On the various mathematical applications and possible connections between Heterotic String Theory E8 x E8 and some sectors of Number Theory

This paper is dedicated to the memory of Dazzeglio Servi, proud that the mathematical talent of his beloved son Roberto leads to new and important results.
(Michele Nardelli)

Michele Nardelli[1,2], Roberto Servi, Francesco Di Noto

[1] Dipartimento di Scienze della Terra
Università degli Studi di Napoli Federico II, Largo S. Marcellino, 10
80138 Napoli, Italy

[2] Dipartimento di Matematica ed Applicazioni "R. Caccioppoli"
Università degli Studi di Napoli "Federico II" – Polo delle Scienze e delle Tecnologie
Monte S. Angelo, Via Cintia (Fuorigrotta), 80126 Napoli, Italy

Abstract

In the present paper we have described some interesting mathematical applications of the Number Theory to the Heterotic String Theory E8 x E8. In the **Chapter 1**, we have described various theoretic arguments and equations concerning the Lie's Group E8, E8xE8 gauge fields and the Heterotic String Theory. In the **Chapter 2**, we have described the link between the subsets of odd natural numbers and of squares, some equations concerning the Theorem that: "every sufficiently large odd positive integer can be written as the sum of three primes", and the possible method of factorization of a number. In the **Chapter 3**, we have described some classifications of the numbers: perfect, defective, abundant. Furthermore, we have described an infinite set of integers, each of which has many factorizations. In the **Chapter 4**, we have described some interesting mathematical applications concerning the possible method of factorization of a number to the number of dimensions of the Lie's Group E8. In conclusion, in the Appendix, we have described some mathematical connections between various series of numbers concerning the Chapter 1 and some sectors of Number Theory.

1. On the various theoretic arguments and equations concerning the Lie's Group E8, E8xE8 gauge fields and the Heterotic String Theory. [1] [2] [3]

The theory of Lie groups is now the language that allows to express the TOE's (unified field theories) of physics of particle. More precisely, it was discovered that the forces electromagnetic, nuclear weak and nuclear strong comply with particular symmetries of phase rotation of the fields, of the exchange of charge of the particles and of exchange of colors of quark, and that the properties of these symmetries are described by Lie groups U(1), SU(2) and SU(3). The relative sizes of these groups are 1, 3 and 8, and correspond to the number of bosons that transmit the three forces:

1 photon, 3 weak bosons and 8 gluons.

[...]

The final step towards the final unification of physical forces, then, passes through the determination of an appropriate Lie group that contains the product U(1) x SU(2) x SU(3). The minimal simple group of Lie that mathematically satisfy the requirement is SU(5), in 24 dimensions, but it does not seems appropriate physically: the great unification based on it, expected phenomena doubts as the too fast proton decay and the existence of magnetic monopoles. The group that today seems most appropriated, for the so-called theory of everything, that includes also the gravity, is instead a dual pair of the maximum sporadic group E8: having double size of **248**, it predicts the existence of **496** bosons field, but for which we know, currently, only the 12 already mentioned.

To deepen, from Wikipedia we have that:

"There is a unique complex Lie algebra of type E_8, corresponding to a complex group of complex dimension **248**. The complex Lie group E_8 of complex dimension **248** can be considered as a simple real Lie group of real dimension **496**. This is simply connected, has maximal compact subgroup the compact form (see below) of E_8, and has an outer automorphism group of order 2 generated by complex conjugation.

The E_8 Lie group has applications in theoretical physics, in particular in String Theory and supergravity. $E_8 \times E_8$ is the gauge group of one of the two types of heterotic string and is one of two anomaly-free gauge groups that can be coupled to the N = 1 supergravity in 10 dimensions. E_8 is the U-duality group of supergravity on an eight-torus (in its split form).

One way to incorporate the Standard Model of particle physics into heterotic string theory is the symmetry breaking of E_8 to its maximal subalgebra $SU(3) \times E_6$".

We let us recall how E_8 is built from $Spin(16)/Z_2$ in ten dimensions. The adjoint representation of E_8 decomposes as

$$\mathbf{248} = \mathbf{120} + \mathbf{128} \qquad (1.1)$$

under $Spin(16)/Z_2$. At the level of ordinary Lie algebras, we get the elements of the E_8 Lie algebra from the adjoint plus a spinor representation of $Spin(16)/Z_2$, and assigning them suitable commutation relations. At the level of WZW conformal families, we could write

$$[\mathbf{1}] = [\mathbf{1}] + [\mathbf{128}]$$

which implicitly includes equation (1.1) as a special case, since the (adjoint-valued) currents are non-primary descendants of the identity operator. That statement about conformal families implies a statement about characters of the corresponding affine Lie algebras, namely that

$$\chi_{E_8}(1,q) = \chi_{Spin(16)}(1,q) + \chi_{Spin(16)}(128,q) \qquad (1.2)$$

where
$$\chi_{E_8}(1,q) = \frac{E_2(q)}{\eta(q)^8}$$

and where $E_2(q)$ is the degree four Eisenstein modular form

$$E_2(q) = 1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^m = 1 + 240\left[q + \left(1^3 + 2^3\right)q^2 + \left(1^3 + 3^3\right)q^3 + \ldots\right] =$$

$$= 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + 60480q^6 + \ldots \qquad (1.3)$$

with $$\sigma_3(m) = \sum_{d|m} d^3 \qquad (1.4)$$

The central charge of a bosonic WZW model at level k is

$$\frac{k \dim G}{k + C} \quad (1.5)$$

where C is the dual Coxeter number. For the case of $G = SU(N)$, $\dim G = N^2 - 1$ and $C = N$, hence the central charge of the bosonic $SU(N)$ WZW is

$$\frac{k\left(N^2 - 1\right)}{k + N} \quad (1.6).$$

For $k = 1$, this reduces to $N - 1$. Thus the $SU(5)$ current algebra at level 1 has central charge 4, and the $SU(9)$ current algebra has central charge **8**. In particular, this means that the $SU(5)\times SU(5)$ current algebra at level 1 has central charge 4 + 4 = **8**, just right to be used in critical heterotic strings to build an E_8. Similarly, the $SU(9)$ current algebra at level 1 has central charge **8**, also just right to be used in critical heterotic strings to build an E_8. Similarly, for E_6, E_7, E_8, the dual Coxeter numbers are 12, 18, 30, respectively, and it is easy to check that at level 1, each current algebra has central charge equal to 6, 7, **8**, respectively. For $SU(5)$, the integrable representations are **5**, **10** = Λ^2**5**, $1\overline{0}$ = Λ^3**5**, and $\overline{5}$ = Λ^4**5**. The fusion rules obeyed by the WZW conformal families have the form

[**5**] × [**5**] = [**10**]; [**5**] × [$\overline{5}$] = [**1**]; [$1\overline{0}$] × [$\overline{5}$] = [**10**]; [**10**] × [$\overline{5}$] = [**5**];

[$1\overline{0}$] × [$1\overline{0}$] = [**5**]; [$1\overline{0}$] × [**10**] = [**1**]; (1.7)

The adjoint representation of E_8 decomposes under $SU(5)^2/Z_5$ as

248 = (**1**, **24**) + (**24**, **1**) + (**5**, $1\overline{0}$) + ($\overline{5}$, **10**) + (**10**, **5**) + ($1\overline{0}$, $\overline{5}$) (1.8)

(indeed we have that: 24 + 24 + 50 + 50 + 50 + 50 = 248) from which one would surmise that the corresponding statement about conformal families is

[**1**] = [**1**, **1**] + [**5**, $1\overline{0}$] + [$\overline{5}$, **10**] + [**10**, **5**] + [$1\overline{0}$, $\overline{5}$] (1.9)

which can be checked by noting that the right-hand side above squares into itself under the fusion rules.

The character of the identity representation of $SU(5)$ is

$$\chi_{SU(5)}(1,q)=\frac{1}{\eta(\tau)^4}\sum_{\vec{m}\in Z^4} q^{\left(\sum m_i^2+\left(\sum m_i\right)^2\right)/2} . \quad (1.10)$$

Taking modular transformations, the characters of the other needed integrable representations are

$$\chi_{SU(5)}(5,q)=\frac{1}{\eta(\tau)^4}\sum_{\vec{m}\in Z^4,\sum m_i=1 \bmod 5} q^{\left(\sum m_i^2-\frac{1}{5}\left(\sum m_i\right)^2\right)/2} \quad (1.11)$$

and

$$\chi_{SU(5)}(10,q)=\frac{1}{\eta(\tau)^4}\sum_{\vec{m}\in Z^4,\sum m_i=2 \bmod 5} q^{\left(\sum m_i^2-\frac{1}{5}\left(\sum m_i\right)^2\right)/2} . \quad (1.12)$$

The remaining two characters are equal to these, by taking $\vec{m}\to-\vec{m}$. Now, we need to verify that

$$\chi_{E_8}(1,q)=\chi_{SU(5)}(1,q)^2+4\chi_{SU(5)}(5,q)\chi_{SU(5)}(10,q) \quad (1.13)$$

which corresponds to equation (1.9) for the conformal families. The E_8 character is given by

$$\chi_{E_8}(1,q)=\frac{E_2(q)}{\eta(\tau)^8} \quad (1.14)$$

where $E_2(q)$ denotes the relevant Eisenstein series. The Z_5 orbifold is implicit here – $\chi(1,q)^2$ arises from the untwisted sector, and each of the four $\chi(5,q)\chi(10,q)$'s arises from a twisted sector. We find:

$$\eta(\tau)^8\left(\chi_{SU(5)}(1,q)^2+4\chi_{SU(5)}(5,q)\chi_{SU(5)}(10,q)\right)=1+240q+2160q^2+6720q^3+17520q^4+$$

$$+30240q^5+60480q^6+\ldots \quad (1.15)$$

which are precisely the first few terms of the appropriate Eisenstein series $E_2(q)$, numerically verifying the prediction (1.13). The equivalence can be proved as follows. We need to relate the theta function of the E_8 lattice to a product of theta functions for $SU(5)$ lattices. Briefly, first one argues that

$$\Theta(E_8)=\Theta(\{A_4,A_4\}[1,2]). \quad (1.16)$$

This can be written as

$$\Theta\left(\bigcup_{i=1}^{5}[ig]\{A_4,A_4\}\right)=\sum_{i=1}^{5}\Theta([ig]\{A_4,A_4\}) \quad (1.17)$$

where g denotes the generator of the Z_5 action. This can be written as

$$\sum_{i=1}^{5}\Theta([ig]A_4)\Theta([ig]A_4)=\sum_{i=1}^{5}\Theta([i]A_4)\Theta([2i]A_4). \quad (1.18)$$

Using the symmetry

$$\Theta([5-i]A_4)=\Theta([i]A_4) \quad (1.19)$$

the result then follows after making the identifications

$$\eta(\tau)^4\chi(1,q)=\Theta(A_4), \quad \eta(\tau)^4\chi(5,q)=\Theta([1]A_4), \quad \eta(\tau)^4\chi(10,q)=\Theta([2]A_4). \quad (1.20)$$

The adjoint representation of E_8 decomposes as

$$\mathbf{248} = \mathbf{80} + \mathbf{84} + \overline{84} \quad (1.21)$$

and so the conformal families of E_8, $SU(9)$ should be related by

$$[\mathbf{1}] = [\mathbf{1}] + [\mathbf{84}] + [\overline{84}] \quad (1.22)$$

The relevant $SU(9)$, level 1, characters are given by

$$\chi_{SU(9)}(1,q)=\frac{1}{\eta(\tau)^8}\sum_{\vec{m}\in Z^8}q^{\left(\sum m_i^2+\left(\sum m_i\right)^2\right)/2} \quad (1.23)$$

and

$$\chi_{SU(9)}(84,q)=\frac{1}{\eta(\tau)^8}\sum_{\vec{m}\in Z^8,\sum m_i=3 \bmod 9}q^{\left(\sum m_i^2-\frac{1}{9}\left(\sum m_i\right)^2\right)/2}. \quad (1.24)$$

Then, from the equation (1.22) it should be true that

$$\chi_{E_8}(1,q)=\chi_{SU(9)}(1,q)+2\chi_{SU(9)}(84,q). \quad (1.25)$$

If one takes Spin(16) and splits it into Spin(7)$\times$Spin(9), then $G_2\subset$Spin(7) and $F_4\subset$Spin(9). Under the $g_2\times f_4$ subalgebra, the adjoint representation of e_8 decomposes as

$$\mathbf{248} = (\mathbf{14}, \mathbf{1}) + (\mathbf{1}, \mathbf{52}) + (\mathbf{7}, \mathbf{26}) \quad (1.26)$$

indeed: 248 = 14 + 52 +182. The commutant of $G_2\times F_4$ in E_8 has rank zero. The dual Coxeter number of G_2 is 4 and that of F_4 is 9, so the central charge of the G_2 algebra at level 1 is 14/5 and that of the F_4 algebra at level 1 is 52/10, which sum to 8, the same as the central charge of the E_8 algebra at level 1. Both G_2 and F_4 affine algebras at level one have only two integrable representations:

$$G_2: \quad [\mathbf{1}], [\mathbf{7}] \qquad F_4: \quad [\mathbf{1}], [\mathbf{26}] \qquad (1.27)$$

The conformal weights of the primary fields are, respectively, $h_7 = \frac{2}{5}$ and $h_{26} = \frac{3}{5}$. So, our proposed decomposition of E_8 level 1

$$[\mathbf{1}] = [\mathbf{1}, \mathbf{1}] + [\mathbf{7}, \mathbf{26}] \qquad (1.28)$$

does, indeed, reproduce the correct central charge and the conformal weights and multiplicity of currents. Under modular transformations,

$$\chi_{E_8}(1,q) = \chi_{G_2}(1,q)\chi_{F_4}(1,q) + \chi_{G_2}(7,q)\chi_{F_4}(26,q) \qquad (1.29)$$

transform identically. To see this, note that the fusion rules of G_2 and F_4 at level 1 are, respectively,

$$G_2: \quad [\mathbf{7}] \times [\mathbf{7}] = [\mathbf{1}] + [\mathbf{7}]; \qquad F_4: \quad [\mathbf{26}] \times [\mathbf{26}] = [\mathbf{1}] + [\mathbf{26}]. \qquad (1.30)$$

The modular S-matrix (for both G_2 and F_4) is

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1-1/\sqrt{5}} & \sqrt{1+1/\sqrt{5}} \\ \sqrt{1+1/\sqrt{5}} & -\sqrt{1-1/\sqrt{5}} \end{pmatrix} \qquad (1.31)$$

which, in both cases, satisfies $S^2 = (ST)^3 = 1$ and $N_{ijk} = \sum_m \frac{S_{im}S_{jm}S_{km}}{S_{1m}}$. Using this modular S-matrix, the particular combination of characters on the right-hand side of (1.29) is invariant, as it should be.

