On some possible mathematical connections concerning Noncommutative Minisuperspace Cosmology, Noncommutative Quantum Cosmology in low-energy String Action, Noncommutative Kantowsky-Sachs Quantum Model, Spectral Action Principle associated with a Noncommutative Space and some aspects concerning the Loop Quantum Gravity

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Abstract

This paper is a review of some interesting results that has been obtained in various sectors of noncommutative cosmology, string theory and loop quantum gravity.
In the **Section 1**, we have described some results concerning the noncommutative model of the closed Universe with the scalar field. In the **Section 2**, we have described some results concerning the low-energy string effective quantum cosmology. In the **Section 3**, we have showed some results regarding the noncommutative Kantowsky-Sachs quantum model. In **Section 4**, we have showed some results regarding the spectral action principle associated with a noncommutative space and applied to the Einstein-Yang-Mills system. **Section 5** is a review of some results regarding some aspects of loop quantum gravity. In **Section 6**, we've described some results concerning the dynamics of vector mode perturbations including quantum corrections based on loop quantum gravity. In **Section 7**, we've described some equations concerning matrix models as a non-local hidden variables theories. In **Section 8**, we have showed some results concerning the quantum supergravity and the role of a "free" vacuum in loop quantum gravity. In **Section 9**, we've described various results concerning the unifying role of equivariant cohomology in the Topological Field Theories.
In conclusion, in **Section 10** we have showed the possible mathematical connections between the arguments above mentioned and some relationship with some equations concerning some sectors of Number Theory.

1. On some equations concerning the Noncommutative model of the closed Universe with the scalar field. [1]

We remember that for any nD minisuperspace model the ordering parameter ξ in the conformally invariant Wheeler-De Witt (WDW) equation (Planck's constant $\hbar = 1$):

$$H\Psi\left(q^A\right) = \left[-\frac{1}{2}\Delta + \xi R + U(q)\right]\Psi\left(q^A\right) = 0\,, \quad (1.1)$$

(where the Laplace-Beltrami operator is $\Delta \equiv \frac{1}{\sqrt{-G(q)}}\frac{\partial}{\partial q^A}\left[\sqrt{-G(q)}G^{ab}(q)\frac{\partial}{\partial q^B}\right]$, R is the scalar Riemannian curvature in minisuperspace M, $U(q)$ is the minisuperspace potential and $\Psi(q^A)$ is the wave function of the universe), is equal to $\xi = (2-n)/[8(1-n)]$ for $n \geq 2$.
The Non-commutative Wheeler-De Witt equation (NWDW) is given by the following expression:

$$\left[4\left(\partial^2_{x_C} - \partial^2_{y_C}\right) + \frac{i\theta}{2}\left(\alpha\partial_{y_C} - \beta\partial_{x_C}\right) + \alpha x_C + \beta y_C - 1\right]\psi(x_C, y_C) = 0. \quad (1.2)$$

It is possible to find the particular solutions of the NWDW equation by applying the Hartle-Hawking condition and using the θ-modified method of path integrals.
Under the Hartle-Hawking (H-H) condition and the gauge condition $\dot{N} = 0$, the non-commutative quantum mechanical propagator $G^{HH}_{\theta E}\left(\tilde{q}^{''A}_C, N|0,0\right)$ can be exactly found from the Pauli formula

$$G^{HH}_{\theta E}\left(\tilde{q}^{''A}_C, N|00\right) = \frac{1}{(2\pi)^{n/2}}\sqrt{-\det\left(\frac{\partial^2 \bar{\tilde{I}}_\theta}{\partial\tilde{q}^{''A}_C \partial\tilde{q}^{'A}_C}\right)}\exp\left(-\bar{\tilde{I}}_\theta\right)\Big|_{\tilde{q}^{'A}_C = 0}, \quad (1.3)$$

where $n = 2$ is the dimension of minisuperspace. The propagator is:

$$G^{HH}_{\theta E}\left(x^{''}_C, y^{''}_C, N|0,0,0\right) = \frac{1}{8\pi N}\exp\left\{-\frac{1}{4}\left\{\frac{-x^{''2}_C + y^{''2}_C}{2N} + \frac{N^3}{6}\left(\alpha^2 - \beta^2\right) - \right.\right.$$
$$\left.\left. - N\left[2 - \alpha x^{''}_C - \beta y^{''}_C + \frac{\theta^2}{32}\left(\alpha^2 - \beta^2\right)\right]\right\}\right\} \times \exp\left[\frac{i\theta}{16}\left(\beta x^{''}_C + \alpha y^{''}_C\right)\right]. \quad (1.4)$$

When $\alpha = \beta$ the propagator (1.4) does not contain the θ^2 dependence so that integration over the complex lapse parameter by applying the method of fastest descent yields the particular solutions of the NWDW equation:

$$\psi_{\theta NB}\left(x^{''}_C, y^{''}_C\right) = \int dN\ \frac{1}{8\pi N}\exp\left\{-\frac{1}{4}\left\{\frac{-x^{''2}_C + y^{''2}_C}{2N} + \frac{N^3}{6}\left(\alpha^2 - \beta^2\right) - \right.\right.$$
$$\left.\left. - N\left[2 - \alpha x^{''}_C - \beta y^{''}_C + \frac{\theta^2}{32}\left(\alpha^2 - \beta^2\right)\right]\right\}\right\} \times \exp\left[\frac{i\theta}{16}\left(\beta x^{''}_C + \alpha y^{''}_C\right)\right] = \exp\left[\frac{i\theta\alpha}{16}\left(x^{''}_C + y^{''}_C\right)\right]\psi_{NB}\left(x^{''}_C, y^{''}_C\right). \quad (1.5)$$

where $\psi_{NB}\left(x^{''}_C, y^{''}_C\right)$ are all particular solutions of the WDW equation.
Now, we consider the noncommutative geometry of the minisuperspace of the quantum model of the closed universe. It is important to note that the WDW equation (1.1) of the standard (commutative) nD minisuperspace model may be obtained from the action:

$$S_n\left[G_{AB}(q), \Psi, \overline{\Psi}\right] = -\int_M d^n q\sqrt{-G}\left\{\frac{1}{2}G^{AB}\nabla_A\overline{\Psi}\nabla_B\Psi + \overline{\Psi}[\xi R + U(q)]\Psi\right\} =$$
$$= -\int_M d^n q\sqrt{-G}\,\overline{\Psi}\left[-\frac{1}{2}\Delta + \xi R + U(q)\right]\Psi = -\int_M d^n q\sqrt{-G}\,\overline{\Psi}H\Psi, \quad (1.6)$$

by the Lagrange variation of $\overline{\Psi}(q^A)$. Physically, action (1.6) is related to the expectation value of the energy of the Universe, which is invariant to the conformal transformation for the fixed value of the ordering parameter ξ. In the case when nD minisuperspace $M^{\bullet}$ possesses Weyl geometry, we have the following action:

$$\overset{w'}{S_n^{'}}\left[G_{AB}^{'}(q),\Psi',\overline{\Psi}',w_A^{'}\right]=-\int_{M^{\bullet}}d^n q\sqrt{-G'}\left\{\frac{1}{2}G^{'AB}\overset{\bullet}{\nabla}_B\Psi'+\overline{\Psi}'\left[\xi R'\left(\overset{\bullet}{\Gamma}'\right)+U'(q)\right]\Psi'\right\},\left(R'\left(\overset{\bullet}{\Gamma}'\right)=R'\left(\overset{w'}{\Gamma}'\right)\right). \quad (1.7)$$

The action (1.7) is invariant on Weyl rescaling:

$$G_{AB}(q)=\Omega^2(q)G_{AB}^{'}(q),\Psi(q^A)=\Omega^{1-\frac{n}{2}}(q)\Psi'(q^A),\overline{\Psi}(q^A)=\Omega^{1-\frac{n}{2}}(q)\overline{\Psi}'(q^A),U(q)=\Omega^{-2}(q)U'(q), \quad (1.8)$$

where n is the dimension of the minisuperspace and the rescaling

$$w_A=w_A^{'}+2\Omega^{-1}\partial_A\Omega, \quad (1.9)$$

for any value of the ordering parameter ξ due to the validity of the following relation:

$$R\left(\overset{\bullet}{\Gamma}\right)=R\left(\overset{w}{\Gamma}\right)=R\left(\overset{w'}{\Gamma}'\right)=R\left(\overset{\bullet}{\Gamma}'\right)=G^{AB}R_{AB}\left(\overset{\bullet}{\Gamma}'\right)=G^{AB}R_{AB}^{'}\left(\overset{\bullet}{\Gamma}'\right)=\Omega^{-2}(q)R'\left(\overset{\bullet}{\Gamma}'\right). \quad (1.9b)$$

