R.H.

① Edwards (Pg:- 15)

$$Z(s) = \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s)$$

$$Z(s) = \int_0^\infty \psi(x)\, x^{s/2-1}\, dx$$

where $\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$ is the well known Jacobi Theta function.

② Neukirch, J. (1999) Algebraic Number Theory Pg: 425 (Theorem 1.6) Completed Zeta function $Z(s)$ has an Analytic Continuation to $\mathbb{C}\setminus\{0,1\}$ & Satisfies $Z(s) = Z(1-s)$.

$$Z(s) = \int_0^1 \psi(x)\, x^{\frac{s}{2}-1}\, dx + \int_1^\infty \psi(x)\, x^{\frac{s}{2}-1}\, dx$$

$s = \sigma + it$

$$Z(\sigma+it) = 0$$

$$\int_0^1 \psi(x)\, x^{\frac{\sigma+it}{2}-1}\, dx + \int_1^\infty \psi(x)\, x^{\frac{\sigma+it}{2}-1}\, dx$$

$$\Rightarrow \int_0^1 \psi(x)\, x^{\frac{\sigma-2}{2}} x^{\frac{it}{2}}\, dx + \int_1^\infty \psi(x)\, x^{\frac{\sigma-2}{2}} x^{\frac{it}{2}}\, dx = 0$$

$$\int_0^1 \psi(x)\, e^{\left(\frac{\sigma-2}{2}\right)\ln x}\, e^{\frac{it}{2}\ln x}\, dx + \int_1^\infty \psi(x)\, e^{\frac{\sigma-2}{2}\ln x}\, e^{\frac{it}{2}\ln x}\, dx = 0 \quad \text{— (1)}$$

$$Z(1-s) = 0$$

$$Z(1-\sigma-it) = 0$$

$$\int_0^1 \psi(x)\, e^{-\left(\frac{\sigma+1}{2}\right)\ln x}\, e^{-\frac{it}{2}\ln x}\, dx + \int_1^\infty \psi(x)\, e^{-\left(\frac{\sigma+1}{2}\right)\ln x} \cdot e^{\frac{it \ln x}{2}}\, dx = 0 \quad \text{— (2)}$$

(1) − (2) gives,

$$\int_0^1 \psi(x)\left[e^{\left(\frac{\sigma-2}{2}\right)\ln x}\, e^{\frac{it}{2}\ln x} - e^{-\left(\frac{\sigma+1}{2}\right)\ln x}\, e^{-\frac{it}{2}\ln x} \right] + \int_1^\infty \psi(x)\left[e^{\left(\frac{\sigma-2}{2}\right)\ln x}\, e^{\frac{it \ln x}{2}} - e^{-\left(\frac{\sigma+1}{2}\right)\ln x}\, e^{-\frac{it \ln x}{2}} \right] = 0 \quad \text{— (3)}$$

$$I + J = 0$$

where $$I = \int_0^1 \psi(x)\left[e^{\left(\frac{\sigma-2}{2}\right)\ln x} e^{\frac{it}{2}\ln x} - e^{-\left(\frac{\sigma+1}{2}\right)\ln x} e^{-\frac{it\ln x}{2}}\right]$$

$$\cancel{\& \; J = \int_1^\infty \psi(x)\left[e^{\left(\frac{\sigma-2}{2}\right)\ln x} e^{\frac{it}{2}\ln x} - e^{-\left(\frac{\sigma+1}{2}\right)\ln x} e^{-it}\right]}$$

$$\& \; J = \int_1^\infty \psi(x)\left[e^{\left(\frac{\sigma-2}{2}\right)\ln x} e^{\frac{it\ln x}{2}} - e^{-\left(\frac{\sigma+1}{2}\right)\ln x} e^{-\frac{it\ln x}{2}}\right]$$

$$x = \frac{1}{y} \quad \Rightarrow dx = -\frac{1}{y^2}\,dy$$

$$J = \int_0^1 \frac{\psi\left(\frac{1}{y}\right)}{y^2}\left[e^{-\left(\frac{\sigma-2}{2}\right)\ln y} e^{-\frac{it\ln x}{2}} - e^{\left(\frac{\sigma+1}{2}\right)\ln y} e^{\frac{it\ln y}{2}}\right]dy$$

$$J = \int_0^1 \frac{\psi\left(\frac{1}{y}\right)}{e^{2\ln y}}\left[e^{-\left(\frac{\sigma-2}{2}\right)\ln y} e^{-\frac{it\ln y}{2}} - e^{\left(\frac{\sigma+1}{2}\right)\ln y} e^{\frac{it\ln y}{2}}\right]dy$$

$$J = \int_0^1 \psi\left(\frac{1}{y}\right)\left[e^{-\left(\frac{\sigma+2}{2}\right)\ln y} e^{-\frac{it\ln y}{2}} - e^{\left(\frac{\sigma-3}{2}\right)\ln y} e^{\frac{it\ln y}{2}}\right]dy$$

