$$\boldsymbol{L}_n(t) = e^t \frac{d^n}{dt^n}(e^{-t}t^n) \tag{1.32}$$

Definition of the Bessel's Polynomial of the first kind:

$$\boldsymbol{J}_n(t) = \left(\frac{t}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{2k}}{\Gamma(k+n+1)\, k!} \tag{1.33}$$

Definition of the Hermite's Polynomial

$$\boldsymbol{H}_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}\left(e^{-t^2}\right) \tag{1.34}$$

PROPERTIES OF THE TRANSFORM V		
PF N.	***Function*** $y(k) \quad k \in \mathbb{N}_0$	***Definition*** $V(t) := T(y(k), t) \quad t \in \mathbb{R}$
1	$\alpha y_1(k) + \beta y_2(k)$	$\alpha V_1(t) + \beta V_2(t)$
2	$y(k)\|_{k=0}$	$V(t)\|_{t=0}$
3	$y(k)\|_{k=n}$	$\left.\frac{d^n}{dt^n} V(t)\right\|_{t=0}$
4	$y(k)\delta^{\Delta}(k-n)$	$\frac{t^n}{n!} y(n)$
5	$y(k)u^{\Delta}(k-n)$	$V(t) - \sum_{k=0}^{n-1} y(k) \frac{t^k}{k!}$
6	$y(k)k$	$t \frac{d}{dt} V(t)$
7	$y(k)k^n$	$\Phi^n \circ V(t) \;\; con \;\; \Phi := t \frac{d}{dt}$
8	$y(k)\alpha^k$	$V(\alpha t)$
9	$y(k)e^{\alpha k}$	$V(te^{\alpha})$
10	$y(k+n)$	$\frac{d^n}{dt^n} V(t)$
11	$y(k-1)$	$y_\tau(-1) + \int_0^t V(\xi) d\xi$
12	$y(k) \sin(\alpha k)$	$\Re(V(te^{i\alpha}))$
13	$y(k) \cos(\alpha k)$	$\Im(V(te^{i\alpha}))$
14	$\binom{k}{n} y(k-n+m)$	$\frac{t^n}{n!} \frac{d^m}{dt^m} V(t)$
15	$\sum_{\tau=0}^{+\infty} \binom{k}{\tau} y(\tau)$	$V(t)e^t \;\; ove \;\; V(t) := T(y(\tau)\|_{\tau=k}, t)$
16	$\frac{y(1+k)}{1+k}$	$\frac{V(t) - V(0)}{t}$
17	$y_1(k) * y_2(k)$	$V_1(t) V_2(t)$
18	$T\left(T^{-1}(e^t T(y(k) y(\xi - k), t), k)\|_{\xi=k}, t\right)$	$V(t)^2$
19	$y(k)\Gamma(k+1)$	$\left.\frac{1}{t} \mathcal{L}(V(t), s)\right\|_{s=\frac{1}{t}} \;\; with \;\; \mathcal{L}(V(t), s) := \int_0^{\infty} e^{-ts} V(t) \, dt$

TRANSFORMATION V OF SOME KNOWN FUNCTIONS		
TF N.	***Function*** $y(k) \quad k \in \mathbb{N}_0$	***Definition*** $V(t) := T(y(k), t) \quad t \in \mathbb{R}$
1	$\delta^{\Delta}(k-n)$	$\frac{t^n}{n!}$
2	$u^{\Delta}(k)$ (idem for 1)	e^t
3	k	te^t
4	k^n	$\Phi^n \circ e^t = V_n(t)e^t \quad with \quad \Phi := t\frac{d}{dt}$
5	α^k	$e^{\alpha t}$
6	$e^{\alpha k}$	$\exp(te^{\alpha})$
7	$\sin(\alpha k)$	$e^{t\cos(\alpha)}\sin(t\sin(\alpha))$
8	$\cos(\alpha k)$	$e^{t\cos(\alpha)}\cos(t\sin(\alpha))$
9	$\alpha^k \sin(k\pi/2)$	$\sin(\alpha t)$
10	$\alpha^k \cos(k\pi/2)$	$\cos(\alpha t)$
11	$\Im((\beta + i\alpha)^k)$	$\sin(\alpha t)e^{\beta t}$
12	$\Re((\beta + i\alpha)^k)$	$\cos(\alpha t)e^{\beta t}$
13	$k!$	$\frac{1}{1-t}$
14	$(-1)^k k!$	$\frac{1}{1+t}$
15	$\frac{(k+n)!}{k!}$	$e^t \boldsymbol{L}_n(-t) \quad with \quad \boldsymbol{L}_n(t) = e^t \frac{d^n}{dt^n}(e^{-t}t^n)$
16	$\binom{k}{n}$	$\frac{t^n e^t}{n!}$
17	$\frac{1}{1+k}$	$\frac{e^t - 1}{t}$
18	$\frac{1}{k!}$	$\boldsymbol{J}_0(2\sqrt{t}) \quad with \quad \boldsymbol{J}_n(t) = \sum_{k=0}^{+\infty} \frac{(-1)^k (t/2)^{n+2k}}{k!\,\Gamma(k+1-n)}$
19	$\frac{1}{(k+n)!}$	$\frac{\boldsymbol{J}_n(2\sqrt{t})}{t^{n/2}}$
20	$y_\alpha(k) := \boldsymbol{V}_k(\alpha)$	$e^{\alpha(e^t - 1)}$
21	$y_\alpha(k) := \boldsymbol{B}_k(\alpha)$	$\frac{te^{\alpha t}}{e^t - 1}$
22	$y_\alpha(k) := \boldsymbol{E}_k(\alpha)$	$\frac{2e^{\alpha t}}{e^t + 1}$

TRANSFORMATION V OF SOME KNOWN FUNCTIONS		
TF N.	***Function*** $y(k) \quad k \in \mathbb{N}_0$	***Definition*** $V(t) := T(y(k), t) \quad t \in \mathbb{R}$
23	$y_\alpha(k) := \boldsymbol{L}_k(\alpha)$	$\frac{1}{1-t}\exp\left(\frac{\alpha t}{t-1}\right)$
24	$y_\alpha(k) := \frac{1}{k!}\boldsymbol{L}_k(\alpha)$	$\boldsymbol{J}_0(2\sqrt{\alpha t})e^t \quad with \quad \boldsymbol{J}_0(t) = \sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{4^k (k!)^2}$
25	$\frac{a+bk}{c+k}$	$(-t)^{-c}\big(a\Gamma(c) - b\Gamma(c+1) - a\Gamma(c,-t) + b\Gamma(c+1,-t)\big)$
26	$\zeta(k+2)\Gamma(k+1)$	$-\frac{\Psi^{(0)}(1-t)}{t} - \frac{\gamma}{t} \quad where \quad \gamma = 0.5772156\ldots \quad with \quad \Psi^{(0)}(t) := \frac{\Gamma'(t)}{\Gamma(t)}$
27	$\Gamma(k+\alpha)$	$\frac{\Gamma(\alpha)}{(1-t)^\alpha}$
28	$y(k) := f_N(k)$ *with* $f_N(k) = f_N(k+N)$	$\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y(k)e^{2\pi n/N}\cos\left(\frac{2\pi nk}{N} - t\sin\left(\frac{2\pi n}{N}\right)\right)$
29	$\frac{2^{2k}(2^{2k}-1)\boldsymbol{B}_{k+1}}{k+1}\sin\left(\frac{\pi k}{2}\right)$	$\tan(t)$

FUNDAMENTAL PROPERTIES OF THE INVERSE TRANSFORM V		
PA N.	**Function** $V(t) \quad t \in \mathbb{R}$	**Definition** $y^{\Delta}(k) := T^{-1}(V(t), k) \quad k \in \mathbb{N}_0$
1	$\alpha V_1(t) + \beta V_2(t)$	$\alpha y_1(k) + \beta y_2(k)$
2	$V(t)\|_{t=0}$	$V(t) := T(y(k), t) \quad t \in \mathbb{R} \quad y(k)\|_{k=0}$
3	$V(\alpha t)$	$\alpha^k y(k)$
4	$tV(t)$	$ky(k-1)$
5	$t^n V(t)$	$\frac{k!}{(k-n)!} y(k-n)$
6	$\alpha^t V(t)$	$\sum_{\tau=0}^{+\infty} (\ln \alpha)^{\tau} \binom{k}{\tau} y(k-\tau)$
7	$e^{\alpha t} V(t)$	$\sum_{\tau=0}^{+\infty} \alpha^{\tau} \binom{k}{\tau} y(k-\tau)$
8	$\sin(\alpha t) V(t)$	$\sum_{\tau=0}^{+\infty} (-1)^{\tau} \alpha^{2\tau+1} \binom{k}{2\tau+1} y(k-2\tau-1)$
9	$\cos(\alpha t) V(t)$	$\sum_{\tau=0}^{+\infty} (-1)^{\tau} \alpha^{2\tau} \binom{k}{2\tau} y(k-2\tau)$
10	$\frac{d^n}{dt^n} V(t)$	$y(k+n)$
11	$\frac{d^n}{dt^n} V(t) \Big\|_{t=0}$	$y(k)\|_{k=0}$
12	$\int_0^t V(\xi) d\xi$	$y(k-1) - y(-1)$
13	$V_1(t) V_2(t)$	$y_1(t) * y_2(t)$
14	$V^2(t)$	$T\left(T^{-1}(T(y(k) y(\xi - k), t), k)\|_{\xi=k}, t\right)$

INVERSE TRANSFORM OF KNOWN FUNCTIONS		
AF N.	**Function** $V(t) \quad t \in \mathbb{R}$	**Definition** $y(k) := T^{-1}(V(t), k) \quad k \in \mathbb{N}_0$
1	*1*	$\delta^{\Delta}(k) \quad with \quad \delta^{\Delta}(k) := f(x) = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$
2	t^n	$n!\,\delta^{\Delta}(k-n)$
3	α^t	$(\log \alpha)^k$
4	$e^{\alpha t}$	α^k
5	$t^n e^t$	$\frac{k!}{(k-n)!}$
6	$\sin(t)$	$\sin(k\,\pi/2)$
7	$\cos(t)$	$\cos(k\,\pi/2)$
8	$\sin(\alpha t)$	$\alpha^k \sin(k\,\pi/2)$
9	$\cos(\alpha t)$	$\alpha^k \cos(k\,\pi/2)$
10	$\sin(\alpha t)e^{\beta t}$	$\Im\left((\beta + i\alpha)^k\right)$
11	$\cos(\alpha t)e^{\beta t}$	$\Re\left((\beta + i\alpha)^k\right)$
12	$\frac{1}{1+t}$	$(-1)^k k!$
13	$\frac{1}{1-t}$	$k!$
14	e^{-t^2}	$\boldsymbol{H}_k(0) \quad with \quad \boldsymbol{H}_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}\left(e^{-t^2}\right)$
15	$\ln(t+1)$	$(\delta^{\Delta}(k) - 1)(-1)^k \Gamma(k)$

GENERALIZED TRANSFORMATION OF NOTE FUNCTIONS		
GTF N.	**Function** $y(k) \quad k \in \mathbb{N}_0$	**Definition** $V(t) := T_x(y(k), t) \quad t \in \mathbb{R}$
1	$\delta^{\Delta}(k-n)$	$\frac{(t-x)^n}{n!}$
2	1	e^{t-x}
3	k	$(t-x)e^{t-x}$
4	k^n	$\Phi_x^n \circ e^{t-x} = V_n(t-x)e^{t-x} \quad with \quad \Phi_x := (t-x)\frac{d}{dt}$
5	α^k	$e^{\alpha(t-x)}$
6	$e^{\alpha k}$	$\exp((t-x)e^{\alpha})$
7	$\sin(\alpha k)$	$\sin((t-x)\sin(\alpha))e^{(t-x)\cos(\alpha)}$
8	$\cos(\alpha k)$	$\cos\left((t-x)\sin(\alpha)\right)e^{(t-x)\cos(\alpha)}$