The heterotic $E_8 \times E_8$ - string is described by a 10d $\mathcal{N} = 1$ supergravity coupled to super-Yang-Mills theory with gauge group $E_8 \times E_8$. The corresponding 10-dimensional heterotic string supergravity effective action is

$$S_{het,eff}[G,\Phi,A,\chi,\varphi,\lambda,H] = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d\,\mathrm{vol}_G\, \Big[e^{-2\Phi}\Big(R_G + 4(\nabla\Phi)^2 - \bar{\chi}_\alpha\gamma^{[\alpha}\gamma^{\beta}\gamma^{\gamma]}\nabla_\beta\chi_\gamma - \bar{\varphi}\mathcal{D}\varphi\Big) +$$

$$+ e^{-2\Phi}\left(\frac{\alpha'}{30}Tr(F_A \wedge F_A) - Tr(\bar{\lambda}\mathcal{D}\lambda)\right) + e^{-2\Phi}\left(-\frac{1}{3}\tilde{H}^2\right) + \ldots\Big]. \qquad (1.32)$$

The term involving $\tilde{H}$ describes the Kalb-Ramond B-field coupled to the space-time metric and the gauge field. This so-called **axion 3-form** is described by the property

$$d\tilde{H} = \alpha'\left(tr(R_G \wedge R_G) - \frac{1}{30}Tr(F_A \wedge F_A)\right), \qquad (1.33)$$

where Tr refers to the "gauge trace" in the adjoint representation and tr to the "normal trace" in the fundamental representation (if applicable). Locally, $\tilde{H}$ can be described by

$$\tilde{H} = dB + \alpha' \left(CS\left(\nabla^G\right) - \frac{1}{30} CS(A) \right), \quad (1.34)$$

involving the **Chern-Simons 3-form** of the metric Levi-Civita connection and the gauge connection, which can be interpreted as secondary characteristic classes involving ideas from topological quantum field theory. Consider functions on Y that transform as $e^{-ik\theta}$ under rotations of the $\mathbf{S}^1$, for some integer k. In their X-dependence, they can be interpreted as sections of $\mathcal{L}^k$. Thus we have a decomposition

$$\mathrm{Fun}(Y) = \oplus_{k \in Z} \Gamma\left(X, \mathcal{L}^k\right). \quad (1.35)$$

Here $\mathrm{Fun}(Y)$ is the space of functions on Y, and $\Gamma\left(X, \mathcal{L}^k\right)$ the space of sections of $\mathcal{L}^k$. Consider an $\mathbf{S}^1$-invariant Dirac operator D_Y on Y with real eigenvalues λ_i. The APS function

$$\eta(s) = \sum_i \left|\lambda_i\right|^{-s} \mathrm{sign}(\lambda_i), \quad (1.36)$$

where the sum runs over all nonzero λ_i, can be written

$$\eta(s) = \sum_{k \in Z} \eta_k(s), \quad (1.37)$$

where $\eta_k(s)$ is the contribution from states that transform as $e^{-ik\theta}$ under rotation of the circle. We write the spin bundle S of Y as $S = \pi^*(S_+) \oplus \pi^*(S_-)$, where S_+ and S_- are the positive and negative chirality spin bundles of X. We can pick a basis of eleven-dimensional gamma matrices such that the Dirac operator reads

$$D_Y = \begin{pmatrix} \frac{i}{R}\frac{\partial}{\partial\theta} & \overline{D} \\ D & -\frac{i}{R}\frac{\partial}{\partial\theta} \end{pmatrix}, \quad (1.38)$$

where we have written the Dirac equation in 16×16 blocks, and we have arranged the spinors as a column vector

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad (1.39)$$

with $\psi_\pm$ being sections of $\pi^*(S_\pm)$. D and $\overline{D}$ are the ten-dimensional Dirac operators for positive and negative chirality. On spinors that transform as $e^{-ik\theta}$ under rotations of the circle, the Dirac equation $D_Y \psi = \lambda \psi$ becomes

$$\begin{pmatrix} \frac{k}{R} & \overline{D} \\ D & -\frac{k}{R} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \lambda \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad (1.40)$$

with $\psi_\pm$ being sections of $S_\pm \otimes \mathcal{L}^k$. Suppose we have a pair of states $\psi_\pm$, which are sections of $S_\pm \otimes \mathcal{L}^k$, with $D\psi_+ = w\psi_-$, $D\psi_- = \overline{w}\,\psi_+$ for some complex number w. Then for these two states, the eleven-dimensional Dirac operator becomes

$$\begin{pmatrix} \frac{k}{R} & \overline{w} \\ w & -\frac{k}{R} \end{pmatrix}. \quad (1.41)$$

Suppose now that ψ is a section of $S_+ \otimes \mathcal{L}^k$ or $S_- \otimes \mathcal{L}^k$ that is a zero mode of D or $\overline{D}$. We set $\chi(\psi)$ to be 1 or -1 depending on whether ψ has positive or negative chirality. ψ is an eigenstate of the eleven-dimensional Dirac operator with eigenvalue $k\chi(\psi)/R$. Its contribution to $\eta_k(s)$ is hence $|k/R|^{-s}\,\mathrm{sign}(k\chi) = |k/R|^{-s}\,\mathrm{sign}(k)\mathrm{sign}(\chi)$. When we sum the quantity $\mathrm{sign}(\chi)$ over all zero modes, we get the index of the ten-dimensional Dirac operator with values in $\mathcal{L}^k$; we denote this as $I(\mathcal{L}^k)$. So we have

$$\eta_k(s) = \left|\frac{k}{R}\right|^{-s} \mathrm{sign}(k) I(\mathcal{L}^k). \quad (1.42)$$

The function $\eta(s)$ is obtained by summing this expression over k. In doing so, we can observe that $I(\mathcal{L}^{-k}) = -I(\mathcal{L}^k)$. So we can express $\eta(s)$ as a sum over positive k only:

$$\frac{\eta(s)}{2} = \sum_{k=1}^{\infty} \left|\frac{k}{R}\right|^{-s} I(\mathcal{L}^k). \quad (1.43)$$

Now, the Atiyah-Singer index theorem gives a formula that in ten dimensions reads

$$I(\mathcal{L}^k) = \alpha k + \beta k^3 + \gamma k^5 \quad (1.44)$$

for certain rational numbers α, β, and γ. In particular, $I(\mathcal{L}^k)$ is a topological invariant. Together with the fact that the factor $|R|^s$ in (1.43) will play no role, this means that η will be a topological invariant. Using (1.44), we have

$$\frac{\eta(s)}{2} = |R|^s \sum_{k=1}^{\infty} \left(\alpha k^{-(s-1)} + \beta k^{-(s-3)} + \gamma k^{-(s-5)}\right). \quad (1.45)$$

As expected, the series converges for sufficiently large $\mathrm{Re}(s)$. In fact, in terms of the Riemann zeta function ζ, we have

$$\frac{\eta(s)}{2} = |R|^s \left(\alpha\zeta(s-1) + \beta\zeta(s-3) + \gamma\zeta(s-5)\right). \quad (1.46)$$

This has the expected analytic continuation to $s=0$. Since $\zeta(s)$ is regular at $s=-1,-3,-5,$ the factor $|R|^s$ can be dropped. Using the values of $\zeta(-1), \zeta(-3),$ and $\zeta(-5)$, we get

$$\frac{\eta}{2} = -\frac{\alpha}{12} + \frac{\beta}{120} - \frac{\gamma}{252}. \quad (1.47)$$

If V is an E_8 bundle with characteristic class a, and if we set $e = c_1(\mathcal{L})$, then we have

$$I(V \otimes \mathcal{L}^k) = \int_X \left(248 + 60a + 6a^2 + \frac{1}{3}a^3\right)\hat{A}(X)e^{ke}. \quad (1.48)$$

Here, $\hat{A}(X)$ can be expanded

$$\hat{A}(X) = 1 + \hat{A}_4 + \hat{A}_8 = 1 - \frac{\lambda}{12} + \left(\frac{7\lambda^2 - p_2}{1440}\right). \quad (1.49)$$

The index formula (1.48) can be written as $\alpha k + \beta k^3 + \gamma k^5$ with

$$\alpha = e\left(6a^2 + 60a\hat{A}_a + 248\hat{A}_8\right); \quad \beta = \frac{e^3}{6}\left(60a + 248\hat{A}_4\right); \quad \gamma = 248\frac{e^5}{5!}. \quad (1.50)$$

The Rarita-Schwinger operator on an eleven-manifold Y is equivalent to the Dirac operator coupled to $TY - 3\mathrm{O}$, and for Y a circle bundle over X, it is equivalent to the Dirac operator coupled to $TX - 2\mathrm{O}$. (In string theory terms, $-2\mathrm{O}$ is the contribution of the ghosts plus the dilatino). The appropriate index formula is therefore

$$I((TX - 2\mathrm{O}) \otimes \mathcal{L}^k) = \int_X \left(\sum_{i=1}^{5} 2\cosh(x_i) - 2\right)\hat{A}(X)e^{ke}, \quad (1.51)$$

where x_i are the Chern roots of TX, so $\lambda = p_1/2 = \sum_i x_i^2/2$ and $p_2 = \sum_{i<j} x_i^2 x_j^2$. We can evaluate the index formula as $\alpha' k + \beta' k^3 + \gamma' k^5$, with

$$\alpha' = e\left(248\hat{A}_8 - \lambda^2\right); \quad \beta' = \frac{2}{9}\lambda e^3; \quad \gamma' = 8\frac{e^5}{5!}. \quad (1.52)$$

These formulas can be used to evaluate the following phase

$$\Phi = \exp\left(2\pi i\left(\left(h_{E_8} + \eta_{E_8}\right)/4 + \left(h_{RS} + \eta_{RS}\right)/8\right)\right). \quad (1.53)$$

The RR fields of Type IIA are expressed in terms of a K-theory class x by $G/2\pi = \sqrt{\hat{A}}chx$. In comparing to M-theory, we will assume that $G_0 = 0$ and hence to evaluate G_2 and G_4, we can set $\hat{A}$ to 1. We then get

$$\frac{G_0}{2\pi}=0\,;\quad \frac{G_2}{2\pi}=c_1(x);\quad \frac{G_4}{2\pi}=\frac{1}{2}c_1(x)^2-c_2(x).\quad (1.54)$$

Let ω be a one-form on Y that is $\mathbf{S}^1$ invariant and restricts on each fiber of $Y\rightarrow X$ to $d\theta/2\pi$. The normalization is picked so that

$$\int_{S^1}\omega=1,\quad (1.55)$$

where $\mathbf{S}^1$ is any fiber of $Y\rightarrow X$. If we set $C'=\pi\omega\wedge d\omega$, and $G'=dC'$, then $G'/2\pi=\frac{1}{2}F\wedge F/(2\pi)^2$. Adding C' to the C-field on Y has the effect, therefore, of shifting $G/2\pi$ by $\frac{1}{2}c_1(\mathcal{L})^2$. Since C' is topologically trivial, the effect of the transformation $C\rightarrow C+C'$ on the phase of the M-theory effective action can be worked out from the form of the Chern-Simons coupling in a completely naive way. The Chern-Simons coupling is

$$L_{CS}=\frac{1}{6}\int_Y C\wedge\left(\left(\frac{G}{2\pi}\right)^2-\frac{1}{8}\left(p_2(Y)-\lambda^2\right)\right).\quad (1.56)$$

If C is shifted by $C\rightarrow C+C'$ with C' topologically trivial, we can calculate directly that

$$L_{CS}\rightarrow L_{CS}+\frac{1}{2}\int_Y C'\wedge\left[\left(\frac{G}{2\pi}\right)^2-\frac{1}{24}\left(p_2(Y)-\lambda^2\right)\right]+\frac{1}{2}\int_Y C'\wedge\frac{dC'}{2\pi}\wedge\frac{G}{2\pi}+\frac{1}{6}\int_Y C'\wedge\frac{dC'}{2\pi}\wedge\frac{dC'}{2\pi}.\quad (1.57)$$

We note that this equation can be connected with the equation concerning the physical vibrations of the bosonic strings, i.e. the following Ramanujan function:

$$24=\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Thence, we obtain:

$$L_{CS}\rightarrow L_{CS}+\frac{1}{2}\int_Y C'\wedge\left[\left(\frac{G}{2\pi}\right)^2-\frac{1}{24}\left(p_2(Y)-\lambda^2\right)\right]+\frac{1}{2}\int_Y C'\wedge\frac{dC'}{2\pi}\wedge\frac{G}{2\pi}+\frac{1}{6}\int_Y C'\wedge\frac{dC'}{2\pi}\wedge\frac{dC'}{2\pi}\Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.57b)$$