When minisuperspace is (conformally) flat, then action (1.7) acquires the following form:

$$\overset{w'}{S_n^{'}}\left[G_{AB}^{'}(q),\Psi',\overline{\Psi}',w_A^{'}\right]=-\int_{M^{\bullet}}d^n q\sqrt{-G'}\left[\frac{1}{2}G^{'AB}\overset{\bullet}{\partial}_A\overline{\Psi}'\overset{\bullet}{\partial}_B\Psi'+\overline{\Psi}'U'(q)\Psi'\right], \quad (1.10)$$

and by applying the Weyl rescaling (1.8) and the rescaling

$$w_A=w_A^{'}+2\Omega^{-1}\partial_A\Omega=0, \quad (1.11)$$

one obtains the action:

$$S_n\left[G_{AB}(q),\Psi,\overline{\Psi},w_A=0\right]=-\int_M d^n q\sqrt{-G}\left[\frac{1}{2}G^{AB}\partial_A\overline{\Psi}\partial_B\Psi+\overline{\Psi}U(q)\Psi\right], \quad (1.12)$$

which is same as action (1.6) for which the Riemann scalar curvature vanishes.
If we apply the Weyl rescaling to the following noncommutative Hamiltonian

$$H^\theta\equiv NH_\theta=\frac{N}{2}\left\{-\frac{p_{a_C}^2}{a_C^2}+\frac{p_{\phi_C}^2}{a_C^4}+a_C^2(\alpha ch2\phi_C+\beta sh2\phi_C)-1-\right.$$
$$\left.-\frac{\theta}{4a_C^2}\left[-p_{a_C}a_C(\beta ch2\phi_C+\alpha sh2\phi_C)+p_{\phi_C}(\alpha ch2\phi_C+\beta sh2\phi_C)\right]\right\}\approx 0, \quad (1.12b)$$

we obtain the following noncommutative Hamiltonian:

$$H'^{\Theta} \equiv N' H'_{\Theta} = \frac{N'}{2}\left[-\frac{p_a^2}{a} + \frac{p_\phi^2}{a^3} + \alpha a^3 ch2\phi - a - \frac{\Theta\alpha}{4}\left(-p_a a sh2\phi + p_\phi ch2\phi \right) \right] \approx 0 . \quad (1.13)$$

Furthermore, with regard the eq. (1.12b), we remember that:

$$\phi = \phi_C + \frac{1}{4}\ln\left| \frac{1 - \frac{\theta}{4a_C^4}\left(p_{\phi_C} - a_C p_{a_C} \right)}{1 - \frac{\theta}{4a_C^4}\left(p_{\phi_C} + a_C p_{a_C} \right)} \right| = \phi_C + \frac{\theta p_{a_C}}{8a_C^3} + \frac{\theta^2 p_{a_C} p_{\phi_C}}{32 a_C^7} + O\left(\theta^3\right). \quad (1.13b)$$

Applying the Legendre transformations to (1.13) we obtain the noncommutative Lagrangian:

$$L'_{\Theta}\left[a, \frac{da}{dt}, N' \right] = \frac{N'}{2}\left[-\frac{a}{N'^2}\left(\frac{da}{dt} \right)^2 - \left(\alpha - \frac{\Theta^2\alpha^2}{64} \right) a^3 + a \right]. \quad (1.14)$$

Solving the equations of motion obtained from (1.14), taking care of the gauge condition $N' = 1$ and the initial conditions $\frac{da}{dt}\big|_{t=0} = 0,\ \frac{d^2 a}{dt^2}\big|_{t=0} = \alpha a(0)$ we obtain the Lorentz 4-metric determined by the space-time interval:

$$ds^2 = -dt^2 + \left(\frac{1}{\alpha} + \frac{\theta^2\alpha}{64} \right) ch^2\left(\sqrt{\alpha} t \right) d\Omega_3^2 , \quad (\sigma = 1). \quad (1.15)$$

The scalar curvature of the θ-deformed de Sitter space-time is:

$${}^4R(t) = 4\Lambda\left[1 - \frac{\theta^2\Lambda^2}{1152\left(1 + \frac{\theta^2\Lambda^2}{576} \right) ch^2\left(\sqrt{\Lambda/3} t \right)} \right], \quad (1.16)$$

while the scalar curvature of the 3D subspace is:

$${}^3R(t) = \frac{2\Lambda}{\left(1 + \frac{\theta^2\Lambda^2}{576} \right) ch^2\left(\sqrt{\Lambda/3} t \right)}, \quad (1.17)$$

where $\Lambda = 3\alpha$.

After applying the Wick rotation to eq. (1.14), and obtaining the corresponding equations of motion from this non-commutative Lagrangian, we obtain the Euclidean 4-metric determined by the space-time interval:

$$ds_E^2 = d\tau^2 + \left(\frac{1}{\alpha} + \frac{\theta^2\alpha}{64} \right) \sin^2\left(\sqrt{\alpha}\tau \right) d\Omega_3^2 , \quad (\sigma = 1). \quad (1.18)$$

From eq. (1.18), the semiclassical non-commutative Hartle-Hawking (H-H) wave functions that corresponds to this 4-geometry are of the form:

$$\psi'^{\pm}_{\theta}(a''=a(1)) \approx \exp\left\{ \pm \frac{1}{3\alpha}\left(1+\frac{\theta^2\alpha^2}{64}\right)^{3/2}\left[1-\alpha a''^2\left(1+\frac{\theta^2\alpha^2}{64}\right)^{-1}\right]^{3/2}\right\}. \quad (1.19)$$

From eq. (1.19) we see that the non-commutativity parameter θ increases (for the "+" sign) or decreases (for the "–" sign) the two corresponding standard semiclassical H-H tunnelling amplitudes. So, from this consideration we may conclude that the canonical non-commutativity prefers the creation of the theta deformed de Sitter universe rather by ψ'^{+}_{θ} than by ψ'^{-}_{θ}.

Hence, at the Hartle-Hawking condition and for different choices of the gauge condition $\dot{N}=0$ (N is the lapse function) the θ^2 term either decreases or increases the semiclassical probability amplitude for tunnelling from nothing to the closed universe with the stable matter potential.

Furthermore, under the Hartle-Hawking condition and when $\alpha > 0$ and $\beta = 0$ the canonical noncommutativity of the minisuperspace prefers as the most probable the creation of the closed universe with $\phi = 0$ by the semiclassical wave function which for $\theta = 0$ corresponds to the geometry of filling in the three-sphere with more than half of a four-sphere of radius $\sqrt{(1/\alpha)}$.

2. On some equations concerning the low-energy string effective quantum cosmology.[2]

At low energy, the tree-level, (3+1)-dimensional string effective action can be written as

$$S = -\frac{1}{2\lambda_s}\int d^4x\sqrt{-g}\,e^{-\phi}\left(R+\partial_\mu\phi\partial^\mu\phi+V\right). \quad (2.1)$$

Here ϕ is the dilaton field, $a(t)=\exp\left[\beta(t)/\sqrt{3}\right]$, λ_s is the fundamental string length parameter governing the high-derivative expansion of the action and V is a possible dilaton potential. When we consider this theory in the metric of isotropic and homogeneous spacetime, after integrating by parts, and using the convenient time parametrization $dt = e^{-\bar{\phi}}d\tau$, reduces to

$$S = -\frac{\lambda_s}{2}\int d\tau\left(\bar{\phi}'^2-\bar{\beta}'^2+Ve^{-2\bar{\phi}}\right), \quad (2.2) \quad \text{where} \quad \bar{\phi} = \phi - \ln\int\left(d^3x/\lambda_s^3\right)-\sqrt{3}\beta. \quad (2.3)$$

The Hamiltonian of the system is

$$H = \frac{1}{2\lambda_s}\left(\Pi_\beta^2-\Pi_\phi^2+\lambda_s^2 Ve^{2\bar{\phi}}\right), \quad (2.4)$$

where the canonical conjugate momenta are,

$$\Pi_\beta = \lambda_s\beta', \qquad \Pi_\phi = -\lambda_s\bar{\phi}'. \quad (2.5)$$

The corresponding Wheeler-DeWitt equation, in a particular factor ordering is

$$\frac{1}{2\lambda_s}\left[\frac{\partial^2}{\partial\bar{\phi}^2}-\frac{\partial^2}{\partial\beta^2}+\lambda_s^2 V\left(\bar{\phi},\beta\right)e^{-2\bar{\phi}}\right]\psi\left(\phi,\beta\right)=0\,. \quad (2.6)$$

We shall assume $V=V(\phi)$ in order to separate variables. We consider two simple cases of the potential as toy models that allow us to obtain exact solutions

1) Case: $V=-V_0 e^{4\bar{\phi}}$

Therefore the solution of the WDW equation (2.6) is

$$\psi_\nu\left(\bar{\phi},\beta\right)=Ce^{i\nu\beta}Y_{i\nu}\left(\lambda_s\sqrt{V_0}e^{\bar{\phi}}\right),$$

where $Y_{i\nu}$ is the second class Bessel function.

2) Case: $V=-V_0$

Now the wave function is

$$\psi_\nu\left(\bar{\phi},\beta\right)=Ce^{i\nu\beta}K_{i\nu}\left(\lambda_s\sqrt{V_0}e^{-2\bar{\phi}}\right), \quad (2.7)$$

where $K_{i\nu}$ is the modified Bessel function. We can construct wormhole type solutions by means integrating over the separation constant ν,

$$\psi_{WH}\left(\bar{\phi},\beta\right)=\int_{-\infty}^{+\infty}e^{i\nu(\beta+\mu)}K_{i\nu}\left(\lambda_s\sqrt{V_0}e^{\bar{\phi}}\right)d\nu=e^{-\lambda_s\sqrt{V_0}e^{\bar{\phi}}\cosh[2\beta+\mu]}\,, \quad (2.8)$$

where $\mu=$const. For the noncommutative quantum cosmology model, we will assume the "cartesian coordinates" $\bar{\phi}$ and β of the Robertson-Walker minisuperspace obey a kind of commutation relation,

$$\left[\bar{\phi},\beta\right]=i\theta\,. \quad (2.9)$$

This is a particular ansatz in these configuration coordinates. The deformation of minisuperspace can be studied in terms of Moyal product,

$$f\left(\bar{\phi},\beta\right)\star g\left(\bar{\phi},\beta\right)=f\left(\bar{\phi},\beta\right)\exp\left[i\frac{\theta}{2}\left(\frac{\overleftarrow{\partial}}{\partial\bar{\phi}}\frac{\overrightarrow{\partial}}{\partial\beta}-\frac{\overleftarrow{\partial}}{\partial\beta}\frac{\overrightarrow{\partial}}{\partial\bar{\phi}}\right)\right]g\left(\bar{\phi},\beta\right). \quad (2.10)$$