GENERALIZED TRANSFORMATION OF NOTE FUNCTIONS		
GTF N.	***Function*** $y(k) \quad k \in \mathbb{N}_0$	***Definition*** $V(t) := T_x(y(k), t) \quad t \in \mathbb{R}$
9	$\alpha^k \sin(k\pi/2)$	$\sin(\alpha(t-x))$
10	$\alpha^k \cos(k\pi/2)$	$\cos(\alpha(t-x))$
11	$\Im((\beta + i\alpha)^k)$	$\sin(\alpha(t-x))e^{\beta(t-x)}$
12	$\Re((\beta + i\alpha)^k)$	$\cos(\alpha(t-x))e^{\beta(t-x)}$
13	$k!$	$\frac{1}{1-t+x}$
14	$(-1)^k k!$	$\frac{1}{1+t-x}$
15	$\frac{(k+n)!}{k!}$	$e^{t-x} \boldsymbol{L}_n(-t+x)$ *with* $\boldsymbol{L}_n(t) = e^t \frac{d^n}{dt^n}(e^{-t}t^n)$
16	$\binom{k}{n}$	$\frac{(t-x)^n e^{t-x}}{n!}$
17	$\frac{1}{1+k}$	$\frac{e^{t-x}-1}{t-x}$
18	$\frac{1}{k!}$	$\boldsymbol{J}_0(2\sqrt{t-x})$ *with* $\boldsymbol{J}_n(t) = \sum_{k=0}^{+\infty} \frac{(-1)^k (t/2)^{n+2k}}{k!\,\Gamma(k+1-n)}$
19	$\frac{1}{(k+n)!}$	$\frac{\boldsymbol{J}_n(2\sqrt{t-x})}{(t-x)^{n/2}}$
20	$y_\alpha(k) := \boldsymbol{V}_k(\alpha)$	$\exp(\alpha(e^{t-x}-1))$
21	$y_\alpha(k) := \boldsymbol{B}_k(\alpha)$	$\frac{(t-x)e^{\alpha(t-x)}}{e^{t-x}-1}$
22	$y_\alpha(k) := \boldsymbol{E}_k(\alpha)$	$\frac{2e^{\alpha(t-x)}}{e^{t-x}+1}$
23	$y_\alpha(k) := \boldsymbol{L}_k(\alpha)$	$\frac{1}{1-t+x}\exp\left(\frac{\alpha(t-x)}{t-x-1}\right)$
24	$y_\alpha(k) := \frac{1}{k!}\boldsymbol{L}_k(\alpha)$	$\boldsymbol{J}_0(2\sqrt{\alpha(t-x)})e^{t-x}$ *with* $\boldsymbol{J}_0(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{4^k (k!)^2}$
25	$\frac{a+bk}{c+k}$	$(-t+x)^{-c}(a\Gamma(c) - b\Gamma(c+1) - a\Gamma(c, -t+x) + b\Gamma(c+1, -t+x))$
26	$\zeta(k+2)\Gamma(k+1)$	$-\frac{\Psi^{(0)}(1-t+x)}{t-x} - \frac{\gamma}{t-x}$ *where* $\gamma = 0.5772156\ldots$ *with* $\Psi^{(0)}(t) := \frac{\Gamma'(t)}{\Gamma(t)}$
27	$\Gamma(k+\alpha)$	$\frac{\Gamma(\alpha)}{(1-t+x)^\alpha}$
28	$y(k) := f_N(k)$ *with* $f_N(k) = f_N(k+N)$	$\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y(k)e^{2\pi n/N}\cos\left(\frac{2\pi nk}{N} - (t-x)\sin\left(\frac{2\pi n}{N}\right)\right)$

GENERALIZED TRANSFORMATION OF NOTE FUNCTIONS		
GTF N.	***Function*** $y(k) \quad k \in \mathbb{N}_0$	***Definition*** $V(t) := T_x(y(k),t) \quad t \in \mathbb{R}$
29	$\frac{2^{k+1}\left(2^{k+1}-1\right)\boldsymbol{B}_{k+1}}{k+1}\sin\left(\frac{\pi k}{2}\right)$	$\tan(t-x)$
30	$\frac{k!}{(c+k)!}$	$\frac{e^{t-x}(\Gamma(c)-\Gamma(c,t-x))}{\Gamma(c)\,(t-x)^c}$

	GENERALIZED INVERSE TRANSFORMATION OF NOTE FUNCTIONS		
GAF N.	***Function*** $V(t) \quad t \in \mathbb{R}$	***Definition*** $y^{\Delta}(k) := T_x^{-1}(V(t), k) \quad k \in \mathbb{N}_0$	
1	1	$\frac{x^{-k}}{(-k)!}$	$x \neq 0$
2	t	$\frac{x^{1-k}}{(1-k)!}$	$x \neq 0$
3	t^n	$\frac{n!\, x^{n-k}}{(n-k)!}$	$x \neq 0$
4	e^t	e^x	$\forall x$
5	α^t	$(\log \alpha)^k \alpha^x$	$\forall x \; with \; \alpha \neq 0$
6	$e^{\alpha t}$	$\alpha^k e^{\alpha x}$	$\forall x$
7	$t^n e^t$	$\begin{cases} \frac{k!}{(k-n)!} & x = 0 \\ n!\, e^x x^n \sum_{\tau=0}^{k} \binom{k}{\tau} \frac{x^{-\tau}}{(n-\tau)!} & x \neq 0 \end{cases}$	$\forall x$
8	$\sin(t)$	$\sin(x + k\,\pi/2)$	$\forall x$
9	$\cos(t)$	$\cos(x + k\,\pi/2)$	$\forall x$
10	$\sin(\alpha t)$	$\alpha^k \sin(\alpha x + k\,\pi/2)$	$\forall x$
11	$\cos(\alpha t)$	$\alpha^k \cos(\alpha x + k\,\pi/2)$	$\forall x$
12	$\sin(\alpha t) e^{\beta t}$	$\Im((\beta + i\,\alpha)^k e^{x(\beta + i\,\alpha)})$	$\forall x$
13	$\cos(\alpha t) e^{\beta t}$	$\Re((\beta + i\,\alpha)^k e^{x(\beta + i\,\alpha)})$	$\forall x$
14	$\frac{1}{1+t}$	$\frac{(-1)^k k!}{(1+x)^{k+1}}$	$x \neq -1$
15	$\frac{1}{1-t}$	$\frac{k!}{(1-x)^{k+1}}$	$x \neq 1$
16	$\log t$	$\log(x)\, \delta^{\Delta}(k) + (\delta^{\Delta}(k) - 1)(-1)^k x^{-k} \Gamma(k)$	$x \neq 0$

From the GAF n.16, we observe:

$$T_x^{-1}(\log t, k) = \log(x)\,\delta^{\Delta}(k) + (\delta^{\Delta}(k) - 1)(-1)^k x^{-k}\Gamma(k) \qquad 1.35$$

Where, especially, if we want to use this property we must remember to run the rest of the calculations with a seed equivalent based on the properties (1.6), (1.7) and (1.8).

$$T_1^{-1}(\log t, k) = \left(\delta^{\Delta}(k) - 1\right)(-1)^k\Gamma(k) \qquad 1.36$$

$$T_e^{-1}(\log t, k) = \delta^{\Delta}(k) + (\delta^{\Delta}(k) - 1)(-1)^k e^{-k}\Gamma(k) \qquad 1.37$$

Example

PROBLEM

Solve the following equation to the finite difference of the 2nd order.

$$Y_{k+2} - 3Y_{k+1} + 2Y_k = 0 \; Y_0 = 2 \; Y_1 = 3 \tag{1.38}$$

Now, to solve such a simple equation to the finite difference of the second order homogeneous with constant coefficients may be used various methods, including the method of the generating function and the method using the transform realized here.

* * * * *

SOLUTION

a) METHOD OF THE GENERATING FUNCTION

We consider the following generating function:

$$G(t) := \sum_{k=0}^{+\infty} Y_k t^k \tag{1.39}$$

$$\sum_{k=0}^{+\infty} Y_{k+2} t^k - 3\sum_{k=0}^{+\infty} Y_{k+1} t^k + 2\sum_{k=0}^{+\infty} Y_k t^k = 0 \tag{1.40}$$

$$(Y_2 + Y_3 t + Y_4 t^2 + \cdots) - 3(Y_1 + Y_2 t + Y_3 t^2 + \cdots) + 2G(t) = 0 \tag{1.41}$$

$$\frac{G(t) - Y_0 - Y_1 t}{t^2} - 3\frac{G(t) - Y_0}{t} + 2G(t) = 0 \tag{1.42}$$

$$\mathrm{G}(t) = \frac{2 - 3t}{1 - 3t + 2t^2} = \frac{2 - 3t}{(1 - t)(1 - 2t)} = \frac{1}{1 - t} + \frac{1}{1 - 2t} \tag{1.43}$$

taking into account the following observations:

$$\sum_{k=0}^{+\infty} t^k = \frac{1}{1 - t} \; |t| < 1 \; \sum_{k=0}^{+\infty} (2t)^k = \frac{1}{1 - 2t} \; |2t| < 1 \tag{1.44}$$

From which we get the new generating function (have already been considered the initial conditions):

$$\mathrm{G}(t) = \sum_{k=0}^{+\infty} \left(1 + 2^k\right) t^k \tag{1.45}$$

b)RESOLVING METHOD BY TRANSFORMED V

Calling with T this transformation from the variable $k \in \mathbb{N}$ to the variable $t \in \mathbb{R}$ and placing $Y_k = y(k)$ with $V(t) := T(y(k), t)$, we obtain:

$$\mathrm{T}(y(k+2), t) - 3T(y(k+1), t) + 2T(y(k), t) = 0 \tag{1.46}$$

$$\frac{d^2}{dt^2}V(t) - 3\frac{d}{dt}V(t) + 2V(t) = 0 \; V(0) = 2 \; \dot{V}(t) = 3 \tag{1.47}$$

The characteristic equation associated is:

$$r^2 - 3r + 2 = 0 \;\rightarrow r_1 = 1 \quad r_2 = 2 \tag{1.48}$$

i.e., the solution of the differential equation with the initial settings is:

$$\mathrm{V}(t) = e^t + e^{2t} \tag{1.49}$$

where now anti-transform, obtaining the solution searched:

$$Y_k = y(k) = T^{-1}(e^t + e^{2t}, k) = 1 + 2^k \tag{1.50}$$

Transformed V of a discrete periodic function:

Let $y_N(k)$ be a particular discrete function in which is valid the following relation:

$$y_N(k) = y_N(k+N)\ \forall k \in \mathbb{N}_0\ e\ N \in \mathbb{N} \tag{1.51}$$

Thanks to the Fourier series in the discrete domain:

$$y_N(k) = \frac{1}{N}\sum_{n=0}^{N-1} C_n e^{i\frac{2\pi nk}{N}} \quad con \quad C_n = \sum_{k=0}^{N-1} y_N(k) e^{-i\frac{2\pi nk}{N}} \tag{1.52}$$

we can apply the transformation V to the (1.52) as in (1.53):

$$\mathrm{Tv}(y_N(k),t) = \frac{1}{N}\sum_{n=0}^{N-1} C_n Tv(exp(i\,2\pi nk/N),t) \tag{1.53}$$

for the transformations tables, we obtain:

$$Y_N(t) = \frac{1}{N}\sum_{n=0}^{N-1} C_n e^{\left(t exp\left(i\frac{2\pi n}{N}\right)\right)} \tag{1.54}$$

$$Y_N(t) = \frac{1}{N}\sum_{n=0}^{N-1} C_n exp\left(t\cos\left(\frac{2\pi n}{N}\right) + i\,t\sin\left(\frac{2\pi n}{N}\right)\right) \tag{1.55}$$

$$Y_N(t) = \sum_{n=0}^{N-1} C_N e^{t cos\left(\frac{2\pi n}{N}\right)}\left(cos\left(t sin\left(\frac{2\pi n}{N}\right)\right) + i sin\left(t sin\left(\frac{2\pi n}{N}\right)\right)\right) \tag{1.56}$$