Using $C' = \pi\omega\wedge d\omega$, together with (1.55) and the fact that $d\omega$ represents $e = c_1(\mathcal{L})$, we can evaluate the integral over the fibers of $Y \to X$ and find that the shift in L_{CS} due to C' is

$$\Delta L_{CS} = 2\pi\int_X\left\{\frac{1}{4}e\left[(a-\lambda/2)^2-\frac{1}{24}\left(p_2-\lambda^2\right)\right]+\frac{1}{8}e^3(a-\lambda/2)+\frac{1}{48}e^5\right\}. \quad (1.58)$$

Combining the contributions of the η invariants, which give phase factors according to the following equation $\Phi = \exp\left(2\pi i\left(\left(h_{E_8}+\eta_{E_8}\right)/4+\left(h_{RS}+\eta_{RS}\right)/8\right)\right)$, with the phase we have just found in (1.58), the phase with which a configuration with specified $e = c_1(\mathcal{L})$ and characteristic class a of the M-theory four-form contributes to the partition function is

$$\Omega_M(e,a) = (-1)^{f(a)}\exp\left[2\pi i\int_X\left(\frac{e^5}{60}+\frac{e^3 a}{6}-\frac{11e^3\lambda}{144}-\frac{ea\lambda}{24}+\frac{e\lambda^2}{48}-\frac{e\hat{A}_8}{2}\right)\right]. \quad (1.59)$$

We note that this equation can be connected with the equation concerning the physical vibrations of the bosonic strings, i.e. the following Ramanujan function:

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Thence, we obtain:

$$\Omega_M(e,a) = (-1)^{f(a)}\exp\left[2\pi i\int_X\left(\frac{e^5}{60}+\frac{e^3 a}{6}-\frac{11e^3\lambda}{144}-\frac{ea\lambda}{24}+\frac{e\lambda^2}{48}-\frac{e\hat{A}_8}{2}\right)\right]\Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \quad (1.59b)$$

With regard the parity symmetry, the discussion concerning the $\mathbf{S}^1$ bundles Y over X, using the following bilinear identity $f(a+a') = f(a) + f(a') + \int_X a \cup Sq^2 a'$, we have

$$f(\lambda - e^2) = f(\lambda - e^2 - a) + f(a) + \int_X (\lambda - e^2 - a) \cup Sq^2 a . \quad (1.60)$$

The expression $\int_X e^2 \cup Sq^2 a$, with e, a integral classes, vanishes as a consequence of this equation $\int Sq^2 a \cup a' = \int a \cup Sq^2 a'$ and the following Cartan formula $Sq^i(y \cup y') = \sum_{j=0}^{i} Sq^j(y) \cup Sq^{i-j}(y')$, taking into account the fact that Sq^1 annihilates integral classes:

$$\int_X e^2 \cup Sq^2 a = \int_X Sq^2 e^2 \cup a = \int_X \left(\left(Sq^2 e\right) \cup e + e \cup \left(Sq^2 e\right)\right) \cup a = 0 . \quad (1.61)$$

Moreover, the following Stong's result $\int a \cup Sq^2 a = \int a \cup Sq^2 \lambda$, implies that

$$\int_X (\lambda - a) \cup Sq^2 a = 0 . \quad (1.62)$$

Therefore, the last term in (1.60) vanishes. Repeating these steps for $f(\lambda - e^2)$, we find

$$f(\lambda) = f(\lambda - e^2 - a) + f(e^2) + f(a) . \quad (1.63)$$

The variation of the additional phase factor in $\Omega_M(e, a)$, written in (1.59), can be evaluated by direct computation. Upon doing so and using (1.61), we find that $\Omega_M(e, a)$ transforms under parity by

$$\Omega_M(-e, \lambda - e^2 - a) = (-1)^{f(\lambda) + f(e^2)} \exp\left[2\pi i \int_X \left(\frac{2e^5}{15} - \frac{\lambda e^3}{18} + e\hat{A}_8\right)\right] \Omega_M(e, a) . \quad (1.64)$$

The phase factor written as an exponential in (1.64) is in fact half the index density of the Dirac operator on X coupled to the K-theory class $\mathcal{L}^2 - \mathcal{O}$, where $\mathcal{O}$ denotes a trivial complex line bundle and $c_1(\mathcal{L}) = e$. Therefore we can rewrite (1.64) as

$$\Omega_M(-e, \lambda - e^2 - a) = (-1)^{f(\lambda) + f(e^2) + I(\mathcal{L}^2 - \mathcal{O})} \Omega_M(e, a) . \quad (1.65)$$

We have that

$$f\left(e^2\right)=I\left(\mathcal{L}^2-\mathcal{O}\right) \quad (1.66)$$

for all integral two-classes e. Therefore (1.65) reduces to

$$\Omega_M\left(-e,\lambda-e^2-a\right)=(-1)^{f(\lambda)}\Omega_M(e,a). \quad (1.67)$$

The identity (1.66) needed above is part of a more general formula expressing the mod two index of an E$_8$ bundle with characteristic class $a=u\cup v$ in terms of elementary invariants. Here u,v are integral two-classes in $H^2(X;Z)$. Such a formula can be derived by constructing E$_8$ bundles using the embedding $SU(3)\subset E_8$. Let $\mathcal{L},\mathcal{M}$ be complex line bundles with $c_1(\mathcal{L})=u,\ c_1(\mathcal{M})=v$. We first construct the $SU(3)$ bundle

$$W=\mathcal{L}\oplus\mathcal{M}\oplus\overline{\mathcal{L}}\otimes\overline{\mathcal{M}}. \quad (1.68)$$

A direct computation shows that

$$c_2(W)=-\left(u^2+v^2+u\cup v\right). \quad (1.69)$$

Therefore, by embedding $SU(3)$ in E$_8$ (using the chain $SU(3)\subset E_6\times SU(3)\subset E_8$) we obtain an E$_8$ bundle with characteristic class $a=u^2+v^2+u\cup v$. The decomposition of the Lie algebra of E$_8$ in terms of representations of $SU(3)\times E_6$ is

$$\mathbf{248}=(\mathbf{8},\mathbf{1})\oplus(\mathbf{1},\mathbf{78})\oplus(\mathbf{3},\overline{27})\oplus(\overline{3},\mathbf{27}). \quad (1.70)$$

(Indeed, we have that 8*1 + 1*78 + 27*3 + 3*27 = 8 + 78 + 81 + 81 = 248). The mod two index of the E$_8$ bundle constructed above is the same as the mod two index with values in the $\mathbf{8}\oplus\mathbf{3}\oplus\overline{3}$ of $SU(3)$. This can be evaluated using the fact that the mod 2 index with values in $S\oplus\overline{S}$ (for any S) is the mod 2 reduction of the ordinary index with values in S. We get

$$f\left(u^2+v^2+u\cup v\right)=I\left(\mathcal{L}^2\otimes\mathcal{M}\oplus\mathcal{L}\otimes\mathcal{M}^2\right)+I\left(\mathcal{L}\otimes\overline{\mathcal{M}}\oplus\mathcal{L}\otimes\mathcal{M}\right)+I(\mathcal{L}\oplus\mathcal{M}) \quad \bmod 2 \quad (1.71)$$

As an ordinary index, the right hand side of equation (1.71) can be expressed in terms of elementary invariants. Setting $\mathcal{M}=\mathcal{O}$, and working mod two, we obtain

$$f\left(u^2\right)=I\left(\mathcal{L}^2-\mathcal{O}\right) \quad \bmod 2. \quad (1.72)$$

This is the formula (1.66) needed above. We record here a more general identity which is easily obtained from (1.71) and (1.72) using the bilinear identity for f and the index theorem. Applying twice the bilinear identity, and taking into account (1.61), we have

$$f\left(u^2+v^2+u\cup v\right)=f\left(u^2\right)+f\left(v^2\right)+f(u\cup v). \quad (1.73)$$

Combining (1.71) – (1.73), we arrive at

$$f(u \cup v) = I\left(\left(\mathcal{L} - \mathcal{O}\right) \otimes \left(\mathcal{M}^2 - \mathcal{O}\right)\right) + I\left(\left(\mathcal{L}^2 - \mathcal{O}\right) \otimes \left(\mathcal{M} - \mathcal{O}\right)\right) + I\left(\mathcal{L} \otimes \overline{\mathcal{M}} \oplus \mathcal{L} \otimes \mathcal{M}\right) \mod 2 . \quad (1.74)$$

The right hand side of (1.74) can be evaluated using the index theorem, obtaining

$$f(uv) = \int \left[uv(u+v)\left(uv - \frac{1}{4}\lambda\right) + \frac{3}{4}uv\left(u^3 + v^3\right) + \frac{1}{12}\left(uv^4 + 2u^3v^2 - \lambda uv^2\right) \right] \mod 2 . \quad (1.75)$$

The right hand side of this formula is symmetric under exchanging u and v since on a spin 10-manifold $I\left(\left(\mathcal{L} \otimes \overline{\mathcal{M}}\right) = I\left(\overline{\mathcal{L}} \otimes \mathcal{M}\right)\right) \mod 2$.

2. Link between the subsets of odd natural numbers and of squares, some equations concerning the Theorem that: "every sufficiently large odd positive integer can be written as the sum of three primes", and the possible method of factorization of a number. [4] [5]

We recall the following mathematical statement: ***the sum of the first N odd numbers is equal to N-squared*** (from: http://lidimatematici.wordpress.com)

This is a very interesting property, which shows the relationship between the subsets of the odd natural numbers and the squares. We have already seen how to build a geometric representation of the problem which seems to confirm the proposition, but we can not say to have actually demonstrated because we are not sure that the identity is valid for any N. We use the following mathematical tools, namely:

1) a function that builds the set of odd numbers
2) the distributive property of summations
3) the sum of the first natural numbers (Gauss formula)

The first step for the algebraic demonstration of the our proposition is to provide a formal representation. The idea behind the process of algebraic demonstration is the sum of the odd numbers using the tool of summation and verify that the sum leads on to numbers in quadratic form. To do this we need a function that lists the odd numbers. We already know how to build the even numbers, using the function 2n, *but if for any* n, 2n *is even then* 2n-1 *is necessarily odd, as is clear from the following table:*

N	2N-1
1	1
2	3
3	5
4	7
...	...

The difficulty that emerges is clear: we are not able to carry out the procedure explicitly for each value of N and therefore we must use a formal tool more sophisticated. For this purpose will use the summation, that we can use to represent the proposition as follows:

$$\sum_{k=1}^{N}(2k-1)=N^2$$

From here, just do the calculations by applying the distributive property:

$$\sum_{k=1}^{N}(2k-1)=2\sum_{k=1}^{N}k-\sum_{k=1}^{N}1$$

We already know the term summation of the natural numbers, is the formula of Gauss, while the sum repeated of the unit for N times is just equal to N. Substituting:

$$\sum_{k=1}^{N}(2k-1)=2\sum_{k=1}^{N}k-\sum_{k=1}^{N}1=2\sum_{k=1}^{N}k-N$$

performing the calculations and simplifying:

$$2\frac{N(N+1)}{2}-N=N^2+N-N=N^2$$

which is precisely our thesis.

With regard the Vinogradov theorem for that: "every sufficiently large odd positive integer can be written as the sum of three primes", recall that the weighted sum, $R_2(N)$, of the number of different ways the number N can be the sum of two primes can be represented by

$$R(N)=\int_0^1 F_N^2(x)e(-Nx)dx=\int_{\mathcal{M}}F_N^2(x)e(-Nx)dx+\int_m F_N^2(x)e(-Nx)dx. \quad (2.1)$$

Vinogradov showed that in the case of the ternary Goldbach problem, the integral over the minor arcs is bounded by

$$\int_m F_M^3(x)e(-Nx)dx<<\frac{M^2}{(\log M)^{(B/2)-5}}. \quad (2.2)$$

From this he showed that for $A>0$

$$R(M)=\int_0^1 F_M^3(x)e(-Mx)dx=\Im(M)\frac{M^2}{2}+O\left(\frac{M^2}{(\log M)^A}\right) \quad (2.3)$$

which is bounded away from 0 since as $M\to\infty$, $\log M\to\infty$ and so

$$\frac{M^2}{(\log M)^A}<<\Im(M)\frac{M^2}{2}. \quad (2.4)$$

Therefore the function $R(M) > 0$. Note that the function

$$R(M) = \sum_{\substack{p_1, p_2 \le M \\ p_1 + p_2 = M}} \log p e(px) \quad (2.5)$$

is greater than 0 if and only if the function

$$C(M) = \sum_{\substack{p_1, p_2 \le M \\ p_1 + p_2 = M}} 1 \qquad (2.6)$$

is greater than 0.
For the ternary case, it was possible to determine bounds for the integral over the minor arcs. If we simply take

$$\int_m F_M^3(x) e(-Nx) dx << \int_m |F_M(x)|^3 dx << \max\{|F_M(x)| : x \in m\} \int_m |F_M(x)|^2 (x) dx \quad (2.7)$$

then it is possible to find a bound for the maximum of the function F_M on the minor arcs and the integral

$$\int_0^1 |F_M(x)|^2 = \sum_{p \le M} (\log p)^2 \le \log M \sum_{p \le M} \log p$$

and by Chebyshev's Theorem,

$$<< M \log M \ .$$

Thence, we have that

$$\int_0^1 |F_M(x)|^2 = \sum_{p \le M} (\log p)^2 \le \log M \sum_{p \le M} \log p << M \log M \ . \quad (2.8)$$

And Vinogradov gives us a bound for the function $F_M(x)$:

$$F_M(x) << \left(\frac{N}{q^{1/2}} + N^{4/5} + N^{1/2} q^{1/2} \right) (\log N)^4 << \frac{N}{(\log N)^{(B/2)-4}} \ . \quad (2.9)$$