Then the noncommutative WDW equation is

$$\frac{1}{2\lambda_s}\star\left[\frac{\partial^2}{\partial\bar{\phi}^2}-\frac{\partial^2}{\partial\beta^2}+\lambda_s^2 V\left(\bar{\phi},\beta\right)e^{-2\bar{\phi}}\right]\star\,\psi\left(\phi,\beta\right)=0\,. \quad (2.11)$$

Now, we take the eq. (2.6) and obtain:

$$\frac{1}{2\lambda_s}\left[\frac{\partial^2}{\partial\bar{\phi}^2}-\frac{\partial^2}{\partial\beta^2}+\lambda_s^2 V\left(\bar{\phi},\beta\right)e^{-2\bar{\phi}}\right]\psi\left(\phi,\beta\right)=0\Rightarrow$$

$$\Rightarrow \frac{1}{2\lambda_s} = \left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\bar{\phi}^2}\right)\frac{1}{\left(\lambda_s^2 V\left(\bar{\phi},\beta\right)e^{-2\bar{\phi}}\right)}. \quad (2.12)$$

Hence, from the eq. (2.1), we have that:

$$S = -\frac{1}{2\lambda_s}\int d^4x\sqrt{-g}e^{-\phi}\left(R + \partial_\mu\phi\partial^\mu\phi + V\right) \Rightarrow$$

$$\Rightarrow -\left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\bar{\phi}^2}\right)\frac{1}{\left(\lambda_s^2 V\left(\bar{\phi},\beta\right)e^{-2\bar{\phi}}\right)}\int d^4x\sqrt{-g}e^{-\phi}\left(R + \partial_\mu\phi\partial^\mu\phi + V\right). \quad (2.13)$$

Furthermore, always from the eq. (2.6), we obtain:

$$\lambda_s = \left(\lambda_s^2 V\left(\bar{\phi},\beta\right)e^{-2\bar{\phi}}\right)\frac{1}{\left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\bar{\phi}^2}\right)} \cdot \frac{1}{2}. \quad (2.14)$$

Hence, from the eqs. (2.2) and (2.3), we have that

$$S = -\frac{\lambda_s}{2}\int d\tau\left(\bar{\phi}'^2 - \bar{\beta}'^2 + Ve^{-2\bar{\phi}}\right) = -\frac{1}{4}\left(\lambda_s^2 V\left(\bar{\phi},\beta\right)e^{-2\bar{\phi}}\right)\frac{1}{\left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\bar{\phi}^2}\right)}\int d\tau\left(\bar{\phi}'^2 - \bar{\beta}'^2 + Ve^{-2\bar{\phi}}\right), \quad (2.15)$$

where $\bar{\phi} = \phi - \ln\int\left(d^3x/\lambda_s^3\right) - \sqrt{3}\beta$.

3. On some equations concerning the noncommutative Kantowski-Sachs quantum model.[3]

The Hamiltonian of General Relativity without matter is

$$H = \int d^3x\left(N\mathrm{H} + N_j\mathrm{H}^j\right), \quad (3.1)$$

where

$$\mathrm{H} = G_{ijkl}\Pi^{ij}\Pi^{kl} - h^{1/2}R^{(3)}, \quad \mathrm{H}^j = 2D_i\Pi^{ij}. \quad (3.2)$$

Hence, we can write the (3.1) as

$$H = \int d^3x\left(NG_{ijkl}\Pi^{ij}\Pi^{kl} - h^{1/2}R^{(3)} + N_j 2D_i\Pi^{ij}\right). \quad (3.2b)$$

Units are chosen such that $\hbar = c = 16\pi G = 1$. The quantity $R^{(3)}$ is the intrinsic curvature of the spacelike hypersurfaces, D_i is the covariant derivative with respect to h_{ij}, and h is the determinant of h_{ij}. The momentum Π_{ij} canonically conjugated to h^{ij}, and the DeWitt metric G_{ijkl} are

$$\Pi_{ij} = -h^{1/2}\left(K_{ij} - h_{ij}K\right), \quad (3.2c) \qquad G_{ijkl} = \frac{1}{2}h^{-1/2}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right), \quad (3.2d)$$

where $K_{ij} = -(\partial_t h_{ij} - D_i N_j - D_j N_i)/(2N)$ is the second fundamental form. The super-Hamiltonian constraint $\mathrm{H} \approx 0$ yields the Wheeler-DeWitt equation

$$\left(G^{ijkl} \frac{\delta}{\delta h^{ij}} \frac{\delta}{\delta h^{kl}} + h^{1/2} R^{(3)} \right) \Psi\left[h^{ij}\right] = 0 . \quad (3.2e)$$

The Kantowski-Sachs line element is

$$ds^2 = -N^2 dt^2 + X^2(t) dr^2 + Y^2(t)\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \quad (3.3)$$

In the Misner parametrization, (3.3) is written as

$$ds^2 = -N^2 dt^2 + e^{2\sqrt{3}\beta} dr^2 + e^{-2\sqrt{3}\beta} e^{-2\sqrt{3}\Omega}\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \quad (3.4)$$

From (3.1) and (3.2), the Hamiltonian of General Relativity for this metric is found to be

$$H = N\mathrm{H} = N \exp\left(\sqrt{3}\beta + 2\sqrt{3}\Omega\right)\left[-\frac{P_\Omega^2}{24} + \frac{P_\beta^2}{24} - 2\exp\left(-2\sqrt{3}\Omega\right)\right]. \quad (3.5)$$

The Poisson brackets for the classical phase space variables are

$$\{\Omega, P_\Omega\} = 1, \quad \{\beta, P_\beta\} = 1, \quad \{P_\Omega, P_\beta\} = 0, \quad \{\Omega, \beta\} = 0. \quad (3.6)$$

For the metric (3.4), the super-Hamiltonian constraint $\mathrm{H} \approx 0$ is reduced to

$$\mathrm{H} = \xi h \approx 0, \quad (3.7)$$

where

$$\xi = \frac{1}{24}\exp\left(\sqrt{3}\beta + 2\sqrt{3}\Omega\right), \quad h = -P_\Omega^2 + P_\beta^2 - 48\exp\left(-2\sqrt{3}\Omega\right) \approx 0. \quad (3.8)$$

Hence, the eq. (3.7) can be rewrite as

$$\mathrm{H} = \frac{1}{24}\exp\left(\sqrt{3}\beta + 2\sqrt{3}\Omega\right)\left[-P_\Omega^2 + P_\beta^2 - 48\exp\left(-2\sqrt{3}\Omega\right)\right] \approx 0. \quad (3.8b)$$

Now, let us introduce a noncommutative classical geometry in the model by considering a Hamiltonian that has the same functional form as (3.5) but is valued on variables that satisfy the deformed Poisson brackets

$$\{\Omega, P_\Omega\} = 1, \quad \{\beta, P_\beta\} = 1, \quad \{P_\Omega, P_\beta\} = 0, \quad \{\Omega, \beta\} = \theta. \quad (3.9)$$

The equations of motion in this case are written as

$$\dot{\Omega} = -2P_\Omega, \quad \dot{P}_\Omega = -96\sqrt{3}e^{-2\sqrt{3}\Omega}, \quad \dot{\beta} = 2P_\beta - 96\sqrt{3}\theta e^{-2\sqrt{3}\Omega}, \quad \dot{P}_\beta = 0. \quad (3.10)$$

The solution for $\Omega(t)$ and $\beta(t)$ are

$$\Omega(t)=\frac{\sqrt{3}}{6}\ln\left\{\frac{48}{P_{\beta_0}^2}\cosh^2\left[2\sqrt{3}P_{\beta_0}(t-t_0)\right]\right\},\ \beta(t)=2P_{\beta_0}(t-t_0)+\beta_0-\theta P_{\beta_0}\tanh\left[2\sqrt{3}P_{\beta_0}(t-t_0)\right].\quad (3.11)$$

Now, we may achieve the solutions above by making use of the auxiliary canonical variables Ω_c and β_c, defined as

$$\Omega_c=\Omega+\frac{\theta}{2}P_\beta,\quad \beta_c=\beta-\frac{\theta}{2}P_\Omega,\quad P_{\Omega_c}=P_\Omega,\quad P_{\beta_c}=P_\beta.\quad (3.12)$$

The Poisson brackets for these variables are

$$\{\Omega_c,P_{\Omega_c}\}=1,\quad \{\beta_c,P_{\beta_c}\}=1,\quad \{P_{\Omega_c},P_{\beta_c}\}=0,\quad \{\Omega_c,\beta_c\}=0.\quad (3.13)$$

As the equations of motion in the canonical formalism in the gauge $N=24\exp\left(-\sqrt{3}\beta-2\sqrt{3}\Omega\right)$ we have

$$\dot{\Omega}_c=-2P_{\Omega_c},\quad \dot{P}_{\Omega_c}=-96\sqrt{3}e^{-2\sqrt{3}\Omega},\quad \dot{\beta}_c=2P_{\beta_c}-48\sqrt{3}\theta e^{-2\sqrt{3}\Omega},\quad \dot{P}_{\beta_c}=0.\quad (3.14)$$

whose solutions are

$$\Omega_c(t)=\frac{\sqrt{3}}{6}\ln\left\{\frac{48}{P_{\beta_0}^2}\cosh^2\left[2\sqrt{3}P_{\beta_0}(t-t_0)\right]\right\}+\frac{\theta}{2}P_{\beta_0},$$

$$\beta_c(t)=2P_{\beta_0}(t-t_0)+\beta_0-\frac{\theta}{2}P_{\beta_0}\tanh\left[2\sqrt{3}P_{\beta_0}(t-t_0)\right],\ P_{\Omega_c}(t)=-P_{\beta_0}\tanh\left[2\sqrt{3}P_{\beta_0}(t-t_0)\right],$$

$$P_{\beta_c}(t)=P_{\beta_0}.\quad (3.15)$$

Finally, from (3.12) and (3.15) we can recover the solution (3.11).
The Wheeler-DeWitt equation, for the Kantowski-Sachs universe, is

$$\left[-\hat{P}_\Omega^2+\hat{P}_\beta^2-48\exp\left(-2\sqrt{3}\Omega\right)\right]\Psi(\Omega,\beta)=0,\quad (3.16)$$

where $\hat{P}_\Omega=-i\partial/\partial\Omega$ and $\hat{P}_\beta=-i\partial/\partial\beta$. Hence, the eq. (3.16) can be rewritten also

$$\left[-\left(-\frac{i\partial}{\partial\Omega}\right)^2+\left(-\frac{i\partial}{\partial\beta}\right)^2-48\exp\left(-2\sqrt{3}\Omega\right)\right]\Psi(\Omega,\beta)=0.\quad (3.16b)$$