Now replacing C_n of the (1.52) in the eq. (1.56) we obtain the eq. (1.57):

$$Y_N(t) = \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{k=0}^{N-1} y_N(k) e^{-i\frac{2\pi nk}{N}}\right) e^{t\,cos\left(\frac{2\pi n}{N}\right)}\left(cos\left(t\,sin\left(\frac{2\pi n}{N}\right)\right) + i\,sin\left(t\,sin\left(\frac{2\pi n}{N}\right)\right)\right) \tag{1.57}$$

$$Y_N(t) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k) e^{t\,cos\left(\frac{2\pi n}{N}\right)} e^{-i\frac{2\pi nk}{N}}\left(cos\left(t\,sin\left(\frac{2\pi n}{N}\right)\right) + i\,sin\left(t\,sin\left(\frac{2\pi n}{N}\right)\right)\right) \tag{1.58}$$

thence expanding the term $e^{-i2\pi nk/N}$,we obtain:

$$Y_N(t) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k)e^{t\cos\left(\frac{2\pi n}{N}\right)}\left(\cos\left(\frac{2\pi nk}{N}\right) - i\sin\left(\frac{2\pi nk}{N}\right)\right)\left(\cos\left(t\sin\left(\frac{2\pi n}{N}\right)\right) + i\sin\left(t\sin\left(\frac{2\pi n}{N}\right)\right)\right) \quad 1.59$$

Now, for the same definition of transformed V we have:

$$Y_N(t) = \sum_{k=0}^{+\infty} y_N(k)\frac{t^k}{k!} \quad 1.60$$

i.e. for $t \in \mathbb{R} \Rightarrow Y_N(t) \in \mathbb{R}$ thence $\Re(Y_N(t)) = Y_N(t)$ e $\Im(Y_N(t)) = 0$ namely:

$$\Re(Y_N(t)) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k)e^{t\cos\left(\frac{2\pi n}{N}\right)}\cos\left(\frac{2\pi n}{N} - t\sin\left(\frac{2\pi n}{N}\right)\right) \quad 1.61$$

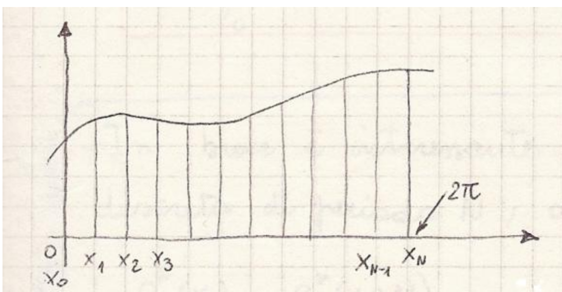
$$\Im(Y_N(t)) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k)e^{t\cos\left(\frac{2\pi n}{N}\right)}\sin\left(\frac{2\pi n}{N} - t\sin\left(\frac{2\pi n}{N}\right)\right) \equiv 0 \quad 1.62$$

from which we deduce the following relations:

$$Y_N(t) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k)e^{t\cos\left(\frac{2\pi n}{N}\right)}\cos\left(\frac{2\pi n}{N} - t\sin\left(\frac{2\pi n}{N}\right)\right) \quad 1.63$$

$$\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k)e^{t\cos\left(\frac{2\pi n}{N}\right)}\sin\left(\frac{2\pi nk}{N} - t\sin\left(\frac{2\pi n}{N}\right)\right) \equiv 0 \quad 1.64$$

In the particular case where the period N of the periodic discrete function is very large or even tending to infinity, we proceed as follows:



$X_n = \frac{2\pi n}{N}$ where the step is $h = \frac{2\pi}{N}$ namely $h = \Delta x_n$thence rewrite the (1.63) as follows:

$$Y_N(t) = \sum_{n=0}^{N-1} y_N(k) \frac{1}{2\pi}\left(\frac{2\pi}{N}\sum_{k=0}^{N-1} e^{tcos\left(\frac{2\pi n}{N}\right)} cos\left(\frac{2\pi n}{N} - tsin\left(\frac{2\pi n}{N}\right)\right)\right) \tag{1.65}$$

and for $N \to +\infty$ we have $\lim_{N\to+\infty} y_N(k) = y(k)$ thence rewrite the eq. (1.65) as follows:

$$Y_N(t) = \lim_{N\to\infty}\left(\sum_{n=0}^{N-1} y_N(k) \frac{1}{2\pi}\left(\sum_{k=0}^{N-1} h\, e^{tcos(x_n)} cos\left(kx_n - tsin(x_n)\right)\right)\right) \tag{1.66}$$

i.e.:

$$Y(t) = \frac{1}{2\pi}\sum_{n=0}^{\infty} y(k) \int_0^{2\pi} e^{tcos(x)} cos\left(kx - tsin(x)\right) dx \tag{1.67}$$

Given that $V(t)$ is the transformed V of $y(k)$, we obtain the equality:

$$\frac{1}{2\pi}\sum_{n=0}^{+\infty} y(k) \int_0^{2\pi} e^{tcos(x)} cos\left(kx - tsin(x)\right) dx = \sum_{k=0}^{+\infty} y(k) \frac{t^k}{k!} \tag{1.68}$$

from which, we have that:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{tcos(x)} cos\left(kx - tsin(x)\right) dx = \frac{t^k}{k!} \ \forall t \in \mathbb{C}, \forall k \in \mathbb{N} \tag{1.69}$$

$$\sum_{k=0}^{+\infty} y_N(k) \frac{t^k}{k!} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} y_N(k) e^{tcos\left(\frac{2\pi n}{N}\right)} cos\left(\frac{2\pi n}{N} - tsin\left(\frac{2\pi n}{N}\right)\right) \ solo\ e\ solo\ se\ y_N(k+N) = y_N(k) \tag{1.70}$$

SERIES DEVELOPMENT OF THE TYPE Nk+M

Assumption

The intent is to find a form equivalent to the infinite sum below with $\mathrm{N} \in \mathbb{N}$ and $\mathrm{M} \in \mathbb{Z}$:

$$W_N^M(y,t) := \sum_{k=0}^{+\infty} y(k) \frac{t^{kN+M}}{(kN+M)!} \tag{2.1}$$

We will solve this problem by using the Volonterio's Transform.
In these proofs and examples we will consider the convergence problem inherent in the solutions and criteria adopted.

Proof

Consider the following examples with $N = 3$ and $\mathrm{M} = 4$ needed to understand the proof that will follow, where $C_N^M(k)$ is a discrete periodic function of value 1, of period N shifted by *M*.

Table 1

n	0	1	2	3	4	5	6	7	8	9	10
$3n+4$	4	7	10	13	16	19	22	25	28	31	34

Table 2

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
$C_3^4(k)$	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0

Thence for the (2.0) and to the Tables 1 and 2 where the relation between n and k is $n = (k-4)/3$ we obtain the following Table:

Table 3

k	0	1	2	3	4	5	6	7	8	9	10
$n = \frac{k-4}{3}$	-4/3	-1	-2/3	-1/3	0	1/3	2/3	1	4/3	5/3	2
$C_3^4(k)$	0	0	0	0	1	0	0	1	0	0	1
$y(n)$	$y\left(\frac{-4}{3}\right)$	$y(-1)$	$y\left(\frac{-2}{3}\right)$	$y\left(\frac{-1}{3}\right)$	$y(0)$	$y\left(\frac{1}{3}\right)$	$y\left(\frac{2}{3}\right)$	$y(1)$	$y\left(\frac{4}{3}\right)$	$y\left(\frac{5}{3}\right)$	$y(2)$
$y(n)C_3^4(k)$	0	0	0	0	$y(0)$	0	0	$y(1)$	0	0	$y(2)$

With the Table 3 is easy to understand the following equivalence:

$$W_N^M(y,t) = \sum_{k=0}^{+\infty} y(k)\frac{t^{kN+M}}{(kN+M)!} \equiv \sum_{k=0}^{+\infty} y\left(\frac{k-M}{N}\right) C_N^M(k)\frac{t^k}{k!} \tag{2.2}$$

Now for the definition (1) of the Volonterio's Transform, we can write the following relation (see tables and definitions attached):

$$W_N^M(y,t) = T\left(y\left(\frac{k-M}{N}\right)C_N^M(k), t\right) \tag{2.3}$$

Of course the relation (2.3) is the one that will lead us to obtain the generalized solution of the expression (2.1).

We consider the following known relationship [A.V. Oppenheim R.W. Schafer – Elaborazione numerica dei segnali (Digital Signal Processing) – Franco Angeli Editions]:

$$\phi_N(\nu) := \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{i2\pi m\nu}{N}} = \begin{cases} 1 & \nu = nN \\ 0 & altrove \end{cases} \qquad n \in \mathbb{Z} \tag{2.4}$$

and more in general with $\phi_N^M(\nu)$ we define the following expression:

$$\phi_N^M(k) := \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{i2\pi m(k-M)}{N}} = \begin{cases} 1, & k = nN + M \\ 0, & altrove \end{cases} \qquad n \in \mathbb{Z} \tag{2.5}$$

of the (2.4) we must narrow the field to only non-negative integers, namely the (2.4) there must return the zero for each integer value negative, so we have to rewrite (2.4) as follows:

$$\frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{i2\pi m\nu}{N}} u^{\Delta}(n) = \begin{cases} 1, & \nu = nN \\ 0, & \nu = -nN \\ 0, & altrove \end{cases} \qquad n \in \mathbb{N} \tag{2.6}$$

where $u^{\Delta}(k)$ is a step discrete function of Heaviside defined as:

$$u^{\Delta}(k) := \begin{cases} 1, & k \ \ 0 \\ 0, & k < 0 \end{cases} \qquad k \in \mathbb{Z} \tag{2.7}$$

furthermore, we observe that:

$$u^{\Delta}(k) = u^{\Delta}\left(\frac{k}{a}\right) := \begin{cases} 1, & k \ \ 0a \\ 0, & k < 0a \end{cases} \qquad k \in \mathbb{Z} \tag{2.8}$$

Now placing $\nu = k - M$ we deduce the discrete function $C_N^{M(k)}$:

$$C_N^M(k) := \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{i2\pi m(k-M)}{N}} u^{\Delta}\left(\frac{k-M}{N}\right) = \begin{cases} 1, & k = nN + M \\ 0, & altrove \end{cases} \qquad n \in \mathbb{N} \tag{2.9}$$

and then through the observation (2.8):

$$C_N^M(k) := \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{i2\pi m(k-M)}{N}} u^{\Delta}(k-M) = \begin{cases} 1, & k = nN + M \\ 0, & altrove \end{cases} \qquad n \in \mathbb{N} \tag{2.10}$$

Or the equivalent expression:

$$C_N^M(k) := \phi_N^M(k) u^{\Delta}(k-M) = \begin{cases} 1, & k = nN + M \\ 0, & altrove \end{cases} \qquad n \in \mathbb{N} \tag{2.11}$$

So to utilize the transformed V is useful to rewrite (2.10) in the following way:

$$C_N^M(k) := \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} \cdot e^{\frac{i2\pi mk}{N}} u^{\Delta}(k-M) = \begin{cases} 1, & k = nN + M \\ 0, & altrove \end{cases} \qquad n \in \mathbb{N} \tag{2.12}$$

then we can rewrite the relation (2.3) as follows:

$$W_N^M(t) := T\left(y\left(\tfrac{k-M}{N}\right) C_N^M(k), t\right) = T\left(y\left(\tfrac{k-M}{N}\right) \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} \cdot e^{\frac{i2\pi mk}{N}} u^{\Delta}(k-M), t\right)$$

and thence:

$$W_N^M(t) := T\left(y\left(\tfrac{k-M}{N}\right) C_N^M(k), t\right) = \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} \cdot T\left(y\left(\tfrac{k-M}{N}\right) e^{\frac{i2\pi mk}{N}} u^{\Delta}(k-M), t\right) \tag{2.13}$$

From the PF N. 9 which is given here for convenience:

$$T\left(y(k)e^{\alpha k}, t\right) = V(te^{\alpha}) \tag{2.14}$$

we have:

$$W_N^M(y,t) := T\left(y\left(\tfrac{k-M}{N}\right) C_N^M(k), t\right) = \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} \cdot T\left(y\left(\tfrac{k-M}{N}\right) u^{\Delta}(k-M), te^{\frac{i2\pi m}{N}}\right) \tag{2.15}$$