From above, we have

$$\max\{|F_M(x)| : x \in m\} \int_m |F_M(x)|^2 (x) dx << \frac{N}{(\log N)^{(B/2)-4}} \int_0^1 |F_M(x)|^2 dx << \frac{N^2}{(\log N)^{(B/2)-5}} \ . \quad (2.10)$$

When we try to apply this method to the binary Goldbach problem, we find that we cannot bound $R(N)$ away from 0 since we cannot determine the contribution from the minor arcs to be small enough that were it to be negative, it still would not contribute enough to diminish the value of $R(N)$ to less than 0. We have

$$R(N) = N\Im(N) + O\left(\frac{N}{(\log N)^{(1-\varepsilon)}B}\right) + O\left(\frac{N}{(\log N)^{C-5B}}\right) + \int_m F_N^2(x)e(-Nx)dx \quad (2.11)$$

and we cannot yet show that

$$\left|O\left(\frac{N}{(\log N)^{(1-\varepsilon)}B}\right)\right| + \left|O\left(\frac{N}{(\log N)^{C-5B}}\right)\right| + \left|\int_m F_N^2(x)e(-Nx)dx\right| < N\Im(N). \quad (2.12)$$

If we try to bound the integral over the minor arcs in the same way that we did for the ternary case we find that

$$\int_m F_N^2(x)e(-Nx)dx << \int_m |F_N(x)|^2 dx << \max\{|F_N(x)|\}\int_m |F_N(x)|dx \quad (2.13)$$

but it is not easy to bound $\int_m |F_N(x)|dx$ (2.14) enough. As we saw earlier,

$$\int_0^1 |F_N(x)|dx << N\log N \quad (2.15)$$

but looking back at the contribution from the major arcs we see that the term from the minor arcs is of larger order.

$$N\Im(N) + O\left(\frac{N}{(\log N)^{(1-\varepsilon)}B}\right) + O\left(\frac{N}{(\log N)^{C-5B}}\right) + O(N\log N) \quad (2.16)$$

is not necessarily larger than 0. Let us try another method to bound the contribution from the minor arcs. We employ the Cauchy-Schwartz inequality. Again, we have

$$\max\{|F_N(x)|\}\int_m |F_N(x)|dx = \max\{|F_N(x)|\}\int_m |F_N(x)| * 1dx \le \max\{|F_N(x)|\}\left(\int_m |F_N(x)|^2\right)^{1/2}\left(\int_m |1|^2\right)^{1/2} <<$$

$$<< \frac{N}{(\log N)^{(B/2)-4}}(N\log N)^{1/2} = \frac{N^{3/2}}{(\log N)^{(B/2)-(7/2)}} \quad (2.17)$$

thence:

$$\max\{|F_N(x)|\}\int_m |F_N(x)|dx = \max\{|F_N(x)|\}\int_m |F_N(x)| * 1dx = \frac{N^{3/2}}{(\log N)^{(B/2)-(7/2)}} \quad (2.18)$$

which is still larger than the term contributed by the major arcs.
We know that Vinogradov has proved that every sufficiently large odd positive integer can be written as the sum of three primes. Furthermore, we have the following theorem:
For $N = b_1 + b_2 + b_3 (\bmod k)$ *and an odd* N *sufficiently large, there holds*

$$J(N; k, b_1, b_2, b_3) > 0 \quad (2.19)$$

for all $k \le N^\delta$*, where* δ *is a very small, positive constant.*

Theorem 1.

Let $R = N^{5/48-\varepsilon}$. *Then the inequality* (2.19) *holds for all prime numbers* $k \le R$ *with at most* $O\left((\log N)^B\right)$ *exceptions for a certain* $B > 0$.

Theorem 1 is a direct consequence of Theorem 2 and 3.

Theorem 2.

For a given prime number $k \le N^{5/48-\varepsilon}$, *if none of the integers* $q \in A_k$ *is N-exceptional, then* (2.19) *is true for this* k.

Theorem 3.

There are at most $O\left((\log N)^B\right)$ *prime numbers* k, $1 \le k \le N$, *such that at least one of the integers* $q \in A_k$ *is N-exceptional. Here,* B *is a fixed positive constant.*

We see that:

$$R(N) = \int_{1/Q}^{1+(1/Q)} e(-N\alpha)\prod_{i=1}^{3} S(\alpha,b_i)d\alpha = \left(\sum_{i=1}^{3}\int_{E_i(k)}\right)e(-N\alpha)\prod_{i=1}^{3} S(\alpha,b_i)d\alpha +$$

$$+O\left(\sum_{i=3}^{4}\int_{E_i(k)}\left|\prod_{i=1}^{3} S(\alpha,b_i)\right|d\alpha\right) =: R_1(N)+R_2(N)+O(R_3(N)+R_4(N)). \quad (2.20)$$

Lemma 1.

Let $A > 0$ *be arbitrary and* $\alpha \in E_3(k) \cup E_4(k)$. *If in* $S(\lambda,b_i) = \sum_{\substack{N/4<n\le N \\ n\equiv b_i \,(\mathrm{mod}\, k)}} \Lambda(n)e(n\lambda)$, $\quad S(\lambda,\chi) =$ $= \sum_{N/4<n\le N} \Lambda(n)e(n\lambda)\chi(n)$, **(a)** $\quad G = G(A)$ *is chosen sufficiently large, then*

$$S(\alpha,b) << \frac{N}{kL^{A+1}}. \quad (2.21)$$

We derive from Lemma 1 and Dirichlet's lemma on rational approximation the following estimate:

$$\int_{E_3(k)\cup E_k(4)} |S(\alpha,b_1)S(\alpha,b_2)S(\alpha,b_3)|d\alpha << \max_{\alpha\in E_3(k)\cup E_4(k)} |S(\alpha,b_1)|\left(\int_0^1 |S(\alpha,b_2)|^2 d\alpha\right)^{1/2} \times$$

$$\times\left(\int_0^1 |S(\alpha,b_3)|^2 d\alpha\right)^{1/2} << \frac{N^2}{k^2L^A}. \quad (2.22)$$

Under the condition of Theorem 2, we have that

$$R_1(N)+R_2(N) = \sigma(N,k)\frac{N^2}{32} + O\left(N^2k^{-2}L^{-A}\right), \quad (2.23)$$

for any $A > 0$ and where $\sigma(N,k)$ is defined as in the following expressions

$$\sigma > 1 - \frac{EL_2}{L}, \quad |t| \le N, \quad L(s,\chi) = 0. \quad (2.23b)$$

Using

$$\frac{k}{\phi^3(k)} >> \sigma(N,k) >> \frac{k}{\phi^3(k)}, \quad (2.24)$$

Theorem 2 follows from (2.20), (2.22) and (2.23).

Lemma 2

There exists a positive number J *such that*:

a)

$$\sum_{q=1}^{\infty} \frac{1}{\phi(k/k_q)^3} A(N,q,k_q) = \sigma(N,k) \quad (2.25)$$

b)

$$\sum_{q \ge P} \frac{1}{\phi(k/k_q)^3} |A(N,q,k_q)| << (Pk)^{-1} L^J. \quad (2.26)$$

We first consider the set $E_1(k)$. If $k \dagger q$, we find

$$S\left(\frac{a}{q} + \lambda, b_i\right) = \sum_{g=1}^{q} e\left(\frac{ga}{q}\right) \sum_{\substack{N/4 < n \le N \\ n \equiv b_i (\mathrm{mod}\, k) \\ n \equiv g (\mathrm{mod}\, q)}} \Lambda(n) e(n\lambda) + O(L^2). \quad (2.27)$$

We shall introduce the Dirichlet characters $\xi \bmod k$ and $\chi \bmod q$ and obtain

$$S\left(\frac{a}{q} + \lambda, b_i\right) = \frac{1}{\phi(k)\phi(q)} C(\chi_0, q, 1, b_i, a) T(\lambda) + \frac{1}{\phi(k)\phi(q)} + \sum_{\xi \bmod k} \overline{\xi}(b_i) \sum_{\chi \bmod q} C(\overline{\chi}, q, 1, b_i, a) W(\lambda, \xi\chi) + O(L^2)$$

(2.28).

We obtain from (2.20)

$$R_1(N) = R_1^m(N) + R_1^e(N), \quad (2.29)$$

where

$$R_1^m(N) = \sum_{\substack{q \le P_1 \\ k|q}} \frac{1}{\phi^3(k)\phi^3(q)} \sum_{a=1}^{q} \prod_{i=1}^{3} C(\chi_0, q, 1, b_i, a) e\left(-\frac{a}{q}N\right) \times \int_{-1/qQ}^{1/qQ} T^3(\lambda) e(-N\lambda) d\lambda. \quad (2.30)$$

We evaluate the main term $R_1^m(N)$ using the following expression

$$\sum_{l \le P/r} A(N,l,k_l) << 1, \quad (2.31)$$

with $r=1$,

$$R_1^m(N)=\frac{1}{\phi^3(k)}\sum_{\substack{q\le P_1\\ k|q}}A(N,q,1)\int_{-1/2}^{1/2}T(\lambda)^3e(-N\lambda)d\lambda+O\left[\frac{1}{\phi^3(k)}\sum_{\substack{q\le P_1\\ k|q}}|A(N,q,1)|\int_{1/qQ}^{1/2}\frac{1}{|\lambda|^3}d\lambda\right]=$$

$$=\frac{1}{\phi^3(k)}\sum_{\substack{q\le P_1\\ k|q}}A(N,q,1)\frac{N^2}{32}+O\left[\frac{(P_1Q)^2}{\phi^3(k)}\right]=\frac{1}{\phi^3(k)}\sum_{\substack{q\le P_1\\ k|q}}A(N,q,1)\frac{N^2}{32}+O\left(N^2k^{-4}L^{-A}\right),\quad (2.32)$$

thence:

$$R_1^m(N)=\frac{1}{\phi^3(k)}\sum_{\substack{q\le P_1\\ k|q}}A(N,q,1)\int_{-1/2}^{1/2}T(\lambda)^3e(-N\lambda)d\lambda+O\left[\frac{1}{\phi^3(k)}\sum_{\substack{q\le P_1\\ k|q}}|A(N,q,1)|\int_{1/qQ}^{1/2}\frac{1}{|\lambda|^3}d\lambda\right]=$$

$$=\frac{1}{\phi^3(k)}\sum_{\substack{q\le P_1\\ k|q}}A(N,q,1)\frac{N^2}{32}+O\left(N^2k^{-4}L^{-A}\right),\quad (2.32b)$$

where we have used $T(\lambda)<<1/|\lambda|$ and

$$\int_{-1/2}^{1/2}T(\lambda)^3e(-N\lambda)d\lambda=\sum_{N/4<n_1<N/2}\;\sum_{N/4<n_2<3N/4-n_1}1=\frac{N^2}{32}+O(N).\quad (2.33)$$

Now, we know that:

$$R_2^m(N)=\sum_{\substack{q\le P_2\\ k|q}}A(N,q,k)\frac{N^2}{32}+O\left(N^2k^{-3}L^{-A}\right)\quad (2.34)$$

Using Lemma 2, we see from (2.32) and (2.34) that for a sufficiently large $G=G(A)$

$$R_1^m(N)+R_2^m(N)=\sigma(N,k)\frac{N^2}{32}+O\left(N^2k^{-2}L^{-A}\right).\quad (2.35)$$

Thence, form (2.32b) and (2.34), we obtain the following expression:

$$R_1^m(N)+R_2^m(N)=\frac{1}{\phi^3(k)}\sum_{\substack{q\le P_1\\ k|q}}A(N,q,1)\int_{-1/2}^{1/2}T(\lambda)^3e(-N\lambda)d\lambda+O\left[\frac{1}{\phi^3(k)}\sum_{\substack{q\le P_1\\ k|q}}|A(N,q,1)|\int_{1/qQ}^{1/2}\frac{1}{|\lambda|^3}d\lambda\right]+$$

$$+\sum_{\substack{q\le P_2\\ k|q}}A(N,q,k)\frac{N^2}{32}+O\left(N^2k^{-3}L^{-A}\right)=\sigma(N,k)\frac{N^2}{32}+O\left(N^2k^{-2}L^{-A}\right).\quad (2.36)$$

We note that this expression can be related with the Ramanujan's modular equation concerning the physical vibrations of the superstrings and we obtain the following interesting relationship:

$$R_1^m(N)+R_2^m(N)=\frac{1}{\phi^3(k)}\sum_{\substack{q\le P_1\\k|q}}A(N,q,1)\int_{-1/2}^{1/2}T(\lambda)^3e(-N\lambda)d\lambda+O\left[\frac{1}{\phi^3(k)}\sum_{\substack{q\le P_1\\k|q}}|A(N,q,1)|\int_{1/qQ}^{1/2}\frac{1}{|\lambda|^3}d\lambda\right]+$$

$$+\sum_{\substack{q\le P_2\\k|q}}A(N,q,k)\frac{N^2}{64}+O\left(N^2k^{-3}L^{-A}\right)\frac{1}{2}=\sigma(N,k)\frac{N^2}{64}+O\left(N^2k^{-2}L^{-A}\right)\frac{1}{2}\Rightarrow$$

$$\Rightarrow\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (2.37)$$

On an article in the network of "Gruppo Eratostene", which refers to the number RSA-2048 (*Numero RSA - 2048: una previsione sulla stima approssimativa dei suoi fattori p e q - Francesco Di Noto, Michele Nardelli*), mentions the presumed relationship among the factors p (min) and q (max). Dr. Servi believes that is possible factorize with a few tens of attempts any number of which we know that relationship.

We describe calculations developed by Servi.