Putting the value of J in (3)

$$\int_0^1 \psi(x)\left[e^{\left(\frac{\sigma-2}{2}\right)\ln x} e^{\frac{it\ln x}{2}} - e^{-\left(\frac{\sigma+1}{2}\right)\ln x} e^{-\frac{it\ln x}{2}}\right]$$

$$+ \int_0^1 \psi\left(\frac{1}{x}\right)\left[e^{-\left(\frac{\sigma+2}{2}\right)\ln x} e^{-\frac{it\ln x}{2}} - e^{\left(\frac{\sigma-3}{2}\right)\ln x} e^{\frac{it\ln x}{2}}\right] dy$$

$$= 0 \quad (4) \ \forall t \in \mathbb{R}$$

(4) holds $\forall\ t \in \mathbb{R}$.

So, (4) holds for $t = 0$

$$\int_0^1 \psi(x)\left[e^{\left(\frac{\sigma-2}{2}\right)\ln x} - e^{-\left(\frac{\sigma+1}{2}\right)\ln x}\right] + \psi\left(\frac{1}{x}\right)\left[e^{-\left(\frac{\sigma+2}{2}\right)\ln x} - e^{\left(\frac{\sigma-3}{2}\right)\ln x}\right] dx = 0$$

(3.) ~~From~~ Edwards Pg. 15

$$\frac{1+2\psi(x)}{1+2\psi\left(\frac{1}{x}\right)} = \frac{1}{\sqrt{x}}$$

$$\psi\left(\frac{1}{x}\right) = \frac{\sqrt{x}\left(1+2\psi(x)\right)-1}{2}$$

$$\int_0^1 \psi(x)\left[x^{\frac{\sigma-2}{2}} - x^{-\left(\frac{\sigma+1}{2}\right)}\right]dx$$

$$+\int_0^1 \psi\left(\frac{1}{x}\right)\left[x^{-\left(\frac{\sigma+2}{2}\right)} - x^{\left(\frac{\sigma-3}{2}\right)}\right]dx = 0$$

$$\lim_{\substack{\xi_1\to 0^+\\ \eta\to 0^+}} \int_{\xi_1}^{1-\eta} \psi(x)\left[x^{\frac{\sigma-2}{2}} - x^{-\left(\frac{\sigma+1}{2}\right)}\right]dx$$

$$+\int_{\xi_1}^{1-\eta} \psi\left(\frac{1}{x}\right)\left[x^{-\left(\frac{\sigma+2}{2}\right)} - x^{\frac{\sigma-3}{2}}\right]dx$$

$$= 0 \quad \text{—} \ (*)$$

Let, $f(x) = x^{\frac{\sigma-2}{2}} - x^{-\left(\frac{\sigma+1}{2}\right)}$

$$\int_{\xi_1}^{\eta} f(x)\,dx = \int_{\xi_1}^{1-\eta} x^{\frac{\sigma-2}{2}} - x^{-\left(\frac{\sigma+1}{2}\right)}\,dx$$

$$= \left.\frac{2}{\sigma}x^{\sigma/2} - \frac{2x^{\frac{1-\sigma}{2}}}{1-\sigma}\right|_{\xi_1}^{\eta}$$

$$= \frac{2\eta^{\sigma/2}}{\sigma} - \frac{2\eta^{\frac{1-\sigma}{2}}}{1-\sigma} - \frac{2}{\sigma}\xi_1^{\sigma/2} + \frac{2}{1-\sigma}\xi_1^{\frac{1-\sigma}{2}}$$