While for the PF N.5 which is given here with $y^*(k) := y\left(\frac{k-M}{N}\right)$:

$$T\left(y^*(k)u^{\Delta}(k-M), t\right) = W(t) - \sum_{k=0}^{M-1} y^*(k)\frac{t^k}{k!} \tag{2.16}$$

and thence:

$$T\left(y\left(\tfrac{k-M}{N}\right) u^{\Delta}(k-M), \xi\right) = T\left(y\left(\tfrac{k-M}{N}\right), \xi\right) - \sum_{k=0}^{M-1} y\left(\tfrac{k-M}{N}\right)\frac{\xi^k}{k!} \quad con \quad \xi = te^{\frac{i2\pi m}{N}} \tag{2.17}$$

We obtain finally the following equivalence:

$$W_N^M(y,t) := \sum_{k=0}^{+\infty} y(k)\frac{t^{kN+M}}{(kN+M)!} := \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} \cdot T\left(y\left(\tfrac{k-M}{N}\right), te^{\frac{i2\pi m}{N}}\right) - \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{M-1} y\left(\tfrac{k-M}{N}\right)\frac{t^k e^{\frac{i2\pi m(k-M)}{N}}}{k!} \tag{2.18}$$

To facilitate the understanding of the examples that follow will call with $A_N^M(k)$:

$$A_N^M(t) := \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi m M}{N}} T\left(y\left(\frac{k-M}{N}\right), te^{\frac{i2\pi m}{N}}\right)$$

and with $B_N^M(y,k)$:

$$B_N^M(y,t) := \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{M-1} y\left(\frac{k-M}{N}\right)\frac{t^k e^{\frac{i2\pi m(k-M)}{N}}}{k!}$$

manipulating the $B_N^M(y,t)$ in the following way:

$$\sum_{k=0}^{M-1}\left(\frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{i2\pi m(k-M)}{N}}\right) y\left(\frac{k-M}{N}\right)\frac{t^k}{k!} \tag{2.19}$$

we observe that the inside bracket of the (2.19) is precisely the function $\phi_N^M(k)$

$$\sum_{k=0}^{M-1} \phi_N^M(k) y\left(\frac{k-M}{N}\right)\frac{t^k}{k!} \qquad k = nN + M \tag{2.20}$$

CASE 1 (N < M):

Example 1.1: $(\mathrm{N}=3)<(\mathrm{M}=4)$ where k goes from 0 to 3

n	-2	-1	0	1
$k = n3 + 4$	-2	1	4	7
$\phi_3^4(k)$	1	1	1	1
$\phi_3^4(k)y\left(\frac{k-4}{3}\right)\frac{t^k}{k!}$	/	$y(-1)t$	/	/

thence:

$$\sum_{k=0}^{4-1}\phi_3^4(k)y\left(\frac{k-4}{3}\right)\frac{t^k}{k!} := y(-1)t \tag{2.21}$$

Example 1.2: $(\mathrm{N}=4)<(\mathrm{M}=11)$ with k from 0 to 10:

n	-3	-2	-1	0	1
$k = n4 + 11$	-1	3	7	11	15
$\phi_4^{11}(k)$	1	1	1	1	1
$\phi_4^{11}(k)y\left(\frac{k-11}{4}\right)\frac{t^k}{k!}$	/	$y(-2)\frac{t^3}{3!}$	$y(-1)\frac{t^7}{7!}$	/	/

thence:

$$\sum_{k=0}^{11-1}\phi_4^{11}(k)y\left(\frac{k-11}{4}\right)\frac{t^k}{k!} := y(-2)\frac{t^3}{3!} + y(-1)\frac{t^7}{7!} \tag{2.22}$$

Example 1.3: $(\mathrm{N} = 2) = (\mathrm{M} = 10)$ where k goes from 0 to 9

n	-6	-5	-4	-3	-2	-1	0	1	2	3
$k = n2 + 10$	-2	0	2	4	6	8	10	12	14	16
$\phi_2^{10}(k)$	1	1	1	1	1	1	1	1	1	1
$y\left(\frac{k-10}{2}\right)\phi_2^{10}(k)\frac{t^k}{k!}$	/	$y(-5)$	$y(-4)\frac{t^2}{2!}$	$y(-3)\frac{t^4}{4!}$	$y(-2)\frac{t^6}{6!}$	$y(-1)\frac{t^8}{8!}$	/	/	/	/

thence:

$$\sum_{k=0}^{2-1} \phi_2^{10}(k) \mathrm{y}\left(\frac{k-10}{2}\right)\frac{t^k}{k!} := y(-5) + y(-4)\frac{t^2}{2!} + y(-3)\frac{t^4}{4!} + y(-2)\frac{t^6}{6!} + y(-1)\frac{t^8}{8!} \quad (2.23)$$

CASE 2 (N = M)

Example 2.1: $(N = 5) = (M = 5)$ where k goes from 0 to 4

n	-2	-1	0	1	2	3
$k = n5 + 5$	-5	0	5	10	15	20
$\phi_5^5(k)$	1	1	1	1	/	/
$\phi_5^5(k)\,y\left(\frac{k-5}{5}\right)\frac{t^k}{k!}$	/	$y(-1)t$	/	/	/	/

thence:

$$\sum_{k=0}^{5-1} \phi_5^5(k) y\left(\frac{k-5}{5}\right)\frac{t^k}{k!} := y(-1)t \tag{2.24}$$

Example 2.2: $(\mathrm{N} = 7) = (\mathrm{M} = 7)$ where $\boldsymbol{k}$ goes from 0 to 6

n	-2	-1	0	1	2	3
$k = n7 + 7$	-7	0	7	14	21	28
$\phi_7^7(k)$	1	1	1	1	/	/
$y\left(\frac{k-7}{7}\right)\phi_7^7(k)\frac{t^k}{k}!$	/	$y(-1)$	/	/	/	/

thence:

$$\sum_{k=0}^{7-1} \phi_7^7(k) y\left(\frac{k-7}{7}\right)\frac{t^k}{k!} := y(-1) \tag{2.25}$$

CASE 3 (N > M):

Example 3.1: $(N = 7) > (M = 5)$ where k goes from 0 to 4

n	-2	-1	0	1	2	3
$k = n7 + 5$	-9	-2	5	12	19	26
$\phi_7^5(k)$	1	1	1	1	1	1
$\phi_7^5(k) y\left(\frac{k-5}{7}\right)\frac{t^k}{k!}$	/	/	/	/	/	/

thence:

$$\sum_{k=0}^{5-1} \phi_7^5(k) y\left(\frac{k-5}{7}\right)\frac{t^k}{k!} = 0 \tag{2.26}$$

***

In general from these observations by induction we conclude for deduction the following equivalence:

$$\sum_{k=0}^{M-1} y\left(\frac{k-M}{N}\right)\phi_N^M(k)\frac{t^k}{k!} = \sum_{k=-Ip(M/N)}^{-1} y(k)\frac{t^{kN+M}}{(kN+M)!} \tag{2.27}$$

Where with $Ip(x)$ represent the function that returns the integer part of x, and then we can rewrite the (2.18) in a simpler form:

$$V_N^M(y,t) := \sum_{k=0}^{+\infty} y(k)\frac{t^{kN+M}}{(kN+M)!} = \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} \cdot T\left(y\left(\frac{k-M}{N}\right), te^{\frac{i2\pi m}{N}}\right) - \sum_{k=Ip(-M/N)}^{-1} y(k)\frac{t^{kN+M}}{(kN+M)!} \tag{2.28}$$

where if $M < N$ and in particular for $(M \le 0)$ we have $B_N^M(k) \equiv 0$, thence:

$$B_N^M(y,t) := \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{M-1} y\left(\frac{k-M}{N}\right)\frac{t^k e^{\frac{i2\pi m(k-M)}{N}}}{k!} = \begin{cases} \sum_{k=-Ip\left(\frac{M}{N}\right)}^{-1} y(k)\frac{t^{kN+M}}{(kN+M)!}, & N < M \\ y(-1), & N = M \\ 0, & N > M \end{cases} \tag{2.29}$$

PARTICULAR CASES

Case with $y(k) = 1$

Case with $y(k) := 1, N \in \mathbb{N},\ M \in \mathbb{Z}$:

$$W_N^M(1,t) := \Theta_N^M(t) := \sum_{k=0}^{+\infty} \frac{t^{kN+M}}{(kN+M)!} = \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} \cdot T\left(1, te^{\frac{i2\pi m}{N}}\right) - \sum_{k=-Ip(M/N)}^{-1} \frac{t^{kN+M}}{(kN+M)!} \qquad (2.30)$$

It can be shown easily from (2.2) and (2.3) that the function $\Theta_N^M(t)$ satisfies the following differential equation:

$$\frac{d^M}{dt^M}\left(\frac{d^N}{dt^N}\Theta_N^M(t) - \Theta_N^M(t)\right) = 0 \qquad (2.31)$$

thence:

$$\Theta_N^M(t) := \sum_{k=0}^{+\infty} \frac{t^{kN+M}}{(kN+M)!} = \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} e^{te^{\frac{i2\pi m}{N}}} - \sum_{k=-Ip(M/N)}^{-1} \frac{t^{kN+M}}{(kN+M)!} \qquad (2.32)$$

we have for the TF N.2 $T(1, \xi) = e^{\xi}$, furthermore we have:

$$e^{\frac{i2\pi m}{N}} = \cos\left(\frac{2\pi m}{N}\right) + i\,\sin\left(\frac{2\pi m}{N}\right) \qquad (2.33)$$

thence:

$$\Theta_N^M(t) := \sum_{k=0}^{+\infty} \frac{t^{kN+M}}{(kN+M)!} = \frac{1}{N}\sum_{m=0}^{N-1} e^{\left(t\cos\left(\frac{2\pi m}{N}\right) + i\left(t\sin\left(\frac{2\pi m}{N}\right) - \frac{2\pi mM}{N}\right)\right)} - \sum_{k=-Ip(M/N)}^{-1} \frac{t^{kN+M}}{(kN+M)!} \qquad (2.34)$$

after several steps we obtain:

$$\Theta_N^M(t) := \sum_{k=0}^{+\infty} \frac{t^{kN+M}}{(kN+M)!} = \frac{1}{N}\sum_{m=0}^{N-1} e^{t\cos\left(\frac{2\pi m}{N}\right) + i\left(t\sin\left(\frac{2\pi m}{N}\right) - \frac{2\pi mM}{N}\right)} - \sum_{k=-Ip(M/N)}^{-1} \frac{t^{kN+M}}{(kN+M)!} \qquad (2.35)$$

where here we separate the real part from the imaginary part:

$$\Theta_N^M(t) = \frac{1}{N}\sum_{m=0}^{N-1} e^{t\cos\left(\frac{2\pi m}{N}\right)} \cos\left(t\sin\left(\frac{2\pi m}{N}\right) - \frac{2\pi mM}{N}\right) + \frac{i}{N}\sum_{m=0}^{N-1} e^{t\cos\left(\frac{2m\pi}{N}\right)} \sin\left(t\sin\left(\frac{2\pi m}{N}\right) - \frac{2\pi mM}{N}\right) - \sum_{k=-Ip(M/N)}^{-1} \frac{t^{kN+M}}{(kN+M)!} \qquad (2.36)$$

it follows:

$$\sum_{m=0}^{N-1} e^{t\cos\left(\frac{2m\pi}{N}\right)} \sin\left(t\sin\left(\frac{2\pi m}{N}\right) - \frac{2\pi mM}{N}\right) \equiv 0 \qquad \begin{matrix} \forall t \in \mathbb{C} \\ \forall N \in \mathbb{N} \\ \forall M \in \mathbb{Z} \end{matrix} \tag{2.37}$$

and thence:

$$\Theta_N^M(t) = \frac{1}{N}\sum_{m=0}^{N-1} e^{t\cos\left(\frac{2\pi m}{N}\right)} \cos\left(t\sin\left(\frac{2\pi m}{N}\right) - \frac{2\pi mM}{N}\right) - \sum_{k=-Ip(M/N)}^{-1} \frac{t^{kN+M}}{(kN+M)!} \qquad \begin{matrix} t \in \mathbb{C} \\ N \in \mathbb{N} \\ M \in \mathbb{Z} \end{matrix} \tag{2.38}$$