The following formula is useful for the factorization of all numbers in which the ratio between the two factors is between (x^2) and (x^2+0.65) approximately, with x odd number.

$$F1=\left[4T_ix-4y+4x\pm\sqrt{\left(16y^2-32xy-32T_ixy+64Rx^2\right)}\right]\frac{1}{8x^2} \quad (2.38)$$

where:

F1= factor to find
R = difference between the square (q) immediately following N (Number to factorize) and N itself.
Ti = last term of the summation of odd numbers that form q. [Es. q=49=Som.1/13, from which, Ti =13.]
x^2(with x odd number. There is also a similar formula for x even.) = about the ratio between the two factors, not necessarily prime, of N.

y = all the even numbers included between (2x-2) and 0. [This implies that the possible numbers of y indicate the attempts to do. Only for one of them the formula give as result an integer equal to F1. However is not necessary to put y = 0 because in that case R is a number to the square.]

The formula is absolutely valid, however, the ratio between the two factors, mentioned above, which guarantees the validity varies according to a trend rather complicated to understand and that Dr. Servi deepen later.
The formula is based on some regularities related to the "R" and "Ti", numbers to factorize that are placed in a specific pattern of numbers.
Such regularities can be found easily by taking an any pair of numbers, x and y, and then calculating the product, increasing from time to time x of 1 and y of a square.
The products thus obtained, from a certain point, will give increasing values of "R" and "Ti", according to a clear pattern.

Es:

1*5=5; R=4;Ti=5
2*14=28 R=8;Ti=11
3*23=69 R=12;Ti=17
4*32=128 R=16;Ti=23
...

In this case, taking as square 9, we will have values of R and Ti which follow the particular type of pattern shown.

In the cases illustrated with the square equal to 9, the R and Ti are calculated as follows:

Since "R" is the distance from the number to factorize, N, at the square immediately next to it, q, and being "Ti" the last term of the summation of odd numbers that form q, we calculate:

3^2=9; 9-5=4=R; furthermore: 9=Som.1/5, (9 = 1+3+5) from which Ti=5
6^2=36; 36-28=8=R; furthermore: 36=Som.1/11, (36=1+3+5+7+9+11) from which Ti=11
9^2=81;81-69=12=R; furthermore; 81=Som. 1/17, (81=1+3+5+7+9+11+13+15+17) from which Ti=17
....

Continuing we get from time to time R+4 and Ti+6.

As regards the formula (2.38) one can proceed to its verification as follows.

Let us take a number F1, multiply it by a square q, and we add to the value obtained a number equal to a fifth of the starting number. (The latter additional value can also be greater or less to a fifth of F1, depending on the case, it does not hinder the validity of the formula.)

Es:

F1=531
F2=531*25+68=13343 (25 = 5^2)

(The number 68 is a random number. In this case, being less than 1/5 of starting number there is a greater certainty that the number N can be factorized since the ratio between these two factors is very near to the square, i.e. to 25. Can also be used a number greater than 1/5, fraction taken as an example, the important thing is that the ratio mentioned above is between x^2 and, approximately, depending on the case, x^2 +0.65. A similar formula can be used for cases in which the ratio between the factors will be between x^2 and, approximately, x^2-0.65).

N=F1*F2=7085133 (the root of 7085133 is about 2661,791314)

from which;

R=1111 (2662^2 = 7086244; 7086244 – 7085133 = 1111)
Ti=5323 (the last term in the summation of odd numbers that form q is equal to 2662*2 – 1 from what has been said before).

At this point we use the formula with x = 5 and y = all even numbers between 2 and 8. (That is, all the even numbers between 0 and (2x-2), i.e. between 0 and (2*5) – 2 = 8, thence all the even numbers between 2 and 8).

We have that:

F1=531 with y=2 and x=5.

Indeed:

$$\begin{aligned} F1 &= \left[4\cdot 5323\cdot 5-4\cdot 2+4\cdot 5\pm\sqrt{(16\cdot 4-32\cdot 10-32\cdot 5323\cdot 10+64\cdot 1111\cdot 25)}\right]/8\cdot 25 = \\ &= \left(106460-8+20\pm\sqrt{64-320-1703360+1777600}\right)/200 = \\ &= \left(106460-8+20\pm\sqrt{73984}\right)/200 = \\ &= (106460-8+20\pm 272)/200 = (106460-8+20-272)/200 = 106200/200 = 531. \end{aligned}$$

We note that $13343 / 531 = 25,12806 \cong 25$, thus the ratio between the two factors F1 and F2 will be between $(x^2) = 25$ and, approximately, $(x^2)+0,65 = 25,65$ with x odd number. (We specify that the number to factorize is N = 7085133).

Dr. Roberto Servi believes that the procedure and this formula is more efficient, though at the moment it is valid only to those particular N, of the existing systems.
However he is strongly convinced about the efficiency of the formula, and he hopes that someone may have the patience to validate.
Servi also believe that we can really to get variations of this formula that make it possible to factorize any number whose ratio between the factors is different from about x^2. Already with the above formula, provided that there isn't hugely different from x^2, it is sufficient to use R'=R+Ti+2 and Ti' = Ti +2 to obtain F1, and possibly increase from time to time R and Ti to obtain the desired result.
All this also is based on a scheme of numbers in which stand out the trends of the various R and Ti from which it is possible to derive the formulas (there are more than one) for the factorization.

We have the new version of the formula (2.38). In this formula is present the term N.

We define:

$Par=Ti^2-4Tix+2Ti-4x+1-4R+4y-4N$

The symbols are the usual as above

$F1=[-Par+/-(Par^2-64Nx^2)^{1/2}]/8x^2$,

We can rewrite the formula also as follows:

$F1=[-(Ti^2-4Tix+2Ti-4x+1-4R+4y-4N)+/-\sqrt{(Ti^2-4Tix+2Ti-4x+1-4R+4y-4N)^2-64Nx^2}]/8x^2$

or

$$F1 = \left[-\left(Ti^2 - 4Tix + 2Ti - 4x + 1 - 4R + 4y - 4N\right) \pm \right.$$

$$\left. \pm \sqrt{\left(Ti^2 - 4Tix + 2Ti - 4x + 1 - 4R + 4y - 4N\right)^2 - 64Nx^2}\right]/8x^2 . \quad (2.39)$$

This formula work also for more general cases, in which the ratio between the two factors of the number to factorize deviates further, compared to the tolerance provided, from the square. Furthermore, it is useful to observe that the formula work also with values negative of y, or odd, (-1, -2, -3,…), values that are able to be almost always low.

The following is another formula for the factorization of numbers for which the relationship between the two factors assume any value, thence all the numbers.

It is a first version in which there is the term N.

Finally, the following formula is valid also for the cases in which the ratio between the two factors of the number to factorize is lower than a given number, not necessarily integer, within the tolerance estimate previously of 0,65.

As for the other formulas, will be enough to calcolate R'=R+Ti+2 e Ti'=Ti+2, or, R''=R+(Ti+2)+(Ti+4) e Ti''=Ti+4 etc…, to factorize those numbers in which the ratio between the two factors deviates from the tolerance.

The formula is the following:

We define:

$$Par=4N+4R+4y-Ti^2-4TiAB-2Ti-4AB-1$$

thence:

$$F1=[-Par+/-(Par^2-64NA^2B^2)^{1/2}]/8B^2 \quad (2.40)$$

where:

B^2/A^2, with B>A, is equal or approximately equal to the ratio between the two factors, not necessarily primes, of the number to factorize.

y = all even numbers between (2AB – 2) and 0, when A and B are both odd numbers or even numbers.

y = all odd numbers between (2AB – 1) and 1, when A and B are, respectively, an even number and an odd number, or vice versa.

Example:

F1=687
F2=2369
N=F1*F2=1627503
R=673
Ti=2551

F2/F1=3,448... ;about equal to: $13^2/7^2$=3,448..

Thence, we obtain:

A=7; B=13

y = all the even numbers between 2*13*7 – 2 = 180 and 0;
R = a number about equal to the square root of N raised to the power of two subtracted to N, i.e.:
$\sqrt{N} = 1275{,}33 \cong 1276$; $1276^2 - N = 673$
[It is possible to obtain a number of possible values of y lower than that estimated above, depending on the cases examined.]

The formula give as a result an integer, equal to F1, with y = 48.

The values 13 and 7 are taken in random mode. In fact, knowing that the ratio between the two factors of the number to factorize is about 3,44.., taking up the case shown, we could attribute to A and B any two numbers such that B^2/A^2 is near to that relationship. [with B>A].

We could also choose:

B=28
A=15

where, $28^2/15^2$=3,48..

Or:

B=35
A=19
where, $35^2/19^2$=3,39..

In these cases, however, the possible values of y would have been in greater numbers. Therefore, it is useful to assign to A and B the integers smallest possible.

To obtain other examples will suffice to take a random square, multiply it by the value of the ratio between the two factors, and extract the square root of the number thus obtained. The nearest integer to the result provided, will be the one to be attributed to B

Example:

$F1=579$
$F2=139521$

$F2/F1=240.96..$

we consider:

$A=17^2$ [random number]

thence:

$(A^2*240.96..)^{1/2}=263,89..$

thence:

$A=17$
$B=264$

This factorization method will be used to make the present work even more clear, interesting and remarkable. Trying the formula on the values of E8XE8, this works very good. The formula is true also for the other values concerning this “Lie’s group”, which is fundamental in the superstring theory.

3. On some classifications of the numbers: perfect, defective, abundant and an infinite set of integers, each of which has many factorizations. [6]

The natural numbers, odd or even, were classified by Nicodemo, according to the sum of their divisors excluding the number itself, as defective, abundant or perfect. Perfect numbers are those for which the sum of the divisors is equal to the number itself, for example the number 6. A number slightly excess must have a sum of + 1 with respect to the number itself.
A defective number has a sum less than the number itself, while an abundant number has a sum greater than the number itself.
Euclide has observed for the perfect numbers a simple rule: If we sum an odd number of numbers, starting from 1, each of which is the double of the previous, obtaining a prime number and then multiply the result of the sum for the last summand of sum, we get a perfect number:

$1 + 2 + 4 = 7$
$7 * 4 = 28$

(the divisors of 28 are 1, 2, 4, 7, 14 and 28. The sum of the divisors, excluding 28, is: 1+2+4+7+14 = 28)

$1 + 2 + 4 + 8 + 16 = 31$
$31 * 16 =$ **496**

(the divisors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, 248 and 496. The sum of the divisors, excluding 496, is: 1+2+4+8+16+31+62+124+248 = 496)

1 + 2 + 4 + 8 + 16 + 32 + 64 = 127
127 * 64 = 8128

Euclid has observed also that a perfect number is the sum of consecutive numbers:

1 + 2 + 3 = 6

1 + 2 + 3 + 4 + 5 + 6 + 7 = 28

1 + 2 + 3 + 4 + 5 + ... + 31 = **496**

An even perfect number follows, in fact, the Euler's rule:

If p is prime number and the Mersenne’s number Mp = 2^p - 1 is prime number, a perfect number is given by:

$$Np = \left[2^{(p-1)}\right] \times \left[\left(2^{p}\right) - 1\right]$$

For example, with the Euler’s rule, we obtain:

p=2 Mp=3 Np=2*3=6
p=3 Mp=7 Np=7*4=28
p=5 Mp=31 Np=16*31=**496**

It can also be observed that the product is between a prime number, not classifiable between the previous ones, and a number power of 2, which is always defective.
Nicodemo has observed that Np is always going to 6 or 8 but the rule is not enough to have a perfect number.
Once that we have found a new Mersenne prime number, we obtain also the perfect number. A multiple of an abundant number is still abundant, because the function “sum of divisors”, including n, is multiplicative, but not completely, if GCD(a, b) = 1 or a and b coprime, then
s(a*b) = s(a)*s(b)

For example, following the definition

GCD(321,992)=1

s(992)=1024+**992**=2016

(Indeed, for the number 992 the divisors are 1, 2, 4, 8, 16, 31, 32, 62, 124, 248, 496 and 992. The sum of proper divisors is 1+2+4+8+16+31+32+62+124+248+496 = 1024)

s(321)=111+321=432

(Indeed, for the number 321 the divisors are 1, 3, 107 and 321. The sum of proper divisors is 1+3+107 = 111) s(992)*s(321)=2016*432=870912

s(318432)=s(992*321)=552480+318432=870912

s(318432)=s(992*321)=s(992)*s(321)

992 is an abundant number (is an abundant number because it is smaller than the sum of its proper divisors); and also 318432 is abundant, obtained as a multiple of **992**. We note that the only perfect numbers between 1 and 1000 are 6, 28 and **496.**

(see: Rosario Turco: http://www.scribd.com/doc/46194270/Dai-numeri-multipli-di-6-alla-Riemann-Hypothesis).

Now we explicitly describe an infinite set of integers, each of which has many factorizations.