$0<\sigma<1$

So f is integrable on $[\xi_1, 1-\eta]$

$$\psi(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x} > 0 \quad \forall\ x \in [\xi, \eta]$$

$$n^2 \geq n$$

$$-n^2 \leq -n$$

$$-n^2 \pi x \leq -n \pi x \qquad x \in [\xi, 1-\eta]$$

$$e^{-n^2 \pi x} \leq e^{-n\pi x}$$

$$\psi(x) \leq \sum_{n=1}^{\infty} e^{-n\pi x}$$

$$\leq \frac{e^{-\pi x}}{1-e^{-\pi x}} = \frac{1}{(e^{\pi x}-1)}$$

$$\int_{\xi}^{1} \psi(x)\, dx \leq \int_{\xi}^{1-\eta} \frac{1}{e^{\pi x}-1}\, dx \qquad \frac{1}{z-1} - \frac{1}{z}$$

$$\leq \int_{\xi}^{1-\eta} \frac{e^{\pi x}}{e^{\pi x}(e^{\pi x}-1)}\, dx$$

$$e^{\pi x} = z$$

$$\pi e^{\pi x}\, dx = dz$$

$$\leq \frac{1}{\pi} \int \frac{\cancel{z}}{z(z-1)} \left(\frac{1}{\cancel{z}}\right) dz$$

$$\leq \frac{1}{\pi} \ln \frac{z-1}{z} = \frac{1}{\pi} \ln \frac{e^{\pi x}-1}{e^{\pi x}} \Bigg|_{\xi}^{1}$$

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$$= \frac{1}{\pi} \ln\left(1 - \frac{1}{e^{\pi x}}\right)\Big|_{\epsilon_1}^{\eta}$$

$$= \frac{1}{\pi}\left[\ln\left(1-\frac{1}{e^{\pi\eta}}\right) - \ln\left(1 - \frac{1}{e^{\pi \epsilon_1}}\right)\right]$$

$$\int_{\epsilon_1}^{1-\eta} \psi(x)\,dx < \infty$$

$$f(x) = x^{\frac{\sigma-2}{2}} - x^{-\left(\frac{\sigma+1}{2}\right)} \text{ is Continuous}$$

on $[\epsilon_1, 1-\eta]$

So, By Generalised First mean value theorem

If f & g are integrable on $[a,b]$ and g keeps the same sign over $[a,b]$ ~~for~~ ~~any number~~ Also if f is continuous in $[a,b]$

then $\exists\ c \in (a,b)$ s.t

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

Applying Generalised first mean value theorem to both the integrals in *

$$g(x) = x^{-\left(\frac{\sigma+2}{2}\right)} - x^{\frac{\sigma-3}{2}}$$

$$\psi\left(\frac{1}{x}\right) = \sum_{n\geq 1}^{\infty} e^{-\frac{n^2\pi}{x}} > 0 \quad \forall\ \epsilon_1 < x < \eta$$

$$\int_{\xi}^{1} g(x)\,dx = \int_{\xi}^{1} x^{-\left(\frac{\sigma+2}{2}\right)} - x^{\frac{\sigma-3}{2}}$$

$$= -\frac{2}{\sigma}x^{-\sigma/2} - \frac{2x^{\frac{\sigma-1}{2}}}{\sigma-1}\Bigg|_{\xi}^{1}$$

$$= -\frac{2}{\sigma}\eta^{-\sigma/2} - \frac{2}{\sigma-1}\eta^{\frac{\sigma-1}{2}} + \frac{2}{\sigma}\xi^{-\sigma/2} + \frac{2}{\sigma-1}\xi^{\frac{\sigma-1}{2}}$$

$< \infty \qquad 0 < \sigma < 1$

$g(x)$ is Continuous on ~~$[\xi,\eta]$~~ $[\xi, 1-\eta]$

So $\exists$ ~~$c \in [\xi,\eta]$~~ $c \in [\xi, 1-\eta]$ s.t.

$$\lim_{\substack{\xi\to 0^+\\ \eta\to 0^+}} \left[c^{\frac{\sigma-2}{2}} - c^{-\left(\frac{\sigma+1}{2}\right)}\right]\int_{\xi}^{1-\eta}\psi(x)\,dx + \left[c^{-\left(\frac{\sigma+2}{2}\right)} - c^{\frac{\sigma-3}{2}}\right]\int_{\xi}^{1-\eta}\psi\left(\frac{1}{x}\right)dx = 0$$

$$= \cancel{\lim_{\xi\to 0^+} \left[c^{\frac{\sigma-2}{2}} - c^{-\left(\frac{\sigma+1}{2}\right)}\right]\int_{\xi}^{1}\psi(x)\,dx + \left[c^{-\left(\frac{\sigma+2}{2}\right)} - c^{\left(\frac{\sigma-3}{2}\right)}\right]\int_{\xi}^{1}\frac{\sqrt{x}\,(1+2\psi(x))-1}{2}\,dx = 0}$$