In particular for $M = 0$ we have:

$$\Theta_N^0(t) = \frac{1}{N}\sum_{m=0}^{N-1} e^{t\cos\left(\frac{2\pi m}{N}\right)} \cos\left(t\sin\left(\frac{2\pi m}{N}\right)\right) \qquad t \in \mathbb{C}, N \in \mathbb{N}, M \in \mathbb{Z} \tag{2.39}$$

From 2.33 it is shown that:

$$\frac{d^\tau}{dt^\tau}\left(e^{t\cos\left(\frac{2\pi m}{N}\right)}\left(\cos\left(t\sin\left(\frac{2\pi m}{N}\right) + \frac{2\pi mM}{N}\right) + i\sin\left(t\sin\left(\frac{2\pi m}{N}\right) + \frac{2\pi mM}{N}\right)\right)\right)$$
$$= e^{t\cos\left(\frac{2\pi m}{N}\right)}\left(\cos\left(t\sin\left(\frac{2\pi m}{N}\right) + \frac{2\pi m(M-\tau)}{N}\right) + i\sin\left(t\sin\left(\frac{2\pi m}{N}\right) + \frac{2\pi m(M-\tau)}{N}\right)\right)$$

and

$$\frac{d^M}{dt^M}\left(e^{t\cos\left(\frac{2\pi m}{N}\right)}\left(\cos\left(t\sin\left(\frac{2\pi m}{N}\right) + \frac{2\pi mM}{N}\right) + i\sin\left(t\sin\left(\frac{2\pi m}{N}\right) + \frac{2\pi mM}{N}\right)\right)\right)$$
$$= e^{t\cos\left(\frac{2\pi m}{N}\right)}\left(\cos\left(t\sin\left(\frac{2\pi m}{N}\right)\right) + i\sin\left(t\sin\left(\frac{2\pi m}{N}\right)\right)\right)$$

$$\frac{d^\tau}{dt^\tau}\left(e^{t\cos\left(\frac{2\pi m}{N}\right)}\left(\cos\left(t\sin\left(\frac{2\pi m}{N}\right)\right) + i\sin\left(t\sin\left(\frac{2\pi m}{N}\right)\right)\right)\right) = \frac{d^N}{dt^N}\left(e^{t\cos\left(\frac{2\pi m}{N}\right)} e^{it\sin\left(\frac{2\pi m}{N}\right)}\right) = e^{\frac{2\tau im\pi}{N} + te^{\frac{2im\pi}{N}}}$$

Putting $\tau = N$ we obtain

$$\frac{d^N}{dt^N}\left(e^{t\cos\left(\frac{2\pi m}{N}\right)}\left(\cos\left(t\sin\left(\frac{2\pi m}{N}\right)\right) + i\sin\left(t\sin\left(\frac{2\pi m}{N}\right)\right)\right)\right) = e^{\frac{2Nim\pi}{N} + te^{\frac{2im\pi}{N}}}$$

but $e^{2im\pi} \equiv 1$ and thence

$$\left(\frac{d^N}{dt^N}\left(e^{t\cos\left(\frac{2\pi m}{N}\right)}\left(\cos\left(t\sin\left(\frac{2\pi m}{N}\right)\right) + i\sin\left(t\sin\left(\frac{2\pi m}{N}\right)\right)\right)\right)\right) = e^{te^{\frac{2im\pi}{N}}}$$

Case with $y(\xi) = y(\xi N + M)$

Rewriting the eq. (2.18) as follows:

$$W_N^M(y,t) := \sum_{k=0}^{+\infty} y(\xi)|_{\xi=k} \frac{t^{kN+M}}{(kN+M)!} = \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} \cdot T\left(y(\xi)|_{\xi=\frac{k-M}{N}}, te^{\frac{i2\pi m}{N}}\right) - \sum_{k=Ip(-M/N)}^{-1} y(\xi)|_{\xi=k} \frac{t^{kN+M}}{(kN+M)!} \qquad (2.40)$$

in this eq. we replacing the following expression $y(\xi) := y(\xi N + M)$ and thence

$$W_N^M(y,t) := \sum_{k=0}^{+\infty} y(kN+M) \frac{t^{kN+M}}{(kN+M)!} = \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} \cdot T\left(y(k), te^{\frac{i2\pi m}{N}}\right) - \sum_{k=Ip(-M/N)}^{-1} y(kN+M) \frac{t^{kN+M}}{(kN+M)!} \qquad (2.41)$$

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EXAMPLES

Example 1

Calculate $W_2^1((-1)^k, t)$.

Solution

We consider $y(k) := (-1)^k$ with N=2 and M=1 and we replace in the (2.41) obtaining:

$$W_2^1((-1)^k, t) := \sum_{k=0}^{+\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} = \frac{1}{2}\sum_{m=0}^{2-1} e^{\frac{-i2\pi m}{2}} \cdot T\left(y\left(\frac{k-1}{2}\right), te^{\frac{i2\pi m}{2}}\right) - \sum_{k=Ip(-1/2)}^{-1} y(k) \frac{t^{2k+1}}{(2k+1)!} \qquad (2.42)$$

the third summation of the (2.42) is null because M<M, thence:

$$W_2^1((-1)^k, t) := \sum_{k=0}^{+\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} = \frac{1}{2}\sum_{m=0}^{1} e^{-i\pi m} \cdot T\left(y\left(\frac{k-1}{2}\right), te^{i\pi m}\right)$$

but

$$W_2^1((-1)^k, t) := \sum_{k=0}^{+\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} = \frac{1}{2}\sum_{m=0}^{1} e^{-i\pi m} \cdot T\left(y\left(\frac{k-1}{2}\right), te^{i\pi m}\right)$$

then we proceed in the steps:

$$W_2^1((-1)^k, t) := \sum_{k=0}^{+\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} = \frac{1}{2}\left(e^{-i\pi 0} \cdot T\left(y\left(\frac{k-1}{2}\right), te^{i\pi 0}\right) + e^{-i\pi} \cdot T\left(y\left(\frac{k-1}{2}\right), te^{i\pi}\right)\right)$$

$$W_2^1((-1)^k,t) := \sum_{k=0}^{+\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!} = \frac{1}{2}\left(T\left(y\left(\tfrac{k-1}{2}\right),t\right) - T\left(y\left(\tfrac{k-1}{2}\right),-t\right)\right) \quad (2.43)$$

$$W_2^1((-1)^k,t) := \sum_{k=0}^{+\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!} = \frac{1}{2}\left(T\left((-1)^{\frac{k-1}{2}},t\right) - T\left((-1)^{\frac{k-1}{2}},-t\right)\right) \quad (2.44)$$

$$W_2^1((-1)^k,t) := \sum_{k=0}^{+\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!} = \frac{1}{2}\left(T(-i(i)^k,t) - T(-i(i)^k,-t)\right)$$

$$W_2^1((-1)^k,t) := \sum_{k=0}^{+\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!} = \frac{1}{2}\left(T(-i\cdot i^k,t) - T(-i\cdot i^k,-t)\right)$$

$$W_2^1((-1)^k,t) := \sum_{k=0}^{+\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!} = \frac{1}{2}\left(T(-i\cdot i^k,t) - T(-i\cdot i^k,-t)\right) \quad (2.45)$$

$$W_2^1((-1)^k,t) := \sum_{k=0}^{+\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!} = \frac{1}{2i}\left(T((i)^k,t) - T((i)^k,-t)\right) \quad (2.46)$$

for the TF.5 we have:

$$W_2^1((-1)^k,t) := \sum_{k=0}^{+\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!} = \frac{1}{2i}\left(e^{it} - e^{-it}\right) \quad (2.47)$$

$$W_2^1((-1)^k,t) := \sum_{k=0}^{+\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!} = \sin(t) \quad (2.48)$$

Example 2

Calculate $W_N^0(k^n,t)$. We shall see how in this example the Volonterio's polynomials lend themselves to providing a generalized solution of this problem.

Solution

In this case is M=0 thus the last term of the (2.30) is null:

$$W_N^0(k^n,t) := \sum_{k=0}^{+\infty}k^n\frac{t^{Nk}}{(Nk)!} = \frac{1}{N}\sum_{m=0}^{N-1}e^{\frac{-i2\pi m0}{N}}\cdot T\left(\left(\frac{k-0}{N}\right)^n, te^{\frac{i2\pi m}{N}}\right) \quad (2.49)$$

For the TF. N.4 we obtain:

$$W_N^0(k^n,t) = \frac{1}{N}\sum_{m=0}^{N-1} T\left(\frac{k^n}{N^n}, te^{\frac{i2\pi m}{N}}\right) = \frac{1}{N^{n+1}}\sum_{m=0}^{N-1} V_n\left(te^{\frac{i2\pi m}{N}}\right)e^{te^{\frac{i2\pi m}{N}}} \tag{2.50}$$

where $V_n(t)$ is the Volonterio's polynomial of order n.

Putting $N = 2$ and $n = 3$ we have $V_3(t) = t + 3t^2 + t^3$:

$$W_2^0(k^3,t) = \frac{1}{2^4}\sum_{m=0}^{2-1} V_3(te^{i\pi m})e^{te^{i\pi m}} \tag{2.51}$$

$$W_2^0(k^3,t) = \frac{1}{16}\left(e^{-t}V_3(-t) + e^t V_3(t)\right) \tag{2.52}$$

$$W_2^0(k^3,t) = \frac{1}{8}t\,(3t\cosh(t) + (1+t^2)\sinh(t)) \tag{2.53}$$

Here we can connect with the Ramanujan's equation concerning the number 8, that is a Fibonacci's number and is linked to the physical vibrations of the superstrings, i.e.

$$W_2^0(k^3,t) = \frac{1}{8}t\,(3t\cosh(t) + (1+t^2)\sinh(t)) \Rightarrow$$

$$\Rightarrow W_0^2(k^3,t) = 1/\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}t(3t\cosh(t)+(1+t^2)\sinh(t)). \tag{2.53b}$$

Example 3

Calculate $W_2^3\left(\frac{1}{2k+4},t\right)$.