A solution to equation (3.16) is

$$\Psi_\nu(\Omega,\beta)=e^{i\nu\sqrt{3}\beta}K_{i\nu}\left(4e^{-\sqrt{3}\Omega}\right),\quad (3.16c)$$

where $K_{i\nu}$ is a modified Bessel function and ν is a real constant.
Now, we fix the gauge $N=24\exp\left(-\sqrt{3}\beta-2\sqrt{3}\Omega\right)$ in (3.5). The Wheeler-DeWitt equation for the noncommutative Kantowski-Sachs model is

$$\left[-P_{\Omega_c}^2 + P_{\beta_c}^2 - 48\exp\left(-2\sqrt{3}\Omega_c\right)\right]\star\Psi(\Omega_c,\beta_c) = 0\,, \quad (3.17)$$

which is the Moyal deformed version of (3.16). By using the properties of the Moyal product, it is possible to write the potential term (which we denote by V to include the general case) as

$$V(\Omega_c,\beta_c)\star\Psi(\Omega_c,\beta_c) = V\left(\Omega_c + i\frac{\theta}{2}\partial_{\beta_c}, \beta_c - i\frac{\theta}{2}\partial_{\Omega_c}\right)\Psi(\Omega_c,\beta_c) = V\left(\hat{\Omega},\hat{\beta}\right)\Psi(\Omega_c,\beta_c)\,, \quad (3.18)$$

where

$$\hat{\Omega} = \hat{\Omega}_c - \frac{\theta}{2}\hat{P}_{\beta_c}\,, \qquad \hat{\beta} = \hat{\beta}_c + \frac{\theta}{2}\hat{P}_{\Omega_c}\,. \quad (3.19)$$

Equation (3.19) is nothing but the operatorial version of equation (3.12). The Wheeler-DeWitt equation then reads

$$\left[-\hat{P}_{\Omega_c}^2 + \hat{P}_{\beta_c}^2 - 48\exp\left(-2\sqrt{3}\hat{\Omega}_c + \sqrt{3}\theta\hat{P}_{\beta_c}\right)\right]\Psi(\Omega_c,\beta_c) = 0\,. \quad (3.20)$$

In our time gauge $N = 24\exp\left(-\sqrt{3}\beta - 2\sqrt{3}\Omega\right)$, the Hamiltonian $H = N\xi h$, with ξ and h defined in Eq. (3.8), reduces simply to h. We can therefore use h to generate time displacements and obtain the equations of motion for $\Omega_c(t)$ and $\beta_c(t)$ as

$$\dot{\Omega}_c(t) = B\left(\frac{1}{i}\left[\hat{\Omega}_c,\hat{h}\right]\right)\Big|_{\substack{\Omega_c=\Omega_c(t)\\ \beta_c=\beta_c(t)}} = -2\frac{\partial S}{\partial\Omega_c}\Big|_{\substack{\Omega_c=\Omega_c(t)\\ \beta_c=\beta_c(t)}}\,, \quad (3.21)$$

$$\dot{\beta}_c(t) = B\left(\frac{1}{i}\left[\hat{\beta}_c,\hat{h}\right]\right)\Big|_{\substack{\Omega_c=\Omega_c(t)\\ \beta_c=\beta_c(t)}} = \left[2\frac{\partial S}{\partial\beta_c} - 48\sqrt{3}\theta\,\mathrm{Re}\left\{\frac{\exp\left(-2\sqrt{3}\Omega_c - i\theta\sqrt{3}\partial_{\beta_c}\right)\left(R\cdot e^{iS}\right)}{R\cdot e^{iS}}\right\}\right]\Big|_{\substack{\Omega_c=\Omega_c(t)\\ \beta_c=\beta_c(t)}}\,. \quad (3.22)$$

As long as $\Omega_c(t)$ and $\beta_c(t)$ are known, the minisuperspace trajectories are given by

$$\Omega(t) = \Omega_c(t) - \frac{\theta}{2}\partial_{\beta_c}S[\Omega_c(t),\beta_c(t)]\,, \quad (3.23) \qquad \beta(t) = \beta_c(t) + \frac{\theta}{2}\partial_{\Omega_c}S[\Omega_c(t),\beta_c(t)]\,. \quad (3.24)$$

A solution to (3.17) is

$$\Psi_\nu(\Omega_c,\beta_c) = e^{i\nu\sqrt{3}\beta_c}K_{i\nu}\left\{4\exp\left[-\sqrt{3}\left(\Omega_c - \frac{\sqrt{3}}{2}\nu\theta\right)\right]\right\}. \quad (3.25)$$

Once a quantum state of the universe is given as a super-position of states

$$\Psi(\Omega_c,\beta_c) = \sum_\nu C_\nu e^{i\nu\sqrt{3}\beta_c}K_{i\nu}\left\{4\exp\left[-\sqrt{3}\left(\Omega_c - \frac{\sqrt{3}}{2}\nu\theta\right)\right]\right\} = R\cdot e^{iS}\,, \quad (3.26)$$

the universe evolution can be determined by solving the system of equations constituted by (3.21) and (3.22) and substituting the solution in (3.23) and (3.24).

4. **On some equations concerning the spectral action principle associated with a noncommutative space, applied to the Einstein-Yang-Mills system.** [4]

The basic data of Riemannian geometry consists in a manifold M whose points $x \in M$ are locally labelled by finitely many coordinates $x^{\mu} \in R$, and in the infinitesimal line element ds,

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} . \quad (4.1)$$

The laws of physics at reasonably low energies are well encoded by the action functional,

$$I = I_E + I_{SM} \quad (4.2)$$

where $I_E = \frac{1}{16\pi G} \int R\sqrt{g}\, d^4x$ is the Einstein action, which depends only upon the 4-geometry and where I_{SM} is the standard model action, $I_{SM} = \int L_{SM}$, $L_{SM} = L_G + L_{GH} + L_H + L_{Gf} + L_{Hf}$. The action functional I_{SM} involves, besides the 4-geometry, several additional fields: bosons G of spin 1 such as γ, $W^{\pm}$ and Z, and the eight gluons, bosons of spin 0 such as the Higgs field H and fermions f of spin 1/2 , the quarks and leptons.
To test the following spectral action functional

$$Trace\chi\left(\frac{D}{\Lambda}\right) + \langle \psi, D\psi \rangle, \quad (4.3)$$

we shall first consider the simplest noncommutative modification of a manifold M. Thus we replace the algebra $C^{\infty}(M)$ of smooth functions on M by the tensor product $A = C^{\infty}(M) \otimes M_N(C)$ where $M_N(C)$ is the algebra of $N \times N$ matrices. We shall compare the spectral action functional (4.3) with the following

$$I = \frac{1}{2\kappa^2} \int R\sqrt{g}\, d^4x + I_{YM} \quad (4.4)$$

where $I_{YM} = \int (L_G + L_{Gf})\sqrt{g}\, d^4x$ is the action for an $SU(N)$ Yang-Mills theory coupled to fermions in the adjoint representation. Hence, the eq. (4.4) can be rewritten also

$$I = \frac{1}{2\kappa^2} \int R\sqrt{g}\, d^4x + \int (L_G + L_{Gf})\sqrt{g}\, d^4x . \quad (4.4b)$$

The coupling of the Yang-Mills field A with the fermions is equal to

$$\langle \psi, D\psi \rangle \quad \psi \in H . \quad (4.5)$$

The operator $D = D_0 + A + JAJ^*$ is given by

$$D = e_{\alpha}^{u} \gamma^{a} \left[(\partial_{\mu} + \omega_{\mu}) \otimes 1_N + 1 \otimes \left(-\frac{i}{2} g_0 A_{\mu}^{i} T^{i} \right) \right] \quad (4.6)$$

where ω_μ is the spin-connection on M :

$$\omega_\mu = \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$$

and T^i are matrices in the adjoint representation of $SU(N)$ satisfying $Tr\left(T^i T^j\right) = 2\delta^{ij}$.
With regard to compute the square of the Dirac operator given by (4.6), this can be cast into the elliptic operator form:

$$P = D^2 = -\left(g^{\mu\nu}\partial_\mu\partial_\nu \cdot \mathbb{1} + A^\mu\partial_\mu + B\right) \quad (4.7)$$

where $\mathbb{1}$, A^μ and B are matrices of the same dimensions as D and are given by:

$$A^\mu = \left(2\omega^\mu - \Gamma^\mu\right)\otimes 1_N - ig_0 1_4 \otimes A^{\mu i}T^i$$
$$B = \left(\partial^\mu\omega_\mu + \omega^\mu\omega_\mu - \Gamma^\nu\omega_\nu + R\right)\otimes 1_N - ig_0\omega_\mu \otimes A^{\mu i}T^i . \quad (4.8)$$