$$\lim_{\substack{\xi\to 0^+\\ \eta\to 0^+}} \left[c^{\frac{\sigma-2}{2}} - c^{-\left(\frac{\sigma+1}{2}\right)} \right] \int_{\xi}^{1-\eta} \psi(x)\,dx$$

$$+ \left[c^{-\left(\frac{\sigma+2}{2}\right)} - c^{\frac{\sigma-3}{2}} \right] \int_{\xi}^{1-\eta} \psi\left(\frac{1}{x}\right) dx$$

$$= 0$$

$$\frac{-\sigma-1-\sigma+2}{2}$$

$$\frac{-2\sigma+1}{2}$$

$$-\sigma-2-\sigma+3$$

$$1-2\sigma$$

$$c^{\frac{\sigma-2}{2}} - c^{-\left(\frac{\sigma+1}{2}\right)}$$

$$= c^{\frac{\sigma-2}{2}} \left(1 - \frac{c^{-\left(\frac{\sigma+1}{2}\right)}}{c^{\frac{\sigma-2}{2}}} \right)$$

$$= c^{\frac{\sigma-2}{2}} \left(1 - c^{\frac{1-2\sigma}{2}} \right)$$

$$c^{-\left(\frac{\sigma+2}{2}\right)} - c^{\frac{\sigma-3}{2}} = \cancel{c^{-\left(\frac{\sigma+2}{2}\right)} \left[1 - c^{\frac{\sigma-3}{2}} \right.}$$

$$= c^{\frac{\sigma-3}{2}} \left(\frac{c^{-\left(\frac{\sigma+2}{2}\right)}}{c^{\frac{\sigma-3}{2}}} - 1 \right)$$

$$= c^{\frac{\sigma-3}{2}} \left[c^{\frac{1-2\sigma}{2}} - 1 \right]$$

$$= -c^{\frac{\sigma-3}{2}} \left[1 - c^{\frac{1-2\sigma}{2}} \right]$$

$$c^{\frac{\sigma-2}{2}}\left(1-c^{\frac{1-2\sigma}{2}}\right)\int_0^1 \psi(x)\,dx$$

$$-c^{\frac{\sigma-3}{2}}\left(1-c^{\frac{1-2\sigma}{2}}\right)\int_0^1 \psi\left(\frac{1}{x}\right)dx = 0$$

$$\left(1-c^{\frac{1-2\sigma}{2}}\right)c^{\frac{\sigma-2}{2}}\left[\int_0^1 \psi(x)\,dx - c^{-1/2}\int_0^1 \psi\left(\frac{1}{x}\right)dx\right] = 0$$

$\Rightarrow$ $\because$ $c \in [\xi, 1-\eta]$ — $\#$

$\& \ \eta, \xi \to 0^+ \Rightarrow c \neq 0$

$\#$ $\Rightarrow$ $1-c^{\frac{1-2\sigma}{2}} = 0$ or

$$\int_0^1 \psi(x)\,dx - \frac{1}{\sqrt{c}}\int_0^1 \psi\left(\frac{1}{x}\right)dx = 0$$

$\Rightarrow$ $c^{\frac{1-2\sigma}{2}} = 1$ or $\displaystyle\int_0^1 \psi(x) - \frac{1}{2\sqrt{c}}\left[\sqrt{x}\left(1+2\psi(x)\right)-1\right] = 0$

$\Rightarrow$ $\frac{1-2\sigma}{2}\ln c = 0$ $\quad I = \displaystyle\int_0^1 \left(1-\frac{\sqrt{x}}{\sqrt{c}}\right)\psi(x)\,dx + \frac{1}{6\sqrt{c}}$

$c \in [\xi, 1-\eta]$

$\sigma = \frac{1}{2}$ $\Big(\because$ $\ln c \neq 0 \Big)$

$\eta \to 0^+.$

Claim:- $I \neq 0$.