Solution

$$W_2^3\left(\frac{1}{2k+4},t\right) := \sum_{k=0}^{+\infty}\frac{1}{2k+4}\cdot\frac{t^{2k+3}}{(2k+3)!} \tag{2.54}$$

$$W_2^3\left(\frac{1}{2k+4},t\right) := \frac{1}{2}\sum_{m=0}^{2-1} e^{\frac{-i2\pi m3}{2}}\cdot T\left(\frac{1}{2\xi+4}\bigg|_{\xi=\frac{k-3}{2}}, te^{\frac{i2\pi m}{2}}\right) - \sum_{k=Ip(-3/2)}^{-1}\frac{1}{2k+4}\cdot\frac{t^{2k+3}}{(2k+3)!} \tag{2.55}$$

$$W_2^3\left(\frac{1}{2k+4},t\right) := \frac{1}{2}\sum_{m=0}^{2-1} e^{\frac{-i2\pi m 3}{2}} \cdot T\left(\frac{1}{k+1}, te^{\frac{i2\pi m}{2}}\right) - \sum_{k=Ip(-3/2)}^{-1} \frac{1}{2k+4}\cdot\frac{t^{2k+3}}{(2k+3)!} \tag{2.56}$$

$$W_2^3\left(\frac{1}{2k+4},t\right) := -\frac{t}{2}+\frac{1}{2}\sum_{m=0}^{1} e^{-i3\pi m} \cdot T\left(\frac{1}{k+1}, te^{i\pi m}\right) \tag{2.57}$$

Now to calculate the transform, we use the TF No. 17:

$$T\left(\frac{1}{k+1},\xi\right) := \frac{e^t-1}{t} \tag{2.58}$$

Thence:

$$W_2^3\left(\frac{1}{2k+4},t\right) := -\frac{t}{2}+\frac{1}{2}\sum_{m=0}^{1} e^{-i3\pi m} \cdot \frac{e^{te^{i\pi m}}-1}{te^{i\pi m}} \tag{2.59}$$

After some calculations we have:

$$W_2^1\left(\frac{1}{2k+4},t\right) := -\frac{t}{2}-\frac{1}{t}+\frac{\cosh(t)}{t} \tag{2.60}$$

Example 4

Calculate $W_2^3\left(\frac{\Gamma(Nk+M+1)}{\Gamma(Nk+M+1)},t\right)$

Solution

We'll see how this example is related to the Laguerre's polynomials.

$$W_N^M\left(\frac{\Gamma(Nk+2M+1)}{\Gamma(Nk+M+1)},t\right) := \sum_{k=0}^{+\infty} \frac{\Gamma(Nk+2M+1)}{\Gamma(Nk+M+1)}\cdot\frac{t^{Nk+M}}{(Nk+M)!} \tag{2.61}$$

$$W_N^M\left(\frac{\Gamma(Nk+2M+1)}{\Gamma(Nk+M+1)},t\right) := \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} \cdot T\left(\left.\frac{\Gamma(N\xi+2M+1)}{\Gamma(N\xi+M+1)}\right|_{\xi=\frac{k-M}{N}}, te^{\frac{i2\pi m}{N}}\right) - \sum_{k=Ip(-M/N)}^{-1} \frac{\Gamma(Nk+2M+1)}{\Gamma(Nk+M+1)}\frac{t^{kN+M}}{(kN+M)!} \tag{2.62}$$

$$W_N^M\left(\frac{\Gamma(Nk+2M+1)}{\Gamma(Nk+M+1)},t\right) := \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} \cdot T\left(\frac{\Gamma(N+M+1)}{\Gamma(k+1)}, te^{\frac{i2\pi m}{N}}\right) - \sum_{k=Ip(-M/N)}^{-1} \frac{\Gamma(Nk+2M+1)}{\Gamma(Nk+M+1)}\cdot\frac{t^{kN+M}}{(kN+M)!} \tag{2.63}$$

from the TF N. 15 we have:

$$T\left(\frac{\Gamma(N+M+1)}{\Gamma(k+1)},\xi\right) := e^{\xi}\,\boldsymbol{L}_M(-\xi) \quad with \quad \boldsymbol{L}_M(t) = e^t\frac{d^M}{dt^M}\left(e^{-t}t^M\right) \tag{2.64}$$

then replacing:

$$W_N^M\left(\frac{\Gamma(Nk+2M+1)}{\Gamma(Nk+M+1)},t\right) := \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{-i2\pi mM}{N}} \cdot e^{te^{\frac{i2\pi m}{N}}} L_M\left(-te^{\frac{i2\pi m}{N}}\right) - \sum_{k=Ip(-M/N)}^{-1} \frac{\Gamma(Nk+2M+1)}{\Gamma(Nk+M+1)} \cdot \frac{t^{kN+M}}{(kN+M)!} \tag{2.65}$$

Putting N=2 and M=3 we have $L_3(t) = 6 - 18t + 9t^2 - t^3$:

$$W_2^3\left(\frac{\Gamma(2k+7)}{\Gamma(2k+4)},t\right) := \frac{1}{2}\sum_{m=0}^{2-1} e^{-i\pi m3} \cdot e^{te^{i\pi m}} L_M(-te^{i\pi m}) - \sum_{k=Ip(-M/N)}^{-1} \frac{\Gamma(2k+7)}{\Gamma(2k+4)} \cdot \frac{t^{2k+3}}{(2k+3)!} \tag{2.66}$$

$$W_2^3\left(\frac{\Gamma(2k+7)}{\Gamma(2k+4)},t\right) := \frac{1}{2}\sum_{m=0}^{2-1} e^{-i\pi m3} \cdot e^{te^{i\pi m}} L_3(-te^{i\pi m}) - \sum_{k=Ip(-M/N)}^{-1} \frac{\Gamma(2k+7)}{\Gamma(2k+4)} \cdot \frac{t^{2k+3}}{(2k+3)!} \tag{2.67}$$

$$W_2^3\left(\frac{\Gamma(2k+7)}{\Gamma(2k+4)},t\right) := -\frac{\Gamma(5)}{\Gamma(2)} \cdot t + \frac{1}{2}\sum_{m=0}^{2-1} e^{-i\pi m3} \cdot e^{te^{i\pi m}} L_3(-te^{i\pi m}) \tag{2.68}$$

$$W_2^3\left(\frac{\Gamma(2k+7)}{\Gamma(2k+4)},t\right) := -24 \cdot t + \frac{1}{2}\sum_{m=0}^{2-1} e^{-i\pi m3} \cdot e^{te^{i\pi m}} L_3(-te^{i\pi m}) \tag{2.69}$$

$$W_2^3\left(\frac{\Gamma(2k+7)}{\Gamma(2k+4)},t\right) := -24t + (18t + t^3)\cosh(t) + (6 + 9t^2)\sinh(t) \tag{2.70}$$

This last equation can be connected with the Euler Gamma Function and with the number linked to the physical vibrations of the bosonic strings, i.e. 24.

Indeed, with regard the number 24, this is related to the "modes" that correspond to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Thence, we have the following expression:

$$W_{\frac{3}{2}}\left(\frac{\Gamma(2k+7)}{\Gamma(2k+4)},t\right) := -24t + (18t+t^3)\cosh(t) + (6+9t^2)\sinh(t) \Rightarrow$$

$$\Rightarrow W_{2}^{3}\left(\frac{\Gamma(2k+7)}{\Gamma(2k+4)},t\right) := -\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} t + \left(18t+t^3\right)\cosh(t) + \left(6+9t^2\right)\sinh(t).$$

(2.70b)

ON SOME MATHEMATICAL CONNECTION WITH SOME SECTORS OF STRING THEORY

In 1968 Veneziano proposed the following heuristic answer

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-(\alpha(s)+\alpha(t)))} \quad (3.1)$$

with $\alpha(s) = \alpha(0) + \alpha' s$.

Euler Gamma function has poles in the negative real axis at integer values $\alpha(s) = n$ with residue

$$\Gamma(-\alpha(s)) = \frac{\Gamma(-\alpha(s)+1)}{-\alpha(s)} = \frac{\Gamma(-\alpha(s)+n+1)}{-\alpha(s)(-\alpha(s)+1)(-\alpha(s)+2)\ldots(-\alpha(s)+(n-1))(-\alpha(s)+n)}$$

$$\overset{\alpha(s)\to n}{\rightarrow} \frac{(-1)^n}{n!}\frac{1}{-\alpha(s)+n} \quad (3.2)$$

Hence, at fixed t, the amplitude has infinitely many poles at $s \in (0,\infty)$ for $\alpha(s) = \alpha(0) + s\alpha' = n$ or

$$s = \frac{n-\alpha(0)}{\alpha'} = M_n^2 \quad (3.3)$$

with residue

$$A^{(4)}(s,t) \overset{\alpha(s)\to n}{=} \frac{(-1)^n}{n!}\frac{\Gamma(-\alpha(t))}{\Gamma(-n-\alpha(t))}\frac{1}{\alpha(s)-n} = \frac{(\alpha(t)+1)(\alpha(t)+2)\ldots(\alpha(t)+n)}{n!}\frac{1}{\alpha(s)-n} \quad (3.4)$$

In the bosonic string the simplest vertex operator is the one for the tachyon state $N=0$ hence $M^2=-4/\alpha'$. We have:

$$\mathcal{V}(0;p) = g_s\int d^2z e^{ip\cdot X} = g_s\int d^2z V(z,\bar{z};p) \quad (3.5)$$

With regard the 4-point tachyon amplitude, we have the following equation:

$$\mathrm{A}^{(m)}(\Lambda_i,p_i) = \delta\left(\sum_i p_i\right)\frac{g_s^{m-2}}{Vol(SL(2,C))}\int\prod_{i=1}^{m} d^2z_i\prod_{j<l}|z_j-z_l|^{\alpha' p_j\cdot p_l} \quad (3.6)$$

Setting $m=4$ we end up with

$$\mathrm{A}^{(4)}(\Lambda_i,p_i) = \delta\left(\sum_i p_i\right)\frac{g_s^{2}}{Vol(SL(2,C))}\int\prod_{i=1}^{4} d^2z_i\prod_{j<l}|z_j-z_l|^{\alpha' p_j\cdot p_l} \quad (3.7)$$

After fixing the $SL(2,C)$ invariance by putting the insertion points at $0,1,z$ and $z_4\to\infty$ we end up with

$$\mathrm{A}^{(4)} \approx g_s^2\delta\left(\sum_i p_i\right)\int d^2z|z|^{\alpha' p_1\cdot p_3}|1-z|^{\alpha' p_2\cdot p_3} \quad (3.8)$$

using Gamma function identities this expression can be given a nice form. One must use the integral representation

$$\int d^2z|z|^{2a-2}|1-z|^{2b-2} = \frac{2\pi\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)} \quad (3.9)$$

where $a+b+c=1$. With this, (3.8) can be shown to be equal to

$$\mathrm{A}^{(4)} \approx g_s^2\delta\left(\sum_i p_i\right)\frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' t/4)\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\Gamma(2+\alpha' t/4)\Gamma(2+\alpha' u/4)} \quad (3.10)$$

in terms of the Mandelstam variables

$$s=-(p_1+p_2)^2\ ;\ t=-(p_2+p_3)^2\ ;\ u=-(p_1+p_4)^2 \quad (3.11)$$

which satisfy on shell (i.e. use the tachyon mass $-p_i^2=M^2=-4/\alpha'$)

$$s+t+u=-\sum_{i=1}^{4}p_i^2=\sum M_i^2=-\frac{16}{\alpha'} \quad (3.12)$$

We can write also the following mathematical connection:

$$A^{(4)}\approx g_s^2\delta\left(\sum_i p_i\right)\int d^2z|z|^{\alpha' p_1\cdot p_3}|1-z|^{\alpha' p_2\cdot p_3}\approx$$

$$\approx g_s^2\delta\left(\sum_i p_i\right)\frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' t/4)\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\Gamma(2+\alpha' t/4)\Gamma(2+\alpha' u/4)}\Rightarrow$$

$$\Rightarrow -\sum_{i=1}^{4}p_i^2=\sum M_i^2=-\frac{16}{\alpha'} \quad (3.13)$$

This expression can be related with the following Ramanujan's modular equation linked with the "modes" (i.e. 8 that is also a Fibonacci's number) that correspond to the physical vibrations of the superstrings:

$$8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \quad . (3.14)$$

Thence, we have the following relationship:

$$A^{(4)}\approx g_s^2\delta\left(\sum_i p_i\right)\int d^2z|z|^{\alpha' p_1\cdot p_3}|1-z|^{\alpha' p_2\cdot p_3}\approx$$

$$\approx g_s^2\delta\left(\sum_i p_i\right)\frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' t/4)\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\Gamma(2+\alpha' t/4)\Gamma(2+\alpha' u/4)}\Rightarrow$$

$$\Rightarrow -\sum_{i=1}^{4}p_i^2=\sum M_i^2=-\frac{16}{\alpha'}\Rightarrow$$

$$\Rightarrow -2\cdot\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\alpha'\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \quad (3.15)$$