We shall now compute the spectral action for this theory given by

$$Tr\chi\left(\frac{D^2}{m_0^2}\right) + \left(\psi, D\psi\right) \quad (4.9)$$

where the trace Tr is the usual trace of operators in the Hilbert space H, and m_0 is a (mass) scale to be specified. The function χ is chosen to be positive and this has important consequences for the positivity of the gravity action.
Using identities:

$$Tr\left(P^{-s}\right) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} Tre^{-tP} dt \qquad \mathrm{Re}(s) \geq 0 \quad (4.10)$$

and the heat kernel expansion for

$$Tre^{-tP} \cong \sum_{n\geq 0} t^{\frac{n-m}{d}} \int_M a_n(x, P) dv(x) \quad (4.11)$$

where m is the dimension of the manifold in $C^\infty(M)$, d is the order of P and $dv(x) = \sqrt{g}\, d^m x$ where $g^{\mu\nu}$ is the metric on M appearing in equation (4.7). If $s = 0, -1, \ldots$ is a non-positive integer then $Tr\left(P^{-s}\right)$ is regular at this value of s and is given by

$$Tr\left(P^{-s}\right) = \mathrm{Res}\,\Gamma(s)\Big|_{s=\frac{m-n}{d}} a_n .$$

From this we deduce that

$$Tr\chi(P) \cong \sum_{n\geq 0} f_n a_n(P) \quad (4.12)$$

where the coefficients f_n are given by

$$f_0 = \int_0^\infty \chi(u) u\, du , \quad f_2 = \int_0^\infty \chi(u) du , \quad f_{2(n+2)} = (-1)^n \chi^{(n)}(0), \quad n \geq 0 \quad (4.13)$$

and $a_n(P)=\int a_n(x,P)dv(x)$. Hence, the eq. (4.12) can be rewritten also

$$Tr\chi(P)\cong\sum_{n\geq 0} f_n \int a_n(x,P)dv(x). \quad (4.13b)$$

The Seeley-de Witt coefficients $a_n(P)$ vanish for odd values of n. The first three a_n's for n even are:

$$a_0(x,P)=(4\pi)^{-m/2}Tr(\mathbb{1}) \qquad a_2(x,P)=(4\pi)^{-m/2}Tr\left(-\frac{R}{6}\mathbb{1}+E\right)$$

$$a_4(x,P)=(4\pi)^{-m/2}\frac{1}{360}Tr\left[\left(-12R;_\mu{}^\mu+5R^2-2R_{\mu\nu}R^{\mu\nu}+2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\mathbb{1}+\right.$$
$$\left.-60RE+180E^2+60E;_\mu{}^\mu+30\Omega_{\mu\nu}\Omega^{\mu\nu}\right], \quad (4.14)$$

where E and $\Omega_{\mu\nu}$ are defined by

$$E=B-g^{\mu\nu}\left(\partial_\mu\omega'_\nu+\omega'_\mu\,\omega'_\nu-\Gamma^\rho_{\mu\nu}\omega'_\beta\right),\;\; \Omega_{\mu\nu}=\partial_\mu\omega'_\nu-\partial_\nu\omega'_\mu+\left[\omega'_\mu\,\omega'_\nu\right],$$
$$\omega'_\mu=\frac{1}{2}g_{\mu\nu}\left(A^\nu-\Gamma^\nu\cdot\mathbb{1}\right). \quad (4.14a)$$

The Ricci and scalar curvature are defined by

$$R_{\mu\rho}=R^{ab}_{\mu\nu}e^\nu_b e_{a\rho}, \qquad R=R^{ab}_{\mu\nu}e^\mu_a e^\nu_b. \quad (4.14b)$$

Now, it is possible evaluate explicitly the spectral action (4.9). Using equations (4.8) and (4.14) we find:

$$E=\frac{1}{4}R\otimes\mathbb{1}_4\otimes\mathbb{1}_N+\frac{i}{4}\gamma^{\mu\nu}\otimes gF^i_{\mu\nu}T^i, \quad \Omega_{\mu\nu}=\frac{1}{4}R^{ab}_{\mu\nu}\gamma_{ab}\otimes 1_N-\frac{i}{2}\mathbb{1}_4\otimes gF^i_{\mu\nu}T^i. \quad (4.15)$$

From the knowledge that the invariants of the heat equation are polynomial functions of R, $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$, E and $\Omega_{\mu\nu}$ and their covariant derivatives, it is then evident from equation (4.15) that the spectral action would not only be diffeomorphism invariant but also gauge invariant. The first three invariants are then

$$a_0(P)=\frac{N}{4\pi^2}\int_M\sqrt{g}\,d^4x, \quad a_2(P)=\frac{N}{48\pi^2}\int_M\sqrt{g}\,R\,d^4x,$$
$$a_4(P)=\frac{1}{16\pi^2}\cdot\frac{N}{360}\int_M d^4x\sqrt{g}\left[\left(12R;_\mu{}^\mu+5R^2-8R_{\mu\nu}R^{\mu\nu}-7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)+\frac{120}{N}g^2F^i_{\mu\nu}F^{\mu\nu i}\right]. \quad (4.16)$$

For the special case where the dimension of the manifold M is four, we have a relation between the Gauss-Bonnet topological invariant and the three possible curvature square terms:

$$R^*R^*=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2 \quad (4.17)$$

where $R^*R^*\equiv\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}_{\mu\nu}R^{\gamma\delta}_{\rho\sigma}$. Moreover, we can change the expression for $a_4(P)$ in terms of

$C_{\mu\nu\rho\sigma}$ instead of $R_{\mu\nu\rho\sigma}$ where

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \left(g_{\mu[\rho}R_{\nu|\sigma]} - g_{\nu[\rho}R_{\mu|\sigma]}\right) + \frac{1}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)R \quad (4.18)$$

is the Weyl tensor. Using the identity:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + 2R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 \quad (4.19)$$

we can recast $a_4(P)$ into the alternative form:

$$a_4(p) = \frac{N}{48\pi^2}\int d^4x\sqrt{g}\left[-\frac{3}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{1}{120}\left(11R^*R^* + 12R;_\mu{}^\mu\right) + \frac{g^2}{N}F^i_{\mu\nu}F^{\mu\nu i}\right] \quad (4.20)$$

and this is explicitly conformal invariant. The Euler characteristic χ_E is related to R^*R^* by the relation

$$\chi_E = \frac{1}{32\pi^2}\int d^4x\sqrt{g}R^*R^* . \quad (4.21)$$

It is also possible to introduce a mass scale m_0 and consider χ to be a function of the dimensionless variable $\chi\left(\frac{P}{m_0^2}\right)$. In this case terms coming from $a_n(P), n > 4$ will be suppressed by powers of $\frac{1}{m_0^2}$:

$$I_b = \frac{N}{48\pi^2}\left\{12m_0^4 f_0\int d^4x\sqrt{g} + m_0^2 f_2\int d^4x\sqrt{g}R + f_4\int d^4x\sqrt{g}\left[-\frac{3}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{1}{10}R;_\mu{}^\mu + \right.\right.$$
$$\left.\left. + \frac{11}{20}R^*R^* + \frac{g^2}{N}F^i_{\mu\nu}F^{\mu\nu i}\right] + 0\left(\frac{1}{m_0^2}\right)\right\}. \quad (4.22)$$

Normalizing the Einstein and Yang-Mills terms in the bare action we then have:

$$\frac{Nm_0^2 f^2}{24\pi^2} = \frac{1}{\kappa_0^2} \equiv \frac{1}{8\pi G_0}, \qquad \frac{f_4 g_0^2}{12\pi^2} = 1, \quad (4.23)$$

and (4.22) becomes:

$$I_b = \int d^4x\sqrt{g}\left[\frac{1}{2\kappa_0^2}R + e_0 + a_0 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + c_0 R^*R^* + d_0 R;_\mu{}^\mu + \frac{1}{4}F^i_{\mu\nu}F^{\mu\nu i}\right], \quad (4.24)$$

where

$$a_0 = \frac{-3N}{80}\frac{1}{g_0^2}, \quad c_0 = -\frac{2}{3}a_0, \quad d_0 = -\frac{11}{3}a_0, \quad e_0 = \frac{Nm_0^4}{4\pi^2}f_0. \quad (4.25)$$

Hence, (4.24) can be rewritten also:

$$I_b = \int d^4x\sqrt{g}\left[\frac{1}{2\kappa_0^2}R + \frac{Nm_0^4}{4\pi^2}f_0 + \frac{-3N}{80}\frac{1}{g_0^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \left(\frac{-N}{40}\frac{1}{g_0^2}\right)R^*R^* + \right.$$
$$\left. -11\left(\frac{-N}{80}\frac{1}{g_0^2}\right)R;_\mu{}^\mu + \frac{1}{4}F_{\mu\nu}^i F^{\mu\nu i}\right]. \quad (4.25b)$$

The action for the fermionic quark sector is given by $(Q, D_q Q)$ (4.26), while the leptonic action have the simple form $(L, D_\ell L)$ (4.27). According to universal formula (4.3) the spectral action for the Standard Model is given by:

$$Tr\left[\chi\left(D^2/m_0^2\right)\right] + (\psi, D\psi), \quad (4.28)$$

where $(\psi, D\psi)$ will include the quark sector (4.26) and the leptonic sector (4.27). Calculating the bosonic part of the above action, we have the following result:

$$I = \frac{9m_0^4}{\pi^2}\frac{5}{4}f_0\int d^4x\sqrt{g} + \frac{3m_0^2}{4\pi^2}f_2\int d^4x\sqrt{g}\left[\frac{5}{4}R - 2y^2H^*H\right] + \frac{f_4}{4\pi^2}\int d^4x\sqrt{g}\left[\frac{1}{40}\frac{5}{4}\left(12R;_\mu{}^\mu + \right.\right.$$
$$+11R^*R^* - 18C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\Big) + 3y^2\left(D_\mu H^* D^\mu H - \frac{1}{6}RH^*H\right) + g_{03}^2 G_{\mu\nu}^i G^{\mu\nu i} + g_{02}^2 F_{\mu\nu}^\alpha F^{\mu\nu\alpha} +$$
$$\left.+\frac{5}{3}g_{01}^2 B_{\mu\nu}B^{\mu\nu} + 3z^2\left(H^*H\right)^2 - y^2\left(H^*H\right);_\mu{}^\mu\right] + 0\left(\frac{1}{m_0^2}\right), \quad (4.29)$$

where we have denoted

$$y^2 = Tr\left(\left|k_0^d\right|^2 + \left|k_0^u\right|^2 + \frac{1}{3}\left|k_0^e\right|^2\right), \quad z^2 = Tr\left(\left(\left|k_0^d\right|^2 + \left|k_0^u\right|^2\right)^2 + \frac{1}{3}\left|k_0^e\right|^4\right),$$
$$D_\mu H = \partial_\mu H - \frac{i}{2}g_{02}A_\mu^\alpha \sigma^\alpha H - \frac{i}{2}g_{01}B_\mu H. \quad (4.30)$$