THEOREM

Let x be large and let

$$\varepsilon = \frac{1}{\log_2 x}\left(\log_3 x + \log_4 x + \frac{\log_4 x}{\log_3 x}\right), \quad t = \left(1+\varepsilon \log_2^2 x\right)^{1/\varepsilon}, \quad k = \log x / \log_2^2 x, \quad n = \prod_{p \le t} p^{\left[kp^{\varepsilon-1}\right]}. \quad (3.1)$$

Then there is an absolute constant C such that

$$f(n) \ge n \cdot \exp\left\{-\frac{\log n}{\log_2 n}\left(\log_3 n + \log_4 n + \frac{\log_4 n - 1}{\log_3 n} + C\frac{\log_4 n}{\log_3^2 n}\right)\right\}. \quad (3.2)$$

Now we want to analyze the proof of this Theorem.
We first show that $\log n$ cannot be too much bigger than $\log x$. In fact, we show

$$\log n \le \log x + O\left(\log x / \log_2^2 x\right). \quad (3.3)$$

To see this, note that

$$\log n \le \sum_{p \le t} kp^{\varepsilon-1} \log p. \quad (3.4)$$

Now if we let $\pi(s) = li(s) + \Delta(s)$, then

$$\sum_{p \le t} p^{\varepsilon-1} \log p = \int_{2^-}^{t} s^{\varepsilon-1} \log s\, d\pi(s) = \int_2^t s^{\varepsilon-1} ds + \int_{2^-}^{t} s^{\varepsilon-1} \log s\, d\Delta(s). \quad (3.5)$$

We note that

$$\log t = \frac{1}{\varepsilon}\left(\log \varepsilon + 2\log_3 x + O\left(\frac{1}{\log_2 x \log_3 x}\right)\right) =$$

$$= \frac{\log_2 x}{\log_3 x + \log_4 x + \log_4 x / \log_3 x} \times \left(\log_3 x + \log_4 x + \frac{\log_4 x}{\log_3 x} - \frac{\log_4^2 x}{2\log_3^2 x} + O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right) =$$

$$= \log_2 x\left(1 - \frac{\log_4^2 x}{2\log_3^3 x} + O\left(\frac{\log_4 x}{\log_3^3 x}\right)\right). \quad (3.6)$$

With this estimate and the fact that $t^e \approx \log_2 x \log_3 x$, we have for $2 \le s \le t$ and t large,

$$s^e = (\log s)^{\varepsilon \log s / \log\log s} \le (\log s)^{\varepsilon \log t / \log\log t} = (\log s)^{1+o(1)}. \quad (3.7)$$

Also using $|\Delta(s)| << s/\log^4 s$, we have

$$\int_{2^-}^t s^{\varepsilon-1} \log s d\Delta(s) = t^{\varepsilon-1} \log t \Delta t - 2^{\varepsilon-1} \log 2\Delta(2) - \int_2^t s^{\varepsilon-2}((\varepsilon-1)\log s + 1)\Delta(s)ds =$$

$$= O(1) + O\left(\int_2^t \frac{s^\varepsilon}{s\log^3 s} ds\right) = O\left(\int_2^t \frac{1}{s\log^{3/2} s} ds\right) = O(1). \quad (3.8)$$

Using (3.8) in (3.5) we have

$$\sum_{p\le t} p^{\varepsilon-1} \log p = \int_2^t s^{\varepsilon-1} ds + O(1) = \frac{1}{\varepsilon} t^\varepsilon - \frac{1}{\varepsilon} 2^\varepsilon + O(1) = \frac{1}{\varepsilon}(t^\varepsilon - 1) + O(1) = \log_2^2 x + O(1). \quad (3.9)$$

Thus (3.3) follows from (3.4) and (3.9).
Recall that the Piltz divisor function $d_l(n)$ counts the number of factorizations of n into exactly l positive factors, where 1 is allowed as a factor and different permutations of a single factorization count separately. It is easily shown that $d_l(n)$ is multiplicative and that

$$d_l(p^a) = \binom{l+a-1}{a-1}. \quad (3.10)$$

Moreover, we evidently have for any choice of l that

$$f(n) \ge d_l(n)/l!. \quad (3.11)$$

Thus

$$\log f(n) \ge \log d_{[k]}(n) - \log[k]! = \sum_{p\le t} \log\binom{[k] + [kp^{\varepsilon-1}] - 1}{[kp^{\varepsilon-1}] - 1} - \log[k]!. \quad (3.12)$$

Now if $a, b \ge 2$, then

$$\log\binom{[a]+[b]-1}{[b]-1} = (a+b)\log(a+b) - a\log a - b\log b + O(\log(a+b)) \quad (3.13)$$

so that

$$\log\begin{pmatrix}[k]+[kp^{\varepsilon-1}]-1\\ [kp^{\varepsilon-1}]-1\end{pmatrix}=k(1+p^{\varepsilon-1})(\log k+\log(1+p^{\varepsilon-1}))-k\log k-kp^{\varepsilon-1}(\log k+(\varepsilon-1)\log p)+O(\log k)=$$
$$=k(1+p^{\varepsilon-1})\log(1+p^{\varepsilon-1})+k(1-\varepsilon)p^{\varepsilon-1}\log p+O(\log k). \quad (3.14)$$

Now

$$\sum_{p\le t}(1+p^{\varepsilon-1})\log(1+p^{\varepsilon-1})=\sum_{p\le t}p^{\varepsilon-1}+O(1)=\int_{2^-}^{t}s^{\varepsilon-1}d\pi(s)+O(1)=\int_{2}^{t}\frac{s^{\varepsilon-1}}{\log s}ds+\int_{2^-}^{t}s^{\varepsilon-1}d\Delta(s)+O(1). \quad (3.15)$$

The last integral is

$$t^{\varepsilon-1}\Delta(t)-2^{\varepsilon-1}\Delta(2)-\int_{2}^{t}(\varepsilon-1)s^{\varepsilon-2}\Delta(s)ds=O(1)+O\left(\int_{2}^{t}\frac{s^{\varepsilon}}{s\log^4 s}ds\right)=O(1), \quad (3.16)$$

by (3.8). Also

$$\int_{2}^{t}\frac{s^{\varepsilon-1}}{\log s}ds=\int_{2^\varepsilon}^{t^\varepsilon}\frac{du}{\log u}-li(t^\varepsilon)+O\left(\int_{2^\varepsilon}^{2}\frac{du}{\log u}\right)=\frac{t^\varepsilon}{\varepsilon\log t-1}\left(1+O\left(\frac{1}{\varepsilon^2\log^2 t}\right)\right)+O(|\log\varepsilon|). \quad (3.17)$$

Thus using (3.6), we have

$$\sum_{p\le t}(1+p^{\varepsilon-1})\log(1+p^{\varepsilon-1})=\frac{\log_2 x(\log_3 x+\log_4 x+\log_4 x/\log_3 x)(1+O(1/\log_3^2 x))}{\log_3 x+\log_4 x+\log_4 x/\log_3 x-1+O(\log_4^2 x/\log_3^2 x)}=$$
$$=\log_2 x\left(1+\frac{1}{\log_3 x}+O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right)\left(1+O\left(\frac{1}{\log_3^2 x}\right)\right)=\log_2 x\left(1+\frac{1}{\log_3 x}+O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right). \quad (3.18)$$

Thence, from (3.15) and (3.17), we can rewrite the eq.(3.15) also as follows:

$$\sum_{p\le t}(1+p^{\varepsilon-1})\log(1+p^{\varepsilon-1})=\sum_{p\le t}p^{\varepsilon-1}+O(1)=\int_{2^-}^{t}s^{\varepsilon-1}d\pi(s)+O(1)=\int_{2^\varepsilon}^{t^\varepsilon}\frac{du}{\log u}-li(t^\varepsilon)+O\left(\int_{2^\varepsilon}^{2}\frac{du}{\log u}\right)+$$
$$+\int_{2}^{t}s^{\varepsilon-1}d\Delta(s)+O(1)=\log_2 x\left(1+\frac{1}{\log_3 x}+O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right). \quad (3.18b)$$

Thus, from (3.3), (3.9) – (3.14), and (3.18), we have

$$\log f(n)\ge\frac{\log x}{\log_2 x}\left(1+\frac{1}{\log_3 x}+O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right)+\log x-\frac{\log x}{\log_2 x}\left(\log_3 x+\log_4 x+\frac{\log_4 x}{\log_3 x}\right)+O\left(\frac{\log x}{\log_2^2 x}\right)+$$
$$-\frac{\log x}{\log_2^2 x}(\log_2 x+O(\log_3 x))=\log x-\frac{\log x}{\log_2 x}\left(\log_3 x+\log_4 x+\frac{\log_4 x-1}{\log_3 x}+O\left(\frac{\log_4 x}{\log_3^2 x}\right)\right)\ge$$
$$\ge\log n-\frac{\log n}{\log_2 n}\left(\log_3 n+\log_4 n+\frac{\log_4 n-1}{\log_3 n}+O\left(\frac{\log_4 n}{\log_3^2 n}\right)\right), \quad (3.19)$$

which proves the theorem.

4. On some interesting mathematical applications concerning the possible method of factorization of a number to the number of dimensions of the Lie's Group E8 [7]

Let us now analyze the number 496 and the other numbers related to the number of dimensions of the Lie's Group E8.
We analyze before the formula (2.38)

$$F1 = \left[4T_i x - 4y + 4x \pm \sqrt{\left(16y^2 - 32xy - 32T_i xy + 64Rx^2\right)}\right]\frac{1}{8x^2} ,$$

1) x = 5 (odd number) ; y = 2 (even number between 2x – 2 = (2 * 5) – 2 and 0)
F1 = 496; F2 = 496 * 25 + 48 = 12448; (we note that 25 = x^2 and that 1/5 * 496 = 99,2 and 48 < 99,2)
N = F1 * F2 = 6174208; R = 1017; (√N = 2484,795; 2485^2 – N = 1017); T_i = 4969 (2485 * 2 – 1).

Thence, we have the following number: N = 6174208. The square root of this number is √N = 2484,795. We take the number immediately following, i.e. 2485 and raise it to the squared and subtracting it to N, we obtain 2485^2 – N = 1017, that we have called R. Instead, 2485 * 2 – 1 = 4969 that we have called T_i. Now, for x we choose an odd number, in this example 5, and y must be between 0 and (2 * 5)– 2 = 8. We take, in this example, y = 2. Furthermore, N/496 = 12448 that we can obtain as follows: 496 * x^2 = 496 * 25 = 12400 to which we add 48 which is smaller than 1/5 * 496 = 99,2. Thence: 496 * 25 + 48 = 12400 + 48 = 12448.
Now, we apply the formula (2.38) and obtain:

$$F1 = \left[4\cdot 4969\cdot 5 - 4\cdot 2 + 4\cdot 5 \pm \sqrt{16\cdot 4 - 32\cdot 10 - 32\cdot 10\cdot 4969 + 64\cdot 1017\cdot 25}\right]\frac{1}{8\cdot 25} =$$
$$= \left[99380 - 8 + 20 \pm \sqrt{64 - 320 - 1590080 + 1627200}\right]\frac{1}{200} = \left(99380 - 8 + 20 \pm \sqrt{36864}\right)/200 =$$
$$= (99380 - 8 + 20 \pm 192)/200 = (99380 - 8 + 20 - 192)/200 = 496 .$$

2) Now we analyze the formula for F1 = 248.

F1 = 248; F2 = 248 * 25 + 48 = 6248; N = F1 * F2 = 1549504; R = 521; (√N = 1244,79; 1245^2 – N = 521); T_i = 2489 (1245 * 2 – 1). For x = 5 and y = 2. We obtain:

$$F1 = \left[4\cdot 2489\cdot 5 - 4\cdot 2 + 4\cdot 5 \pm \sqrt{16\cdot 4 - 32\cdot 10 - 32\cdot 2489\cdot 10 + 64\cdot 521\cdot 25}\right]\frac{1}{8\cdot 25} =$$
$$= \left[49780 - 8 + 20 \pm \sqrt{64 - 320 - 796480 + 833600}\right]/200 = \left[49780 - 8 + 20 \pm \sqrt{36864}\right]/200 =$$
$$= (49780 - 8 + 20 - 192)/200 = 49600/200 = 248 .$$

3) Now we analyze the formula for F1 = 128.

F1 = 128; F2 = 128 * 25 + 48 = 3248; N = F1 * F2 = 415744; R = 281 (√N = 644,78; 645^2 – N = 281); T_i = 1289 (645 * 2 – 1 = 1289). For x = 5 and y = 2, we obtain:

$$F1 = \left[4\cdot 5\cdot 1289 - 4\cdot 2 + 4\cdot 5 \pm \sqrt{16\cdot 4 - 32\cdot 10 - 32\cdot 10\cdot 1289 + 64\cdot 281\cdot 25}\right]\frac{1}{8\cdot 25} =$$
$$= \left[25780 - 8 + 20 \pm \sqrt{64 - 320 - 412480 + 449600}\right]/200 = \left[25780 - 8 + 20 \pm \sqrt{36864}\right]/200 =$$
$$= (25780 - 8 + 20 - 192)/200 = 25600/200 = 128.$$

4) Now we analyze the formula for F1 = 120.

F1 = 120; F2 = 120 * 25 + 48 = 3048; N = F1 * F2 = 365760; R = 265 (√N = 604,780; 605^2 – N = 265); T_i = 1209 (605 * 2 – 1 = 1209). For x = 5 and y = 2, we obtain:

$$F1 = \left[4\cdot 5\cdot 1209 - 4\cdot 2 + 4\cdot 5 \pm \sqrt{16\cdot 4 - 32\cdot 10 - 32\cdot 10\cdot 1209 + 64\cdot 265\cdot 25}\right]\frac{1}{8\cdot 25} =$$
$$= \left[24180 - 8 + 20 \pm \sqrt{64 - 320 - 386880 + 424000}\right]/200 = \left(24180 - 8 + 20 \pm \sqrt{36736}\right)/200 =$$
$$= (24192 \pm 191{,}66)/200 = (24192 - 192)/200 = 24000/200 = 120.$$

5) Now we analyze the formula for F1 = 84.

F1 = 84; F2 = 84 * 25 + 48 = 2148; N = F1 * F2 = 180432; R = 193 (√N = 424,772; 425^2 – N = 193); T_i = 849 (425 * 2 – 1 = 849). For x = 5 and y = 2, we obtain:

$$F1 = \left[4\cdot 5\cdot 849 - 4\cdot 2 + 4\cdot 5 \pm \sqrt{16\cdot 4 - 32\cdot 10 - 32\cdot 10\cdot 849 + 64\cdot 193\cdot 25}\right]\frac{1}{8\cdot 25} =$$
$$= \left[16980 - 8 + 20 \pm \sqrt{64 - 320 - 271680 + 308800}\right]/200 = \left[16980 + 12 \pm \sqrt{36736}\right]/200 =$$
$$= (16980 + 12 \pm 191{,}66)/200 = (16992 - 192)/200 = 16800/200 = 84.$$

6) Now we analyze the formula for F1 = 81.