We note that this relationship ca be related also with the eq. (d), i.e. the inverse transform of $V(t)$, thence we obtain this further mathematical connection:

$$T^{-1}(V(t),k) := \frac{2}{\pi}\Gamma(k+1)\left(\int_0^\pi \mathcal{R}\left(V\left(e^{i\xi}\right)\right)\cos(k\xi)d\xi\right)\Rightarrow$$

$$\Rightarrow g_s^2\delta\left(\sum_i p_i\right)\int d^2z|z|^{\alpha' p_1\cdot p_3}|1-z|^{\alpha' p_2\cdot p_3}\approx$$

$$\approx g_s^2\delta\left(\sum_i p_i\right)\frac{\Gamma(-1-\alpha's/4)\Gamma(-1-\alpha't/4)\Gamma(-1-\alpha'u/4)}{\Gamma(2+\alpha's/4)\Gamma(2+\alpha't/4)\Gamma(2+\alpha'u/4)}\Rightarrow$$

$$\Rightarrow -\sum_{i=1}^4 p_i^2 = \sum M_i^2 = -\frac{16}{\alpha'}\Rightarrow$$

$$\Rightarrow -2\cdot\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\alpha'\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \quad (3.15b)$$

b)the open string scattering

With regard the open string scattering, the amplitude is computed with operator insertions along the boundary of the disk which maps onto the real axis of the complex plane. The equation of the amplitude is:

$$A^{(4)} = \frac{g_s}{Vol(SL(2,R))}\int\prod_{i=1}^4 dx_i\left\langle e^{ip_1\hat{X}(x_1)}\dots e^{ip_1\hat{X}(x_4)}\right\rangle \approx \frac{g_s}{Vol(SL(2,R))}\delta^{26}\left(\delta\sum_i p_i\right)\int\prod_{i=1}^4 dx_i\prod_{j\le l}\left|x_i - x_j\right|^{2\alpha' p_i\cdot p_j} .$$

(3.16)

For a given ordering, the residual symmetry can be used to fix 3 points to $x_1 = 0, x_2 = 0, x_3 = x$ and $x_4 = \infty$. The resulting expression contains a single integration for $0 \le x \le 1$

$$\mathrm{A}^{(4)} \approx g_s \int_0^1 dx |x|^{2\alpha' p_1 \cdot p_2} |1-x|^{2\alpha' p_2 \cdot p_3} . \quad (3.17)$$

This integral is related to the Euler Beta function (thence with the Euler Gamma function)

$$B(a,b) = \int_0^1 dx x^{a-1} (1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} . \quad (3.18)$$

Whence, using now the tachyon mass $M^2 = -1/\alpha'$ one recovers the Veneziano amplitude

$$\mathrm{A}^{(4)} \approx g_s \left(\frac{\Gamma(-1-\alpha' s)\Gamma(-1-\alpha' t)}{\Gamma(-1-\alpha'(s+t))} \right) . \quad (3.19)$$

Thence, we have the following possible mathematical relationship between 3.16), (3.17) and (3.19):

$$\mathrm{A}^{(4)} = \frac{g_s}{Vol(SL(2,R))} \int \prod_{i=1}^{4} dx_i \left\langle e^{ip_1 \hat{X}(x_1)} ... e^{ip_1 \hat{X}(x_4)} \right\rangle \approx \frac{g_s}{Vol(SL(2,R))} \delta^{26} \left(\delta \sum_i p_i \right) \int \prod_{i=1}^{4} dx_i \prod_{j \le l} |x_i - x_j|^{2\alpha' p_i \cdot p_j}$$

$$\Rightarrow g_s \int_0^1 dx |x|^{2\alpha' p_1 \cdot p_2} |1-x|^{2\alpha' p_2 \cdot p_3} \Rightarrow g_s \left(\frac{\Gamma(-1-\alpha' s)\Gamma(-1-\alpha' t)}{\Gamma(-1-\alpha'(s+t))} \right) . \quad (3.20)$$

Also this relationship can be related with eq. (d), thence we obtain this further mathematical connection:

$$T^{-1}(V(t),k) := \frac{2}{\pi} \Gamma(k+1) \left(\int_0^{\pi} \mathcal{R}\left(V\left(e^{i\xi}\right)\right) \cos(k\xi) d\xi \right) \Rightarrow$$

$$\Rightarrow \frac{g_s}{Vol(SL(2,R))} \int \prod_{i=1}^{4} dx_i \left\langle e^{ip_1 \hat{X}(x_1)} ... e^{ip_1 \hat{X}(x_4)} \right\rangle \approx \frac{g_s}{Vol(SL(2,R))} \delta^{26} \left(\delta \sum_i p_i \right) \int \prod_{i=1}^{4} dx_i \prod_{j \le l} |x_i - x_j|^{2\alpha' p_i \cdot p_j}$$

$$\Rightarrow g_s \int_0^1 dx |x|^{2\alpha' p_1 \cdot p_2} |1-x|^{2\alpha' p_2 \cdot p_3} \Rightarrow g_s \left(\frac{\Gamma(-1-\alpha' s)\Gamma(-1-\alpha' t)}{\Gamma(-1-\alpha'(s+t))} \right) . \quad (3.20b)$$

c) Four point amplitude for the tachyons from CFT

The ground state tachyons in the twisted sector corresponds to:

$$M^2 = -\left(1 - \frac{k}{N}\right) \quad (3.21)$$

For the near marginal tachyons, in the large N limit, which are in the $(N-k)th$ sector, the vertex operator in the $(-1,-1)$ representation is,

$$V^{+}_{(-1,-1)}(z,\bar{z}) = e^{-\phi} e^{-\tilde{\phi}} e^{i\left(1-\frac{k}{N}\right)H(z)} e^{-i\left(1-\frac{k}{N}\right)\tilde{H}(\bar{z})} \sigma_{+} e^{ik.x}(z,\bar{z}) . \quad (3.22)$$

The four point amplitude for these lowest lying tachyons can now be computed by taking two vertices in the $(0,0)$ representation and two in the $(-1,-1)$ representation.

$$C\int_C d^2z \left\langle V^{-}_{(-1,-1)}(z_\infty,\bar{z}_\infty) e^{\phi} T_F e^{\tilde{\phi}} \tilde{T}_F V^{+}_{(-1,-1)}(1) V^{-}_{(-1,-1)}(z,\bar{z}) e^{\phi} T_F e^{\tilde{\phi}} \tilde{T}_F V^{+}_{(-1,-1)}(0) \right\rangle . \quad (3.23)$$

The constant $C = g_c^4 C_s^2$, where C_s^2 is related to g_c by

$$C_s^2 = \frac{4\pi}{g_c^2} . \quad (3.24)$$

This amplitude can now be computed and is given by,

$$I = C(k_1.k_3)^2 \int_C d^2z \frac{|z|^{-2-s} |1-z|^{-2-t}}{|F(z)|^2} , \quad (3.25)$$

where $F(z)$ is the hypergeometric function,

$$F(z) \equiv F\left(\frac{k}{N}, 1-\frac{k}{N}; 1; z\right) = \frac{1}{\pi}\int_0^1 dy y^{-\frac{k}{N}} (1-y)^{-\left(1-\frac{k}{N}\right)} (1-yz)^{-\frac{k}{N}} , \quad (3.26)$$

and $s = -(k_1+k_2)^2$, $t = -(k_2+k_3)^2$, $s = -(k_3+k_1)^2$.

In the large N approximation,

$$F(z) \approx 1 + \frac{k}{N}\left(z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \ldots\right) + O\left((k/N)^2\right) \quad . (3.27)$$

Note that the terms proportional to k/N in (3.27) shift the s-channel pole. There is an additional factor of $(k_1.k_2)^2$, due to which the contact term from any of the terms of (3.27) apart from 1, would at least be of $O\left((k/N)^2\right)$. With this observation, the integral can now be performed for $F(z) \to 1$.

$$I = C2\pi(k_1.k_3)^2 \frac{\Gamma\left(-\frac{s}{2}\right)\Gamma\left(-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}+\frac{t}{2}\right)}{\Gamma\left(-\frac{s}{2}-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1+\frac{t}{2}\right)} =$$

$$= -(4\pi)^2 g_c^2 \times \frac{1}{4}\left(u-2m^2\right)^2\left(\frac{1}{s}+\frac{1}{t}\right) \frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}+\frac{t}{2}\right)}{\Gamma\left(1-\frac{s}{2}-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1+\frac{t}{2}\right)} \quad . (3.28)$$

Now using $s+t+u=4m^2$,

$$I = -4\pi^2 g_c^2\left[\frac{\left(t-2m^2\right)^2}{s} + \frac{\left(s-2m^2\right)^2}{t} + 3(s+t) - 8m^2\right] \times \frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}+\frac{t}{2}\right)}{\Gamma\left(1-\frac{s}{2}-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1+\frac{t}{2}\right)} \quad . (3.29)$$

where we have to expand the gamma functions.

Also here we can write the following relationship between (3.25) and (3.29):

$$I = C(k_1.k_3)^2 \int_C d^2z \frac{|z|^{-2-s}|1-z|^{-2-t}}{|F(z)|^2} \Rightarrow$$

$$\Rightarrow -4\pi^2 g_c^2\left[\frac{\left(t-2m^2\right)^2}{s}+\frac{\left(s-2m^2\right)^2}{t}+3(s+t)-8m^2\right]\times\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}+\frac{t}{2}\right)}{\Gamma\left(1-\frac{s}{2}-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1+\frac{t}{2}\right)} \quad (3.30)$$

Also this expression can be related with the eq. (d) and with the Ramanujan's modular equation concerning the number 8 and thence, we obtain this further mathematical connection:

$$T^{-1}(V(t),k):=\frac{2}{\pi}\Gamma(k+1)\left(\int_0^{\pi}\mathcal{R}\left(V\left(e^{i\xi}\right)\right)\cos(k\xi)d\xi\right)\Rightarrow$$

$$\Rightarrow C(k_1.k_3)^2\int_C d^2z\frac{|z|^{-2-s}|1-z|^{-2-t}}{|F(z)|^2}\Rightarrow$$

$$\Rightarrow -4\pi^2 g_c^2\left[\frac{\left(t-2m^2\right)^2}{s}+\frac{\left(s-2m^2\right)^2}{t}+3(s+t)-8m^2\right]\times\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}+\frac{t}{2}\right)}{\Gamma\left(1-\frac{s}{2}-\frac{t}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1+\frac{t}{2}\right)}\Rightarrow$$

$$\Rightarrow\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^{\infty}\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \quad (3.30b)$$

d) expressions concerning the four tachyon amplitude in CSFT

With regard a closed analytical expression for the off-shell four tachyon amplitude in CSFT, Giddings gave an explicit conformal map that takes the Riemann surfaces defined by the Witten diagrams to the standard disc with four tachyon vertex operators on the boundary. This conformal map is defined in terms of four parameters $\alpha,\beta,\gamma,\delta$.