Normalizing the Einstein and Yang-Mills terms gives:

$$\frac{15m_0^2 f_2}{4\pi^2} = \frac{1}{\kappa_0^2}, \quad \frac{g_{03}^2 f_4}{\pi^2} = 1, \quad g_{03}^2 = g_{02}^2 = \frac{5}{3}g_{01}^2. \quad (4.31)$$

Relations (4.31) among the gauge coupling constants coincide with those coming from $SU(5)$ unification. To normalize the Higgs field kinetic energy we have to rescale H by:

$$H \to \frac{2}{3}\frac{g_{03}}{y}H. \quad (4.32)$$

This transforms the bosonic action (4.29) to the form:

$$I_b = \int d^4x\sqrt{g}\left[\frac{1}{2\kappa_0^2}R - \mu_0^2\left(H^*H\right) + a_0 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + b_0 R^2 + c_0{}^* R^* R + d_0 R;_\mu{}^\mu + e_0 + \frac{1}{4}G_{\mu\nu}^i G^{\mu\nu i} + \right.$$
$$\left. + \frac{1}{4}F_{\mu\nu}^\alpha F^{\mu\nu\alpha} + \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \left|D_\mu H\right|^2 - \xi_0 R\left|H\right|^2 + \lambda_0\left(H^*H\right)^2\right], \quad (4.33)$$

where

$$\mu_0^2 = \frac{4}{3\kappa_0^2}, \quad a_0 = -\frac{9}{8g_{03}^2}, \quad b_0 = 0, \quad c_0 = -\frac{11}{18}a_0, \quad d_0 = -\frac{2}{3}a_0, \quad e_0 = \frac{45}{4\pi^2}f_0 m_0^4,$$
$$\lambda_0 = \frac{4}{3}g_{03}^2\frac{z^2}{y^4}, \quad \xi_0 = \frac{1}{6}. \quad (4.34)$$

This action has to be taken as the bare action at some cutoff scale Λ. The renormalized action will have the same form as (4.33) but with the bare quantities $\kappa_0, \mu_0, \lambda_0, a_0$ to e_0 and g_{01}, g_{02}, g_{03} replaced with physical quantities.

5. Review of some equations concerning various aspects of Loop Quantum Gravity. [5]

We introduce the volume associated with a region $\Omega \subset \Sigma$, where Σ is a spatial manifold of fixed topology (and no boundary). We have that:

$$V(\Omega) = \int_\Omega d^3x e = \int_\Omega d^3x\sqrt{\widetilde{E}} \equiv \int_\Omega d^3x\sqrt{\frac{1}{3!}\varepsilon_{mnp}\varepsilon^{abc}\widetilde{E}_a^m \widetilde{E}_b^n \widetilde{E}_c^p}. \quad (5.1)$$

Writing $V \equiv V(\Sigma)$, we first use the substitution

$$e_m{}^a(x) = \varepsilon_{mnp}\varepsilon^{abc}\widetilde{E}^{-1/2}\widetilde{E}_b^n \widetilde{E}_c^p(x) = \frac{1}{4\gamma}\left\{A_m{}^a(x), V\right\} \quad (5.2)$$

to recover the spatial dreibein. The second trick is to eliminate the extrinsic curvature using a doubly nested bracket. The first bracket is introduced by rewriting

$$K_m{}^a(x) = \frac{1}{\gamma}\left\{A_m{}^a(x), \overline{K}\right\} \quad \text{where} \quad \overline{K} \equiv \overline{K}(\Sigma) := \int_\Sigma d^3x K_m{}^a \widetilde{E}_a{}^m. \quad (5.3)$$

The second bracket comes in through identity

$$\overline{K}(x) = \frac{1}{\gamma^{3/2}}\left\{\frac{\widetilde{E}_a{}^m \widetilde{E}_b{}^n}{\sqrt{\widetilde{E}}}\varepsilon^{abc}F_{mnc}(x), V\right\}. \quad (5.4)$$

Canonical quantisation in the "position space representation" now proceeds by representing the dreibein as a multiplication operator, and the canonical momentum by the functional differential operator

$$\Pi_a{}^m(x) = \frac{\hbar}{i}\frac{\delta}{\delta e_m{}^a(x)}. \quad (5.5)$$

With these replacements, the classical constraints are converted to quantum constraint operators which act on suitable wave functionals. The diffeomorphism and Lorentz constraints become

$$H_a(x)\Psi[e]=0\,, \qquad L_{ab}(x)\Psi[e]=0\,. \quad (5.6)$$

They will be referred to as "kinematical constraints" throughout. Dynamics is generated via the Hamiltonian constraint, the *Wheeler-De Witt (WDW) equation*

$$H_0(x)\Psi[e]=0\,. \quad (5.7)$$

It is straightforward to include matter degrees of freedom, in which case the constraint operators and the wave functional $\Psi[e,...]$ depend on further variables (indicated by dots). The functional $\Psi[e,...]$ is sometimes referred to as the "wave function of the Universe", and is supposed to contain the complete information about the Universe "from beginning to end".
Now we represent the connection $A_m{}^a$ by a multiplication operator, and sets

$$\tilde{E}_a^m(x)=\frac{\hbar}{i}\frac{\delta}{\delta A_m{}^a(x)}\,. \quad (5.8)$$

The WDW functional depending on the spatial metric (or dreibein) is replaced by a functional $\Psi[A]$ living on the space of connections (modulo gauge transformations). The spatial metric must be determined from the operator for the inverse densitised metric

$$gg^{mn}(x)=-\hbar^2\frac{\delta}{\delta A_m{}^a(x)}\frac{\delta}{\delta A_n{}^a(x)}\,. \quad (5.9)$$

Furthermore, the spatial volume density is obtained from

$$g(x)=\tilde{E}(x)=\frac{i\hbar^3}{6}\varepsilon^{abc}\varepsilon_{mnp}\frac{\delta}{\delta A_m{}^a(x)}\frac{\delta}{\delta A_n{}^b(x)}\frac{\delta}{\delta A_p{}^c(x)}\,. \quad (5.10)$$

For the quantum constraints the replacement of the metric by connection variables leads to a Hamiltonian which is simpler than the original WDW Hamiltonian. Allowing for an extra factor of e (and assuming $e\neq 0,\infty$) the WDW equation becomes

$$\varepsilon^{abc}F_{mna}(A(x))\frac{\delta}{\delta A_m{}^b(x)}\frac{\delta}{\delta A_n{}^c(x)}\Psi[A]=0\,. \quad (5.11)$$

There is at least one interesting solution if one allows for a non-vanishing cosmological constant Λ. Using an ordering opposite to the one above, and including a term Λg with the volume density (5.10), the WDW equation reads

$$\varepsilon_{abc}\frac{\delta}{\delta A_m{}^a(x)}\frac{\delta}{\delta A_n{}^b(x)}\left[F_{mnc}(A(x))-\frac{i\hbar\Lambda}{6}\varepsilon_{mnp}\frac{\delta}{\delta A_p{}^c(x)}\right]\Psi_\Lambda[A]=0\,. \quad (5.12)$$

This is solved by

$$\Psi_{\Lambda}[A] = \exp\left(\frac{i}{\hbar\Lambda}\int_{\Sigma} d^3x L_{CS}(A)\right), \quad (5.13)$$

with the Chern-Simons Lagrangian $L_{CS} = A \wedge dA + iA \wedge A \wedge A$. Thence, the eq. (5.12) can be also rewritten

$$\exp\left(\frac{i}{\hbar\Lambda}\int_{\Sigma} d^3x A \wedge dA + iA \wedge A \wedge A(A)\right) \varepsilon_{abc} \frac{\delta}{\delta A_m{}^a(x)} \frac{\delta}{\delta A_n{}^b(x)} \left[F_{mnc}(A(x)) - \frac{i\hbar\Lambda}{6} \varepsilon_{mnp} \frac{\delta}{\delta A_p{}^c(x)}\right] = 0 .$$
(5.13b)