F1 = 81; F2 = 81 * 25 + 48 = 2073; N = F1 * F2 = 167913; R = 187 (√N = 409,771; 410^2 – N = 187); T_i = 819 (410 * 2 – 1 = 819). For x = 5 and y = 2, we obtain:

$$F1 = \left[4\cdot 819\cdot 5 - 4\cdot 2 + 4\cdot 5 \pm \sqrt{16\cdot 4 - 32\cdot 10 - 32\cdot 10\cdot 819 + 64\cdot 25\cdot 187}\right]\frac{1}{8\cdot 25} =$$
$$= \left[16380 - 8 + 20 \pm \sqrt{64 - 320 - 262080 + 299200}\right]/200 = \left[16380 - 8 + 20 \pm \sqrt{36736}\right]/200 =$$
$$= (16380 - 8 + 20 \pm 191{,}66)/200 = (16392 - 191{,}66)/200 = (16392 - 192)/200 = 16200/200 = 81.$$

7) Now we analyze the formula for F1 = 80.

F1 = 80; F2 = 80 * 25 + 48 = 2048; N = F1 * F2 = 163840; R = 185 (√N = 404,77; 405^2 – N = 185); T_i = 809 (405 * 2 – 1 = 809). For x = 5 and y = 2, we obtain:

$$F1 = \left[4\cdot 809\cdot 5 - 4\cdot 2 + 4\cdot 5 \pm \sqrt{16\cdot 4 - 32\cdot 10 - 32\cdot 10\cdot 809 + 64\cdot 25\cdot 185}\right]\frac{1}{8\cdot 25} =$$
$$= \left[16180 - 8 + 20 \pm \sqrt{64 - 320 - 258880 + 296000}\right]/200 = \left[16180 + 12 \pm \sqrt{36736}\right]/200 =$$

$$=(16180+12\pm 191{,}66)/200=(16180+12-192)/200=(16192-192)/200=16000/200=80\,.$$

8) Now we analyze the formula for F1 = 78.

F1 = 78; F2 = 78 * 25 + 48 = 1998; N = F1 * F2 = 155844; R = 181 (√N = 394,77; 395^2 – N = 181); T_i = 789 (395 * 2 – 1 = 789). For x = 5 and y = 2, we obtain:

$$F1=\left[4\cdot 789\cdot 5-4\cdot 2+4\cdot 5\pm\sqrt{16\cdot 4-32\cdot 10-32\cdot 10\cdot 789+64\cdot 181\cdot 25}\right]\frac{1}{8\cdot 25}=$$
$$=\left[15780-8+20\pm\sqrt{64-320-252480+289600}\right]/200=\left[15780-8+20\pm\sqrt{36736}\right]/200=$$
$$=(15780+12\pm 191{,}66637)/200\cong(15792-192)/200=15600/200=78\,.$$

9) Now we analyze the formula for F1 = 50.

F1 = 50; F2 = 50 * 25 + 8 = 1258; N = F1 * F2 = 62900; R = 101 (√N = 250,798; 251^2 – N = 101); T_i = 501 (251 * 2 – 1 = 501). For x = 5 and y = 2, we obtain:

$$F1=\left[4\cdot 501\cdot 5-4\cdot 2+4\cdot 5\pm\sqrt{16\cdot 4-32\cdot 10-32\cdot 10\cdot 501+64\cdot 101\cdot 25}\right]\frac{1}{8\cdot 25}=$$
$$=\left[10020-8+20\pm\sqrt{64-320-160320+161600}\right]/200=\left[10020+12\pm\sqrt{1024}\right]/200=$$
$$=(10020+12\pm 32)/200=(10032-32)/200=10000/200=50\,.$$

Now we analyze with the formula (2.39)

$$\text{F1}=\Big[-\left(Ti^2-4Tix+2Ti-4x+1-4R+4y-4N\right)$$
$$\pm\sqrt{\left(Ti^2-4Tix+2Ti-4x+1-4R+4y-4N\right)^2-64Nx^2}\Big]/8x^2,$$

the value 496, thence for F1 = 496. We obtain:

F1 = 496; F2 = 496 * 25 + 48 = 12448; N = F1 * F2 = 6174208; R = 1017; T_i = 4969; for x = 5 and y = 2.

$$\text{F1}=\Big[-\left(Ti^2-4Tix+2Ti-4x+1-4R+4y-4N\right)$$
$$\pm\sqrt{\left(Ti^2-4Tix+2Ti-4x+1-4R+4y-4N\right)^2-64Nx^2}\Big]/8x^2=$$
$$=\Big[-\left(4969^2-4\cdot 4969\cdot 5+2\cdot 4969-4\cdot 5+1-4\cdot 1017+4\cdot 2-4\cdot 6174208\right)\pm$$
$$\pm\sqrt{\left(4969^2-4\cdot 4969\cdot 5+2\cdot 4969-4\cdot 5+1-4\cdot 1017+4\cdot 2-4\cdot 6174208\right)^2-64\cdot 6174208\cdot 25}\Big]\frac{1}{8\cdot 25}=$$
$$=\Big[-\left(24690961-99380+9938-20+1-4068+8-24696832\right)\pm$$
$$\pm\sqrt{\left(24690961-99380+9938-20+1-4068+8-24696832\right)^2-64\cdot 25\cdot 6174208}\Big]/8\cdot 25=$$
$$=\left[-(-99392)\pm\sqrt{9878769664-9878732800}\right]/200=\left(99392\pm\sqrt{36864}\right)/200=$$
$$=(99392\pm 192)/200=(99392-192)/200=99200/200=496\,.$$

Now we analyze with the formula (2.40) the values 496, 128 and 120.

For F1 = 496, and A = 5; B = 13 and y = 48, we obtain:

F1 = 496; F2 = 496* (B^2/A^2) = 496 * (169/25) = 496 * 6,76 = 3352,96 = 3353;
N = F1 * F2 = 496 * 3353 = 1663088; R = 1012 (√N = 1289,607; 1290^2 – N = 1012); T_i = 2579 (1290 * 2 – 1 = 2579).

Now:

$$Par=4N+4R+4y-Ti^2-4TiAB-2Ti-4AB-1$$

thence:

$$F1=[-Par+/-(Par^2-64NA^2B^2)^{1/2}]/8B^2$$, we obtain:

$$\text{Par} = \left(4\cdot 1663088 + 4\cdot 1012 + 4\cdot 48 - 2579^2 - 4\cdot 65\cdot 2579 - 2\cdot 2579 - 4\cdot 65 - 1\right) =$$
$$= \left(6652352 + 4048 + 192 - 6651241 - 670540 - 5158 - 260 - 1\right) = -670608;$$

$$F1 = \left|670608 \pm \sqrt{449715089664 - 64\cdot 1663088\cdot 4225}\right|/8\cdot 169 =$$
$$= \left|670608 \pm \sqrt{449715089664 - 449698995200}\right|/1352 = \left|670608 \pm \sqrt{16094440}\right|/1352 =$$
$$= \left(670608 \pm 4011{,}7876\right)/1352 = 493{,}0445;\ 498{,}9791$$ Ma = (493,0445 + 498,9791) / 2 ≈ 496.

2) For F1 = 128, we obtain:

F1 = 128; F2 = 865; F1 * F2 = 110720; R = 169 (√N = 332,746; 333^2 – N = 169); T_i = 665;
F2 / F1 = 6,7578 ≈ 6,76 = 13^2 / 5^2; A = 5; B = 13 and y = 48. Thence:

$$\text{Par} = \left(4\cdot 110720 + 4\cdot 169 + 4\cdot 48 - 665^2 - 4\cdot 665\cdot 65 - 2\cdot 665 - 4\cdot 65 - 1\right) =$$
$$= \left(442880 + 676 + 192 - 442225 - 172900 - 1330 - 260 - 1\right) = -172968;$$

$$F1 = \left|172968 \pm \sqrt{(29917929024 - 64\cdot 110720\cdot 4225)}\right|/8\cdot 169 =$$
$$= \left|172968 \pm \sqrt{20758976}\right|/8\cdot 169 = \left(172968 \pm 4556{,}2019\right)/1352 = 131{,}3048;\ 124{,}5649;$$

Ma = 127,93≈128.

3) For F1 = 120, we obtain:

F1 = 120; F2 = 811; N = F1 * F2 = 97320; R = 24 (√N = 311,961; 312^2 – N = 24); T_i = 623;
F2 / F1 = 6,7583 ≈ 6,76 = 13^2 / 5^2; A = 5; B = 13 and y = 48. Thence:

$$\text{Par} = \left(4\cdot 97320 + 4\cdot 24 + 4\cdot 48 - 623^2 - 4\cdot 623\cdot 65 - 2\cdot 623 - 4\cdot 65 - 1\right) =$$
$$= \left(389280 + 96 + 192 - 388129 - 161980 - 1246 - 260 - 1\right) = -162048;$$

$$F1 = \left|162048 \pm \sqrt{(26259554304 - 64\cdot 97320\cdot 4225)}\right|/8\cdot 169 = \left|162048 \pm \sqrt{55773696}\right|/1352 =$$
$$= \left(162048 \pm 7468{,}178\right)/1352 = 125{,}38178;\ 114{,}33418;$$ Ma = 119,8579 ≈ 120.

We note that: $(\Phi)^{13} \cdot \frac{5}{3} = 521{,}0019193787 \cdot \frac{5}{3} = 868{,}3365322978$ and $\ln 868{,}3365322978 \cong 6{,}76$;

with $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}61803398...$ Furthermore, 13 and 5 are both Fibonacci's number and y = 48 correspond to the physical vibrations of the bosonic strings.

Furthermore, it is possible note that in order to factorize any number without the knowledge of the relationship between its two factors, is need to find a simple and immediate way to calculate which square is equal to a multiple of the number to factorize, added with a much smaller number linked to it .

Q=XN+N'

The N is divided into four classes.
One of these is:

N=9X+5

Now we see how could be factorized the numbers 496 and 248. (Note that 248 is half of 496, so the reasoning can be done very well also only on 496)
Trying to factorize 496 with the system before mentioned, is obtained, in one case, the following equality:

11*496=88*62.

From here we can to obtain the factors of 496.
The method is simple. The previous factorization method was based on the regularity, referring to the R and Ti present in the columns of products in which the first term increased by 1 and the second of a square.

1*1 R=3 Ti=3
2*10 R=5 Ti=9
3*19 R=7 Ti=15
4*28 R=9 Ti=21
...
...

in the case in which Q is equal to 9.
Now, in this column there will be some products for which the values of R will be of the squares from which one can derive the factors other than those present in the column itself.
For example in the case above of 4 * 28 we obtain R = 9, which yields to 14 * 8.
In these particular cases if the numbers that make up the second term of the product considered, in this case the number 28, are present separately in the product obtained by R, it is possible easily identify. In other words, by exploiting the quadratic form of R, we can obtain two versions of the same number, in this case 112 = 4 * 28 = 14 * 8, from which one can go back to the various primes that compose it.
Let us assume that we want to enter the number to factorize, in this case 496, in the right part of the column taken as an example, i.e. between the terms that increase of the square.

We have that:

1*1   R=3
2*10   R=5
...
55*496 R=113

At this point we have to find a product of this column in which the second term is a multiple of the number to be factorize and yielding an R of quadratic form.
The multiples of 496, present in the right part of the column, are the following:

1*1 R=3
...
55*496 R=113
...
551*(496*10) R=113+992(496*2)
...
1048*(496*19) R=113+992*2
...

So we get an R squared by 113+992+992+992+ ...

We obtain $R=105^2=113+992*11$, thence: 5512*(496*100)=16640*16430=273395200

The method works with the columns obtained by increasing the second term of any square. For example, we can insert 496 in the right part of the column obtained by increasing the second term of 49.

1*6   R=19
2*55   R=34(19+15)
3*104 R=49(34+15)
...
11*496 $R=169=13^2$
...

In this case we get immediately R of quadratic form, from which:

11*496=88*62=5456.

We note that 496 = 62 * 8 and 88 = 11 * 8, where 8 is a Fibonacci's number and regard the physical vibrations of the superstrings.
In conclusion, we observe that **all the numbers that are obtained from F1 * F2**, thence the numbers that must be factorized, (in bleu) **are all new possible mathematical solutions concerning the equations of the string theory** $E8 \times E8$**.**

APPENDIX

In the Section 1, we have various number that in this Appendix have analyzed for to obtain various and interesting mathematical connections with some sectors of Number Theory

Series **14**, *24*, 50, **52**, **78**, 80, 81, 84, 120, 128, 182, **248**, 496

First observation: in bold the Lie's groups **G2 =14, F4=52, E6=78, E8 = 248)**

Second observation : Some numbers of the series are multiples of the **Lie's groups**, for ex. 84=6***14**, , 182=13***14** and 496 = 2***248)**

Third observation: the other numbers of the series are about all multiples of the **Lie's numbers**

Subsequent ratios

TABLE 1

Series	Previous number	Following ratio	Factors (see **observations**)	**Observations** Many factors are Lie's numbers of form $\mathbf{n^2+n+1}$ or exponent (in **red**)
14	---		2***7**	
24	14	**1,71**	2^3 ***3**	
50	24	**2,08**	$2*5^2$	
52	50	**1,04**	2^2***13**	
78	52	**1,50**	2***3*****13**	
80	78	**1,02**	2^4*5	
81	80	**1,01**	$\mathbf{3}^4$	
84 =6***14**	81	**1,03**	2^2***3*****7**	
120 =5**24*	84	**1,42**	2^3***3***5	
128	120	**1,06**	2^7	
182=13***14**	128	**1,42**	2***7*****13**	
248	128	**1,93**	2^3***31**	
496=2***248**	248	**2**	2^4***31**	

Sum following ratios:

1,71+2,08+1,04+1,50+1,02+1,01+1,03+1,42+1,06+1,42+1,93+2= 17,22

Arithmetic mean **17,22/12** =**1,435** about mean between **1,3247** and 1,618 = 1,4713, where 1,3247 is the fixed number for the Padovan's series and 1,618 is phi.