The four parameters are not independent variables. They satisfy the relations

$$\alpha\beta=1,\ \gamma\delta=1 \quad (3.31)$$

and

$$\frac{1}{2}=\Lambda_0(\theta_1,k)-\Lambda_0(\theta_2,k), \quad (3.32)$$

where $\Lambda_0(\theta,k)$ is defined by

$$\Lambda_0(\theta,k)=\frac{2}{\pi}\left(E(k)F(\theta,k')+K(k)E(\theta,k')-K(k)F(\theta,k')\right). \quad (3.33)$$

In (3.33) $K(k)$ and $E(k)$ are complete elliptic functions of the first and second kinds, $F(\theta,k)$ is the incomplete elliptic integral of the first kind. The parameters θ_1,θ_2,k and k' satisfy

$$k^2=\frac{\gamma^2}{\delta^2}, \qquad k'^2=1-k^2, \quad (3.34)$$

$$\sin^2\theta_1=\frac{\beta^2}{\beta^2+\gamma^2}, \quad \sin^2\theta_2=\frac{\alpha^2}{\alpha^2+\gamma^2}. \quad (3.35)$$

By using the integral representations of the elliptic functions it is possible to write the equation (3.32) in a useful form

$$E(\gamma^2)\int_{\alpha\gamma}^{\gamma/\alpha}dt\frac{1}{\sqrt{t^2+\gamma^4}\sqrt{1+t^2}}-(1-\gamma^4)K(\gamma^2)\int_{\alpha\gamma}^{\gamma/\alpha}dt\frac{1}{\sqrt{t^2+\gamma^4}\left(\sqrt{1+t^2}\right)^3}=\frac{\pi}{4}. \quad (3.36)$$

To expand (3.36) for small γ and α we have to divide the integration region into three intervals in such a way that the square roots in the denominators of (3.36) can be consistently expanded and the integrals in t performed. For example consider the integral in the first term of (3.36) , it can be rewritten as

$$\int_{\alpha\gamma}^{\gamma/\alpha}dt\frac{1}{\sqrt{t^2+\gamma^4}\sqrt{1+t^2}}=\int_{\alpha\gamma}^{\gamma^2}dt\frac{1}{\gamma^2\sqrt{1+\frac{t^2}{\gamma^4}}\sqrt{1+t^2}}+\int_{\gamma^2}^{1}dt\frac{1}{t\sqrt{1+\frac{\gamma^4}{t^2}}\sqrt{1+t^2}}+\int_{1}^{\gamma/\alpha}dt\frac{1}{t^2\sqrt{1+\frac{\gamma^4}{t^2}}\sqrt{1+\frac{1}{t^2}}} \quad (3.37)$$

In each integral of the rhs the integration domain is contained in the convergence radius of the Taylor expansions of the square roots containing γ, so that they can be safely expanded and the integrals in t performed. With this procedure one gets the following equation equivalent to (3.36):

$$E(\gamma^2)\sum_{n,k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}-k\right)n!k!}\left\{\frac{2}{2n+2k+1}\left[\gamma^{4k}-\left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right]+(1-\delta_{kn})\right.$$

$$\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n}-\delta_{kn}\gamma^{4n}\ln\gamma^{2}\Bigg\}-\left(1-\gamma^{4}\right)K\left(\gamma^{2}\right)\sum_{n,k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(-\frac{1}{2}-k\right)n!k!}\left\{\frac{1}{2n+2k+1}\left[\gamma^{4k}-\left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right]+\right.$$

$$\left.+\left(1-\delta_{kn}\right)\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n}-\delta_{kn}\gamma^{4n}\ln\gamma^{2}+\frac{1}{2n+2k+3}\left[\gamma^{4n}-\left(\frac{\alpha}{\gamma}\right)^{2k+3}(\alpha\gamma)^{2n}\right]\right\}=\frac{\pi}{4}$$

(3.38)

Thence, from (3.36) and (3.38) we can write the following mathematical relationship:

$$E\left(\gamma^{2}\right)\int_{\alpha\gamma}^{\gamma/\alpha}dt\frac{1}{\sqrt{t^{2}+\gamma^{4}}\sqrt{1+t^{2}}}-\left(1-\gamma^{4}\right)K\left(\gamma^{2}\right)\int_{\alpha\gamma}^{\gamma/\alpha}dt\frac{1}{\sqrt{t^{2}+\gamma^{4}}\left(\sqrt{1+t^{2}}\right)^{3}}=$$

$$=E\left(\gamma^{2}\right)\sum_{n,k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)^{2}}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}-k\right)n!k!}\left\{\frac{2}{2n+2k+1}\left[\gamma^{4k}-\left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right]+\left(1-\delta_{kn}\right)\right.$$

$$\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n}-\delta_{kn}\gamma^{4n}\ln\gamma^{2}\Bigg\}-\left(1-\gamma^{4}\right)K\left(\gamma^{2}\right)\sum_{n,k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(-\frac{1}{2}-k\right)n!k!}\left\{\frac{1}{2n+2k+1}\left[\gamma^{4k}-\left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right]+\right.$$

$$\left.+\left(1-\delta_{kn}\right)\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n}-\delta_{kn}\gamma^{4n}\ln\gamma^{2}+\frac{1}{2n+2k+3}\left[\gamma^{4n}-\left(\frac{\alpha}{\gamma}\right)^{2k+3}(\alpha\gamma)^{2n}\right]\right\}=\frac{\pi}{4}$$

. (3.39)

Also this expression can be related with the eq. (d), and thence we obtain this further mathematical connection:

$$T^{-1}(V(t),k):=\frac{2}{\pi}\Gamma(k+1)\left(\int_{0}^{\pi}\mathcal{R}\left(V\left(e^{i\xi}\right)\right)\cos(k\xi)d\xi\right)\Rightarrow$$

$$\Rightarrow E\left(\gamma^{2}\right)\int_{\alpha\gamma}^{\gamma/\alpha}dt\frac{1}{\sqrt{t^{2}+\gamma^{4}}\sqrt{1+t^{2}}}-\left(1-\gamma^{4}\right)K\left(\gamma^{2}\right)\int_{\alpha\gamma}^{\gamma/\alpha}dt\frac{1}{\sqrt{t^{2}+\gamma^{4}}\left(\sqrt{1+t^{2}}\right)^{3}}=$$

$$= E\left(\gamma^2\right)\sum_{n,k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}-k\right)n!k!}\left\{\frac{2}{2n+2k+1}\left[\gamma^{4k}-\left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right]+(1-\delta_{kn})\right.$$

$$\left.\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n}-\delta_{kn}\gamma^{4n}\ln\gamma^2\right\}-\left(1-\gamma^4\right)K\left(\gamma^2\right)\sum_{n,k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(-\frac{1}{2}-k\right)n!k!}\left\{\frac{1}{2n+2k+1}\left[\gamma^{4k}-\left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right]+\right.$$

$$\left.+(1-\delta_{kn})\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n}-\delta_{kn}\gamma^{4n}\ln\gamma^2+\frac{1}{2n+2k+3}\left[\gamma^{4n}-\left(\frac{\alpha}{\gamma}\right)^{2k+3}(\alpha\gamma)^{2n}\right]\right\}=\frac{\pi}{4} \quad (3.40)$$

e) physical interpretation of the nontrivial zeta zeros in terms of tachyonic string poles

The four-point dual string amplitude obtained by Veneziano was

$$A_4 = A(s,t)+A(t,s)+A(u,s)=\int_R dx|x|^{\alpha-1}|1-x|^{\beta-1}=B(\alpha,\beta) \text{, } (3.41)$$

where the Regge trajectories in the respective s,t,u channels are:

$$-\alpha(s)=1+\frac{1}{2}s \text{, } -\beta(t)=1+\frac{1}{2}t \text{, } -\gamma(u)=1+\frac{1}{2}u \text{. } (3.42)$$

The conservation of the energy-momentum yields:

$$k_1+k_2=k_3+k_4 \Rightarrow k_1+k_2-k_3-k_4=0 \text{. } (3.43)$$

We have also that the sum

$$s+t+u=2\left(k_1^2+k_2^2+k_3^2\right)+2\left(k_1\cdot k_2-k_2\cdot k_3-k_1\cdot k_3\right)=-8 \quad (3.44)$$

in mass units of m Planck = 1, when all the four particles are tachyons and one has the on-shell condition:

$$k_1^2 = k_2^2 = k_3^2 = m^2 =$$

$-2m^2$Planck = – 2 (3.45)

in the natural units LPlanck = 1 such that the string slope parameter in those units is given by L2Planck = 1/2 and the string mass spectrum is quantized in multiplies of the Planck mass mPlanck = 1.

From the conservation of energy-momentum (3.43) and the tachyon on-shell condition eq. (3.45) one can deduce that:

$$(k_1 + k_2)^2 = (k_3 + k_4)^2 \Rightarrow k_1 \cdot k_2 = k_3 \cdot k_4 \text{ . (3.46)}$$

Therefore, from eqs. (3.44) – (3.46) it is straightforward to show:

$$s + t + u = 2(-2-2-2) + 2(k_1 \cdot k_2 - k_3 \cdot (k_1 + k_2)) = -12 + 2(k_1 \cdot k_2 - k_3 \cdot (k_3 + k_4)) =$$
$$= -12 + 2(k_1 \cdot k_2 - k_3 \cdot k_4 - k_3 \cdot k_3) = -12 - 2k_3 \cdot k_3 = -12 + 4 = -8 \quad (3.47)$$

This relationship among $s + t + u = 4m^2 = -8$ will be crucial in what follows next. From eqs. (3.42), (3.44), and (3.47) we learn that:

$$\alpha + \beta + \gamma = 1 \text{ . (3.48)}$$

There exists a well-known relation among the Γ functions (Euler Gamma function) in terms of ζ functions (Riemann zeta function) appearing in the expression for $A(s,t,u)$ when α, β fall inside the critical strip. In this case, the integration region in the real line that defines $A(s,t,u)$ in eq. (3.41) can be divided into three parts and leads to the very important identity

$$A(s,t,u) = B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} + \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)} = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \frac{\zeta(1-\beta)}{\zeta(\beta)} \frac{\zeta(1-\gamma)}{\zeta(\gamma)} \quad (3.49)$$

where $\alpha + \beta + \gamma = 1$ and α, β are confined to the interior of the critical strip.

The derivation behind eq. (3.49) relies on the condition $\alpha + \beta + \gamma = 1$ eq. (3.48) and the identities

$$\sin \pi(\alpha+\beta)+\sin \pi(\alpha+\gamma)+\sin \pi(\beta+\gamma)=4 \cos \frac{\pi \alpha}{2} \cos \frac{\pi \beta}{2} \cos \frac{\pi \gamma}{2}, \quad (3.50)$$

$$\Gamma(\gamma)=\Gamma(1-\alpha-\beta)=\frac{1}{\Gamma(\alpha+\beta)} \frac{\pi}{\sin \pi(\alpha+\beta)}, \quad (3.51)$$

plus the remaining cyclic permutations from which one can infer

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma) \frac{\sin \pi(\alpha+\beta)}{\pi}, \quad (3.52)$$

$$\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}=\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma) \frac{\sin \pi(\alpha+\gamma)}{\pi}, \quad (3.53)$$

$$\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}=\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma) \frac{\sin \pi(\beta+\gamma)}{\pi}. \quad (3.54)$$

Therefore, eqs. (3.50) – (3.54) allow us to recast the left hand side of (3.49) as

$$A(s,t,u)=B(\alpha,\beta)=\frac{4}{\pi} \cos \frac{\pi \alpha}{2} \cos \frac{\pi \beta}{2} \cos \frac{\pi \gamma}{2} \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma). \quad (3.55)$$

And, finally, the known functional relation

$$(2\pi)^{z} \zeta(1-z)=2 \cos \frac{\pi z}{2} \Gamma(z)\zeta(z), \quad (3.56)$$

in conjunction with the condition $\alpha+\beta+\gamma=1$ such that $(2\pi)^{\alpha+\beta+\gamma}=2\pi$ is what establishes the important identity (3.49) expressing explicitly the string amplitude $A(s,t,u)$ either in terms of zeta functions or in terms of Γ functions.

In conclusion, we have the following interesting relationship between the eqs. (3.41), (3.49) and (3.55):

$$A_4=A(s,t)+A(t,s)+A(u,s)=\int_R dx|x|^{\alpha-1}|1-x|^{\beta-1}=B(\alpha,\beta) \Rightarrow$$

$$\Rightarrow A(s,t,u) = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} + \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)} = \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1-\beta)}{\zeta(\beta)}\frac{\zeta(1-\gamma)}{\zeta(\gamma)} =$$

$$= \frac{4}{\pi}\cos\frac{\pi\alpha}{2}\cos\frac{\pi\beta}{2}\cos\frac{\pi\gamma}{2}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma) \quad , (3.57)$$

from which we can to obtain the following equivalent expression:

$$\frac{1}{4}A_4 = \frac{1}{4}\int_R dx|x|^{\alpha-1}|1-x|^{\beta-1} = \frac{1}{4}B(\alpha,\beta) = \frac{1}{\pi}\cos\frac{\pi\alpha}{8}\cos\frac{\pi\beta}{8}\cos\frac{\pi\gamma}{8}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)$$

(3.58)

In this expression there are both π and 8, i.e. the number that is connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \quad . (3.59)$$

Thence the final mathematical connection:

$$\frac{1}{4}A_4 = \frac{1}{4}\int_R dx|x|^{\alpha-1}|1-x|^{\beta-1} = \frac{1}{4}B(\alpha,\beta) = \frac{1}{\pi}\cos\frac{\pi\alpha}{8}\cos\frac{\pi\beta}{8}\cos\frac{\pi\gamma}{8}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma) \Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \quad (3.60)$$

References

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