Loop Quantum Gravity makes use of wave functions which have singular support in the sense that they only probe the gauge connection on one-dimensional networks embedded in the three-dimensional spatial hypersurface Σ. By definition, each network is a graph Γ embedded in Σ and consisting of finitely many edges $e_i \in \Gamma$ and vertices $v \in \Gamma$. The edges are connected at the vertices. Each edge e carries a holonomy $h_e[A]$ of the gauge connection A. The wave function on the spin network over the graph Γ can be written as

$$\Psi_{\Gamma,\psi}[A] = \psi\left(h_{e_1}[A], h_{e_2}[A], \ldots\right), \quad (5.14)$$

where the ψ is some function of the basic holonomies associated to the edges $e \in \Gamma$.
The wave functionals (5.14) are called cylindrical, because they probe the connection A only "on a set of measure zero". With regard the definition of the space of spin network states, we introduce a suitable scalar product. In Loop Quantum Gravity this is the scalar product of two cylindrical functions $\Psi_{\Gamma,\{j\},\{C\}}[A]$ and $\Psi_{\Gamma',\{j'\},\{C'\}}[A]$ and it is defined as

$$\left\langle \Psi_{\Gamma,\{j\},\{C\}} \middle| \Psi'_{\Gamma',\{j'\},\{C'\}} \right\rangle = 0 \qquad \text{if} \quad \Gamma \neq \Gamma'$$
$$\left\langle \Psi_{\Gamma,\{j\},\{C\}} \middle| \Psi'_{\Gamma',\{j'\},\{C'\}} \right\rangle = \int \prod_{e_i \in \Gamma} dh_{e_i} \overline{\psi}_{\Gamma,\{j\},\{C\}}\left(h_{e_1}, \ldots\right) \psi'_{\Gamma',\{j'\},\{C'\}}\left(h_{e_1}, \ldots\right) \quad \text{if} \quad \Gamma = \Gamma', \quad (5.15)$$

where the integrals $\int dh_e$ are to be performed with the $SU(2)$ Haar measure.
With regard the form of the quantum Hamiltonian one starts with the classical expression written in loop variables. Despite the simplifications brought about by the following equation

$$\varepsilon_{abc} \tilde{E}_a^m \tilde{E}_b^n F_{mnc} = -\gamma^2\left(\Pi_{ab}\Pi_{ab} - \frac{1}{2}\Pi^2\right) - e^2 R^{(3)} = -\gamma^2 e \mathrm{H}_0 - \frac{1}{4}\left(1+\gamma^2\right)e^2\left(K_{ab}K_{ab} - K^2\right), \quad (5.16)$$

the Hamiltonian constraint is:

$$H[N] = \int_{\Sigma} d^3x N \frac{1}{\sqrt{\det \tilde{E}}} \tilde{E}_a^m \tilde{E}_b^n \left[\varepsilon^{abc} F_{mnc} - \frac{1}{2}\left(1+\gamma^2\right) K_{[m}{}^a K_{n]}{}^b\right]. \quad (5.16b)$$

In order to write the constraint in terms of only holonomies and fluxes, one has to eliminate the inverse square root as well as the extrinsic curvature factors. This can be done using the relations (5.2) – (5.4). Inserting these into the Hamiltonian constraint one obtains the expression

$$H[N]=\int_\Sigma d^3xN\varepsilon^{mnp}Tr\left[F_{mn}\{A_p,V\}-\frac{1}{2}(1+\gamma^2)\{A_m,\bar{K}\}\{A_n,\bar{K}\}\{A_p,V\}\right]. \quad (5.17)$$

6. On some equations concerning the dynamics of vector mode perturbations including quantum corrections based on Loop Quantum Gravity. [6]

The perturbed densitized triad and Ashtekar connection around a spatially flat Friedmann-Robertson-Walker (FRW) background are given by

$$E_i^a=\bar{p}\delta_i^a+\delta E_i^a, \quad A_a^i=\Gamma_a^i+\gamma K_a^i=\gamma\bar{k}\delta_a^i+\left(\delta\Gamma_a^i+\gamma\delta K_a^i\right), \quad (6.1)$$

where $\bar{p}$ and $\gamma\bar{k}$ are the background densitized triad and Ashtekar connection. In a canonical formulation, the Einstein-Hilbert action can be written equivalently using the Ashtekar connection and densitized triad as

$$S_{EH}=\int dt\int_\Sigma d^3x\left[\frac{1}{8\pi G\gamma}E_i^aL_tA_a^i-\left(\Lambda^iG_i+N^aC_a+NC\right)\right], \quad (6.2)$$

where Λ^i, N^a and N are Lagrange multipliers of the Gauss, diffeomorphism and Hamiltonian constraints. In triad variables, a Gauss constraint appears which generates internal gauge rotations of phase space functions because triads whose legs are rotated at a fixed point correspond to the same spatial metric. This constraint is given by

$$G(\Lambda):=\int_\Sigma d^3x\Lambda^iG_i=\frac{1}{8\pi G\gamma}\int_\Sigma d^3x\Lambda^i\left(\partial_aE_i^a+\varepsilon_{ij}{}^kA_a^jE_k^a\right). \quad (6.3)$$

Using the perturbed form of basic variables (6.1), it can be reduced to

$$G(\Lambda)=\frac{1}{8\pi G}\int_\Sigma d^3x\Lambda^i\left(\varepsilon_{ij}{}^a\bar{p}\delta K_a^j+\varepsilon_{ia}{}^k\bar{k}\delta E_k^a\right). \quad (6.4)$$

The diffeomorphism constraint generates gauge transformations corresponding to spatial coordinate transformations of phase space functions. Its general contribution from gravitational variables is given by

$$D_G[N^a]:=\int_\Sigma d^3xN^aC_a=\frac{1}{8\pi G\gamma}\int_\Sigma d^3xN^a\left[F_{ab}^iE_i^b-A_a^iG_i\right], \quad (6.5)$$

where the subscript "G" stands for "gravity" to separate the term from the matter contribution.
Using the expression of the perturbed basic variables (6.1), one can reduce the diffeomorphism constraint to

$$D_G[N^a]=\frac{1}{8\pi G}\int_\Sigma d^3x\delta N^c\left[-\bar{p}\left(\partial_k\delta K_c^k\right)-\bar{k}\delta_c^k\left(\partial_d\delta E_k^d\right)\right]. \quad (6.6)$$

In a canonical formulation, the Hamiltonian constraint generates "time evolution" of the spatial manifold for phase space functions satisfying the equations of motion. Its gravitational contribution in Ashtekar variables is (see also (5.16b))

$$H_G[N]=\frac{1}{16\pi G}\int_\Sigma d^3xN\frac{1}{\sqrt{|\det E|}}E_j^cE_k^d\left(\varepsilon_i^{\ jk}F_{cd}^i-2\left(1+\gamma^2\right)K_{[c}^jK_{d]}^k\right). \quad (6.7)$$

Using the general perturbed forms of basic variables and the expression of curvature $F_{ab}^i=\partial_aA_b^i-\partial_bA_a^i+\varepsilon^i{}_{jk}A_a^jA_b^k$, one can simplify (6.7). Up to quadratic terms is given by

$$H_G[N]=\frac{1}{16\pi G}\int_\Sigma d^3x\bar{N}\left[-6\bar{k}^2\sqrt{\bar{p}}-\frac{\bar{k}^2}{2\bar{p}^{\frac{3}{2}}}\left(\delta E_j^c\delta E_k^d\delta_c^k\delta_d^j\right)+\sqrt{\bar{p}}\left(\delta K_c^j\delta K_d^k\delta_k^c\delta_j^d\right)-\frac{2\bar{k}}{\sqrt{\bar{p}}}\left(\delta E_j^c\delta K_c^j\right)\right], \quad (6.8)$$

with $\delta N=0$ for vector modes.
With regard the quantum corrected Hamiltonian constraint, this is given by

$$H_G^Q[N]=\frac{1}{16\pi G}\int_\Sigma d^3x\bar{N}\alpha\left(\bar{p},\delta E_i^a\right)\left[-6\bar{k}^2\sqrt{\bar{p}}-\frac{\bar{k}^2}{2\bar{p}^{\frac{3}{2}}}\left(\delta E_j^c\delta E_k^d\delta_c^k\delta_d^j\right)+\sqrt{\bar{p}}\left(\delta K_c^j\delta K_d^k\delta_k^c\delta_j^d\right)-\frac{2\bar{k}}{\sqrt{\bar{p}}}\left(\delta E_j^c\delta K_c^j\right)\right]$$

(6.9),

where $\alpha\left(\bar{p},\delta E_i^a\right)$ is the correct function, now also depends on triad perturbations.
To study quantum gravity effects, we have introduced a quantum correction function $\alpha\left(\bar{p},\delta E_i^a\right)$ which depends on phase space variables. Having a new expression for the Hamiltonian constraint, there could be an anomaly term of quantum origin in the constraint algebra. A non-trivial anomaly in the algebra could occur in the Poisson bracket between $H_G^Q[N]$ and $D_G\left[N^a\right]$. This bracket turns out to be

$$\left\{H_G^Q[N],D_G\left[N^a\right]\right\}=\frac{1}{8\pi G}\int_\Sigma d^3x\bar{p}\left(\partial_j\delta N^c\right)3\bar{N}\bar{k}^2\sqrt{\bar{p}}\left[\frac{\partial\alpha}{\partial\left(\delta E_j^c\right)}+\frac{1}{3\bar{p}}\frac{\partial\alpha}{\partial\bar{p}}\left(\delta E_k^d\delta_c^k\delta_d^j\right)\right]. \quad (6.10)$$

With regard the quantum dynamics, there are also the holonomy corrections. Hence, it is possible write the following expression for the corrected Hamiltonian constraint

$$H_G^Q[N]=\frac{1}{16\pi G}\int_\Sigma d^3x\bar{N}\left[-6\sqrt{\bar{p}}\left(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2-\frac{1}{2\bar{p}^{\frac{3}{2}}}\left(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2\left(\delta E_j^c\delta E_k^d\delta_c^k\delta_d^j\right)+\right.$$
$$\left.+\sqrt{\bar{p}}\left(\delta K_c^j\delta K_d^k\delta_k^c\delta_j^d\right)-\frac{2}{\sqrt{\bar{p}}}\left(\frac{\sin 2\bar{\mu}\gamma\bar{k}}{2\bar{\mu}\gamma}\right)\left(\delta E_j^c\delta K_c^j\right)\right]. \quad (6.11)$$