Thus there exists a weak connection with the **numbers of Padovan**. But the mean 1,444 between 1,2720 = √1,618 and 1,618 is also reliable, since 1,444 is very near also to **1,435.**

However, it should be noted that also the ratio between 248 and 182 (numbers that are part of the series), that is approximately equal to 1,362637, is a number between 1,3247 and 1,375 (the mean factor of partitions).

Now we see any possible connections with triangular numbers T, 2T+1, Fibonacci's numbers, partitions of numbers and Padovan's numbers

TABLE 2

Series to be studied	Triangular numbers T more near	**Lie's numbers = 2T+1 more near**	**Fibonacci's numbers more near**	**Partition's numbers more near**	**Padovan's numbers more near**
14	15	**13**	**13**	**15**	**12**
24	21, 28 (mean **24,**5)	**21**	**21**	**22**	**21**
50	45, 55 (mean **50**)	**57**	**55**	**42, 56 (mean 49)**	**49**
52	55	**57**	**55**	**42, 56 (mean 49)**	**49**
78	**78**	**73**	**55, 89 (mean 72)**	**77**	**65, 86 (mean 75,5)**
80	78	**73**	**55, 89 (mean 72)**	**77**	**65, 86 (mean 75,5)**
81	78	**73**	**55, 89 (mean 72)**	**77**	**65, 86 (mean 75,5)**
84	78, 91 (mean **84**,5	**73, 91 (mean 82)**		**77, 101 (mean 89)**	**86**
120	**120**		**89, 144 (mean 116,5**	**101, 135 (mean 118)**	**114**
128	120, 136 (mean **128)**	**133**	**144**	**101, 135 (mean 118)**	**114, 151 (mean 132,5)**
182	171, 190 (mean 180,5)	**183**	**144, 233 (mean 188,5)**	**171, 190 (mean 180,5)**	**151,200 (mean 175,5)**
248	231, 253 (mean 242)	**241, 273 (mean 257)**	**233**	**231**	**200, 265 (mean 232,5)**
496	**496**	**463, 551 (mean 507)**	**377, 610 (mean 493,5**	**490**	**465, 616 (mean 540,5), or: 351, 616 (mean 483,5)**

The triangular numbers up to 496 are:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496,

Partition numbers up to 490

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490

Padovan's numbers up to 616 :

1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265, 351, 465, 616

As can be see from the Table 2, the numbers of the new series are more or less aligned with the numbers (or their arithmetic means) of the other numerical series notes, with a few oscillations (few units) from the mean value. Usually, the oscillation is generally equal or near to the square root of the first value of the initial series (first column): for example, for the last line the oscillation maximum is 507- 496= 11≈ √496/2 = 22,27/2 = 11,13; or at the most near to the square root, for example for 50, the maximum oscillation is 57-50 = 7 ≈ 7,07 = √50 (if this observation may be useful)

Series 6, 8, 12, 15, 18, 24, 48, 60, 120, 144, 252, 1440

Before the usual table, we see the results of the division by 6, the numbers being almost all multiples of 6

6/6	1	Observations about Fibonacci
8/6	1,33	≈**1**
12/6	2	**2**
15/6	2,5	
18/6	3	**3**
24/6	4	
48/6	8	**8**
60/6	10	
120/6	20	21 ≈ 20
144/6	24	
252/6	42	≈ 44,5 = (34+55/2
1440/6	240	≈233

First observation: the results of the division by 6 of the numbers of the series are near to the **Fibonacci's series**

Subsequent ratios

S(n) /S/(n-1)	value	observations
8/6	1,33	√Φ = 1,27
12/8	1,50	
15/12	1,25	√Φ = 1,27
18/15	**1,2**	√Φ = 1,27
24/18	1,33	√Φ = 1,27
48/24	2	
60/48	1,25	√Φ = **1,2**7
120/60	2	≈ Φ*√Φ= **2**,05
144/120	**1,2**	
252/144	1,75	≈ Φ= 1,618
1440/252	5,71	≈3,52*1,618≈5,69 ≈π*Φ=5,08

Second observation: the subsequent ratios, often repeated two by two (for example, there are three values 1,33 , 1,25, 2) that are connected to the aureo number 1,618… and perhaps also to π (last ratio)

Third observation:

a) $\sqrt{1,75} = 1,322 \approx 1,33$ thus $1,322^2 = 1,7476 \approx$ **1,75**

b) $\sqrt[8]{5,71} = 1,2433 \approx 1,25$, thus $1,2433^8 = 5,7096 \approx$ **5,71**

c) $1,2^2 = 1,44 \approx 1,41 = \sqrt{2}$, thence $1,2 \approx \sqrt[4]{2} = 1,1892 \approx$ **1,2**

d) $1,25^2 = 1,56 \approx 1,50$ = second subsequent ratio.

Thence the subsequent ratios are near to the small powers or roots of other subsequent ratios

Mean ratio 16,44/11 =**1,49** $=1,618 / \sqrt[16]{3,14} = 1,618/1,074 = 1,506283 \approx 1,618/1,06 = 1,50 \approx$ **1,49** **($1,074^{16}$ >= 3,1337 ≈ 3,14)**

Sometimes the ratio between a number and one of the previous, is 10 (three times out of five) or very near to 10:

1440/144 = **10**
252/24 = **10**,5
144/12 = 12
144/15 = 9,6
120/12 = **10**
60/6= **10**

But also 18 *10 = 180 = 60 +120 . Furthermore, 10 is near to the squared of $\pi^2 = 9,85$

First conclusion: the phenomenon from which we obtain the numbers of the series, is governed by the numbers referred in these observations, and partly related to Φ and π.

More definitive conclusions emerge from the table below:

Table classical with the “equation preferred by Nature” (n^2+n+1)

Series: 6, 8, 12, 15, 18, 24, 48, 60, 120, 144, 252, 1440

Triangular numbers T :

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253 … 741 --- 1431,

Series	T more near or their means	2T more near	2T+1 more near	Fibonacci more near
6	6	12	7	5 =6-1
8	6		7	8
12	12,5 =(10+15)/2		13	13=12+1
15	15	12	13	13=15-2
18	18 =(15+21)/2	20	21	
24	24,5=(21+28)/2	20	21	21=24 -3
48	50= (45+55)/2	42	43	
60	60,5 = (55+66)/2	56	57	55 =60 -5
120	120	110	111	133=120 +13
144	144,5 =(136+153)/2	132	133	144 + 0
252	253=252+1	240	241	233=252 -19 (19≈21)
1440	1431= 1440 -9	1406	1407	1597= 1440 +157=144+13

With the usual table, we find that the numbers of the series:

a) coincide with the triangular number T , for example 6, 15, 120;

b) coincide (for example, 18) or are very near to the arithmetic means of two consecutive triangular numbers T, for example:

12, 24, 60, 144

c) are very near to the triangular numbers T, for example 252 =253 -1 and 1440 = 1431+9

d) are very near to the Fibonacci's numbers (fifth column) with differences also Fibonacci's numbers.

Definitive conclusion

The numbers of the new series, are connected with Φ and π (from the first general observations), almost coincide with the triangular numbers T or their

arithmetic means, and are near to the numbers 2T, 2T+1 (Lie's numbers) and to the Fibonacci's numbers (from the final Table).

Let's see now with the partitions of numbers:

Series:

6, 8, 12, 15, 18, 24, 48, 60, 120, 144, 252, 1440

Partitions of numbers up to 1575

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575,

Numbers partitions more near

5, 7, 11, 15, 22, 42, 56, 135, 231

Now we see with the following Table:

Numerical series	Partitions of numbers (or their arithmetic means)	Differences (near to the Fibonacci's numbers
6	5	1
8	7	1
12	11	1
15	15	0
18	18,5 =(15+22)/2/	0
24	22	2
48	49 = (42+56)/2	-1
60	56	4 =3+1
120	118 = (101+135)/2	2
144	135	9=8+1
252	231	21
1440	1415= (1255+1575)/2	25 =21+4

Also here, there is a certain closeness to the number of partitions (only the number 15 coincide) and with differences also here near to the Fibonacci's numbers. Therefore, also with the partitions (or their arithmetic means) there is a certain proximity, with the triangular numbers T, 2T , 2T+1 and the Fibonacci's series. The general connection with the formula L(n) = n^2+n+1 is therefore once again confirmed also for this new series. Also with the Padovan's series (called daughter of Fibonacci) there may be some connections. We continue the series:

6, 8, 12, 15, 18, 24, 48, 60, 120, 144, 252, 1440

Padovan's numbers:

1, 0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265, …1432

TABLE

Series	Padovan's number more near (or their means)	Differences Padovan – series (near to Fibonacci's numbers)
6	6= (5+7)/2	**0**
8	**8**=(7+9)/2	**0**
12	**12**	**0**
15	**16**	**1**
18	**18,5 =(16+21)/2**	**0**
24	24**,5)(21+28)/2**	**0**
48	**49**	**1**
60	**65**	**5**
120	**114**	**- 6 = - 5-1**
144	**151**	**7 = 8-1**
252	**265**	**13**
1440	**1432**	**- 8**

Only the Padovan's number 12 coincide with 12, number of the series, for all the other there is a certain proximity between the numbers of the series, the

Padovan's numbers or their arithmetic means, with differences very near to small Fibonacci's numbers or near to these.

Series N = 6, 8, 12, 15, 18, 24, 48, 60, 120, 144, 252, 1440

Connections with square roots and factors

N	√N	next square	Factors of N
6	2,44	9	2***3**
8	2,82	**9**	2*2*2
12	3,46	16	2*2***3**
15	3,87	**16**	**3***5
18	4,24	25	2***3*****3**
24	4,89	**25**	2*2*2***3**
48	6,92	**49**	2*2*2*2***3**
60	7,74	64	2*2***3***5
120	10,95	**121**	2*2*2***3***5
144	12	**144**	2*2*2*2***3*****3**
252	15,87	256	2*2***3*****3*****7**
1440	37,94	1444	2*2*2*2*2*3*3*5

Observations

The numbers of the series have often, in their square roots, the decimal part near to 1. This means, for Legendre's conjecture (see our demonstration, and that could be used in the future as possible demonstration of the percentage hypothesis on the factorization), that they are very near to the next square (i.e . for 1440, we have that 37 +1 = 38 and 38^2 = 1444), in fact, five of them (8,15,24,48,120) just simply add a single unit for the next square, while only 144 is a square perfect. For the other, the difference is 3 (in the initial case 9-6 = 3), while for the other is 4, as in the cases 16-12 = 4, 64 - 60 = 4, 256 - 252 = 4, 1444 - 1440 = 4, with the only exception of 25 - 18 = 7. Recall that 1 and 4 are perfect squares, for which the numbers of the series which differ by 1 or 4 by the next square, can be seen as differences of two squares, for example 1440 = 1444-4, 252 = 256 - 4, 60 = 64-4, 12 = 16-4, and so even those that differ by 1, because also 1 is a square ($1 = 1^2$), although improper: 8 = 9-1, 15 = 16-1, 24 = 25-1, 120 = 121 -1. This rule isn't respected only from 6 = 9-3 and 18 = 25 - 7, where 3 and 7 are not perfect squares.
With regard the factors, **are all powers of 2 and 3**, and in some cases also the

numbers 5 and 7. Recall that 3 and 7 are also numbers of Lie $L(n) = n^2 + n + 1$, since $3 = 1^1 + 1 + 1$, $7 = 2^2 + 2 + 1$. Furthermore, 2, 3, 5 and are also Fibonacci's numbers.
In other phenomena the emerging numbers (Fibonacci and Partitions) are about halfway between a square and the next, while in this new series, the numbers are generally very near to the next square, as seen above.

Acknowledgments

The co-author Michele Nardelli would like to thank P.A. **Francesco Di Noto** for the useful discussions and fundamental contributes in various sectors of Number Theory, would like to thank **Roberto Servi** for his useful discussions and contributes concerning the various applications concerning the new possible method of factorization of a number. In conclusion, Nardelli would like to thank also Prof. **Branko Dragovich** of Institute of Physics of Belgrade (Serbia) for his availability and friendship.

References

[1] Jacques Distler and Eric Sharpe – "*Heterotic Compactification with Principal Bundles for General Groups and General Levels*" – hep-th/0701244 – January 28, 2007.

[2] Benjamin Helmut Friedrich Jurke – "*Semi-Realistic Orbifold Compactification of Heterotic Strings*" – Diploma thesis in theoretical physics presented to University of Bielefeld, Department of Physics – July 11, 2007.

[3] Duiliu-Emanuel Diaconescu, Gregory Moore and Edward Witten – "E_8 *Gauge Theory, and a Derivation of K-Theory from M-Theory*" – hep-th/0005090 – March 23, 2004.

[4] Jeff Law – "*The Circle Method on the Binary Goldbach Conjecture*" – Mathematics Department of Princeton University – April 3, 2005.

[5] Claus Bauer and Wang Yonghui – "*On the Goldbach Conjecture in Arithmetic Progressions*" – Rocky Mountain Journal of Mathematics – Volume 36, Number 1, 2006.

[6] E. R. Canfield, Paul Erdos and Carl Pomerance – "*On a Problem of Oppenheim concerning "Factorisatio Numerorum"*" – Journal of Number Theory, Vol. 17. No 1 August 1983.

[7] Roberto Servi – "*Link between the subsets of odd natural numbers and of squares and the possible methodof factorization of a number*" http://www.scribd.com/doc/81348290/Possible-Method-of-Factorization