$$I = \int_0^1 \left(1 - \frac{\sqrt{x}}{\sqrt{c}}\right) \psi(x)\, dx + \frac{1}{6\sqrt{c}}$$

$$= \int_0^1 \left(1 - \frac{\sqrt{x}}{\sqrt{c}}\right) \sum_{n=1}^{\infty} e^{-n^2 \pi x}\, dx + \frac{1}{6\sqrt{c}}$$

$$= \sum_{n=1}^{\infty} \int_0^1 \left(1 - \frac{\sqrt{x}}{\sqrt{c}}\right) e^{-n^2 \pi x}\, dx + \frac{1}{6\sqrt{c}}$$

$$0 < \frac{\sqrt{x}}{\sqrt{c}} < \frac{1}{\sqrt{c}}$$

$$1 - \frac{1}{\sqrt{c}} < 1 - \frac{\sqrt{x}}{\sqrt{c}} < 1$$

$$\cancel{I > \left(1 - \frac{1}{\sqrt{c}}\right)}$$

$$I > \sum_{n=1}^{\infty} \left(1 - \frac{1}{\sqrt{c}}\right) \int_0^1 e^{-n^2 \pi x}\, dx + \frac{1}{6\sqrt{c}}$$

$$I > \sum_{n=1}^{\infty} \left(1 - \frac{1}{\sqrt{c}}\right) \left. \frac{e^{-n^2 \pi x}}{-n^2 \pi} \right|_0^1 + \frac{1}{6\sqrt{c}}$$

$$I > \sum_{n=1}^{\infty} \left(1 - \frac{1}{\sqrt{c}}\right) \left(\frac{1 - e^{-n^2 \pi}}{n^2 \pi}\right) + \frac{1}{6\sqrt{c}}$$

$$I > \left(1 - \frac{1}{\sqrt{c}}\right) \sum_{n=1}^{\infty} \frac{1 - e^{-n^2\pi}}{n^2\pi} + \frac{1}{6\sqrt{c}}$$

$$I > \left(1 - \frac{1}{\sqrt{c}}\right) \left[\frac{\pi}{6} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{e^{-n^2\pi}}{n^2}\right] + \frac{1}{6\sqrt{c}}$$

$$n^2\pi \geq \pi$$

$$-n^2\pi \leq -\pi$$

$$\frac{e^{-n^2\pi}}{n^2} \leq \frac{e^{-\pi}}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{e^{-n^2\pi}}{n^2} \leq \sum_{n=1}^{\infty} \frac{e^{-\pi}}{n^2}$$

$$\Rightarrow -\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{e^{-n^2\pi}}{n^2} \geq -\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{e^{-\pi}}{n^2}$$

$$\geq -\frac{e^{-\pi}\pi^2}{6\pi}$$

$$\geq -\frac{\pi}{6e^{\pi}}$$

$$I \geq \left(1 - \frac{1}{\sqrt{c}}\right)\left[\frac{\pi}{6} - \frac{\pi}{6e^{\pi}}\right] + \frac{1}{6\sqrt{c}}$$

$$I \geq \frac{\pi}{6}\left(1 - \frac{1}{\sqrt{c}}\right)\left(1 - \frac{1}{e^{\pi}}\right) + \frac{1}{6\sqrt{c}}$$

$$c \in [\varepsilon_l, 1-\eta]$$

$$\varepsilon_l \leq c \leq 1-\eta$$

$$\sqrt{\varepsilon_l} \leq \sqrt{c} \leq \sqrt{1-\eta}$$

$$\frac{1}{\sqrt{1-\eta}} \leq \frac{1}{\sqrt{c}} \leq \frac{1}{\sqrt{\varepsilon_l}}$$

$$-\frac{1}{\sqrt{\varepsilon_l}} \leq -\frac{1}{\sqrt{c}} \leq -\frac{1}{\sqrt{1-\eta}}$$

$$1-\frac{1}{\sqrt{\varepsilon_l}} \leq 1-\frac{1}{\sqrt{c}} \leq 1-\frac{1}{\sqrt{1-\eta}}$$

$$1-\frac{1}{\sqrt{\varepsilon_l}} > 0$$

$$I \geq \frac{\pi}{6}\underbrace{\left(1-\frac{1}{\sqrt{\varepsilon_l}}\right)}_{>0}\left(1-\frac{1}{e^{\pi}}\right) + \underbrace{\frac{1}{6\sqrt{\varepsilon_l}}}_{>0}$$

$$\boxed{I \neq 0}$$

$$e^{\pi}-1 > 0$$

$$e^{\pi} = 22.9\ldots$$

$$e^{\pi} > 1$$

$$\pi \ln e > 0$$

$$\pi > 0$$