Further, a non-trivial anomaly in the algebra can occur between the Poisson bracket between $H_G^Q[N]$ and $D_G\left[N^a\right]$

$$\left\{H_G^Q[N],D_G\left[N^a\right]\right\}=\frac{\bar{N}}{\sqrt{\bar{p}}}\left(\bar{k}-\frac{\sin 2\bar{\mu}\gamma\bar{k}}{2\bar{\mu}\gamma}\right)D_G\left[N^a\right]+\frac{1}{8\pi G}\int_\Sigma d^3x\bar{p}\left(\partial_c\delta N^j\right)\cdot$$
$$\cdot\frac{\bar{N}}{\sqrt{\bar{p}}}\left[\bar{p}\frac{\partial}{\partial\bar{p}}\left(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2+\left(\frac{\sin\bar{\mu}\gamma\bar{k}}{\bar{\mu}\gamma}\right)^2-\bar{k}^2\right]\left(\frac{\delta E_j^c}{\bar{p}}\right). \quad (6.12)$$

7. **On some equations concerning Matrix models as non-local hidden variables theories.** [7]

In this section we describe some equations that show that the matrix models which give non-perturbative definitions of string and M theory, may be interpreted as non-local hidden variables theories in which the quantum observables are the eigenvalues of the matrices while their entries are the non-local hidden variables.
We study a bosonic matrix model which is the bosonic part of the models used in string and M theory. The degrees of freedom are d $N \times N$ real symmetric matrices X_{ai}^{j}, with $a = 1,...,d$ and $i, j = 1,...,N$. The action is:

$$S = \mu \int dt Tr\left[\dot{X}_a^2 + \omega^2 [X_a, X_b][X^a, X^b]\right]. \quad (7.1)$$

We choose the matrices X^a to be dimensionless. ω is a frequency and μ has dimensions of $mass \cdot length^2$. It is useful to split the matrices into diagonal and off-diagonal pieces,

$$X_{ai}^{j} = D_{ai}^{j} + Q_{ai}^{j} \quad (7.2)$$

where $D^a =$ diagonal $\left(d_1^a,...,d_N^a\right)$ is diagonal and Q_{ai}^{j} has no diagonal elements. Since the Q_{ai}^{j} are dimensionless we will expect them to scale like a power of $T / \mu\omega^2$. We then write the action (7.1) as

$$S = \int dt\left[L^d + L^Q + L^{\text{int}}\right]. \quad (7.3)$$

The theory of the d's alone is free,

$$L^d = \mu \sum_{ai} \left(\dot{d}_i^a\right)^2, \quad (7.4)$$

while the theory of the Q's alone has the same quartic interaction

$$L^Q = \mu\left[\sum_{aij} \left(\dot{Q}_{ai}^{j}\right)^2 + \omega^2 [Q_a, Q_b][Q^a, Q^b]\right]. \quad (7.5)$$

The interaction terms between the diagonal and off-diagonal elements are

$$L^{\text{int}} = 2\mu\omega^2 \sum_{abij} \left[-\left(d_i^a - d_j^a\right)^2 \left(Q_{bi}^{j}\right)^2 - \left(d_i^a - d_j^a\right)\left(d_i^b - d_j^b\right) Q_{ai}^{j} Q_{bj}^{i} + 2\left(d_i^a - d_j^a\right) Q_i^{bj} [Q_a, Q_b]_j^{\ i}\right]. \quad (7.6)$$

Hence, the action (7.3) can be rewritten also as

$$S = \int dt \cdot \{ \ \mu \sum_{ai} \left(\dot{d}_i^a\right)^2 + \mu\left[\sum_{aij} \left(\dot{Q}_{ai}^{j}\right)^2 + \omega^2 [Q_a, Q_b][Q^a, Q^b]\right] +$$
$$+ 2\mu\omega^2 \sum_{abij} \left[-\left(d_i^a - d_j^a\right)^2 \left(Q_{bi}^{j}\right)^2 - \left(d_i^a - d_j^a\right)\left(d_i^b - d_j^b\right) Q_{ai}^{j} Q_{bj}^{i} + 2\left(d_i^a - d_j^a\right) Q_i^{bj} [Q_a, Q_b]_j^{\ i}\right] \}. \quad (7.7)$$

We now derive the Schroedinger equation for the eigenvalues of the matrices. This is a three step process. (i) Formulate the statistical variational principle for the matrix model. (ii) Make assumptions about the statistical ensemble. We assume that the model is in an S-ensemble, heat it to finite temperature T and then study the large N limit with $T \approx 1/N$. (iii) Derive an effective statistical variational principle for the eigenvalues by averaging over the variational principle of the matrix elements and show that when $N \to \infty$ this is equivalent to Schroedinger quantum theory for the eigenvalues.

STEP 1.

We begin by defining an S-ensemble for the matrix elements. That is, we begin with the variational principle

$$I[\rho,S]=\int dt\int (dd)(dQ)\rho(d,Q,t)\left[\dot{S}(d,Q)+\frac{1}{2\mu}\left(\frac{\delta S(d,Q,t)}{\delta d_i^a}\right)^2+\frac{1}{2\mu}\left(\frac{\delta S(d,Q,t)}{\delta Q_{ij}^a}\right)^2+U(d,Q)\right], \quad (7.8)$$

where $U(d,Q)$ is the interaction term L^{int} given by (7.6).

STEP 2.

- The Q system is in a distribution that is to leading order in $1/N$ statistically independent of the distribution of the eigenvalues. This means that to leading order the probability density factorizes
$$\rho(d,Q)=\rho_d(d)\rho_Q(Q)+O(1/N). \quad (7.9)$$

- The Q subsystem is in thermal equilibrium at a temperature T. So we have

$$\rho_Q(Q)=\frac{1}{Z}e^{-H(Q)/T} \quad (7.10)$$

where $H(Q)$ is the Hamiltonian corresponding to the Q system alone

$$H(Q)=\mu\left[\sum_{aij}\left(\dot{Q}_{ai}^j\right)^2-\omega^2[Q_a,Q_b][Q^a,Q^b]\right] \quad (7.11)$$

and

$$Z=\int dQe^{-H(Q)/T}\ . \quad (7.12)$$

As a result of these assumptions our variational principle reads,

$$I[\rho_d,S,T]=\int dt\int (dd)(dQ)\rho_d(d)\rho_Q(Q)\left[\dot{S}(d,Q)+\frac{1}{2\mu}\left(\frac{\delta S}{\delta d_i^a}\right)^2+\frac{1}{2\mu}\left(\frac{\delta S}{\delta Q_{ij}^a}\right)^2+U(d,Q)\right]. \quad (7.13)$$

STEP 3.

Now we want to derive an effective variational principle to describe the evolution of the probability distribution for the eigenvalues. We will do this by averaging the variational principle (7.13) over the values of the matrix elements, and then extracting the leading behaviour for large N and small T. We begin by inserting the factor unity in the form

$$1=\int\prod_{ai} d\lambda_i^a \delta\left(\lambda_i^a - d_i^a - \sum_j \frac{Q_{ij}^a Q_{ji}^a}{d_i^a - d_j^a} + \ldots\right). \quad (7.14)$$

Thus, we have,

$$I[\rho_d, S, T] = \frac{1}{Z}\int dt \int dddQ \int d\lambda \delta\left(\lambda_i^a - d_i^a - \sum_j \frac{Q_{ij}^a Q_{ji}^a}{\lambda_i^a - \lambda_j^a} + \ldots\right)\rho_d e^{-H(Q)/T}$$

$$\left[\dot{S} + \frac{1}{2\mu}\left(\frac{\delta S}{\delta d_i^a}\right)^2 + \frac{1}{2\mu}\left(\frac{\delta S}{\delta Q_{ij}^a}\right)^2 + U(d,Q)\right]. \quad (7.15)$$

We recall that in the theory of stochastic processes the limits which define time derivatives are taken after the averages over probability distributions, not before. So, we must write

$$\int dt \int dddQ \rho_d(d)\rho_Q(Q)\frac{1}{2\mu}\left(\frac{\delta S}{\delta d_i^a}\right)^2 = \int dt dddQ \rho_d(d)\rho_Q(Q))\mu\left(V(d)_i^a\right)^2 =$$

$$= \int dt \lim_{\Delta t\to 0}\int dddQ \rho_d(d)\rho_Q(Q)\mu\left[\frac{d_i^a(t+\Delta t) - d_i^a(t))^2}{\Delta t^2}\right] =$$

$$= \int dt \lim_{\Delta t\to 0}\int dddQ \rho_d(d)\rho_Q(Q)\frac{\mu}{2}\left\{\left[\frac{d_i^a(t+\Delta t) - d_i^a(t))^2}{\Delta t^2}\right] + \left[\frac{d_i^a(t) - d_i^a(t-\Delta t))^2}{\Delta t^2}\right]\right\}. \quad (7.16)$$

Note that the last equation follows trivially, for smooth motion, but it will have non trivial consequences once we have averaged over the Q's because the result for large N is to induce Brownian motion for the off diagonal elements and eigenvalues. Now we perform the integral over the d's. It is useful to write

$$d_i^a = \lambda_i^a + \Delta\lambda_i^a \quad (7.17)$$

where

$$\Delta\lambda_i^a(Q,\lambda) = -\sum_j \frac{Q_{ij}^a Q_{ji}^a}{\lambda_i^a - \lambda_j^a} + \ldots \quad (7.18)$$

has to be treated as a stochastic variable, taking into account its dependence on the Q's which are themselves fluctuating due to the assumption that they are in equilibrium in a potential. We then have, to leading order in $1/N$,

$$I[\rho_d, S_d, T] = \int dt \int d\lambda \rho_d(\lambda,t) \int dQ \rho_Q(Q)\left[\dot{S}(\lambda+\Delta\lambda,Q) + \frac{1}{2\mu}\left(\frac{\delta S(\lambda+\Delta\lambda,Q)}{\delta Q_{ij}^a}\right)^2 + U(\lambda,Q)\right] + K_{en}$$

(7.19)

where to leading order, the kinetic energy terms for the d's have become,

$$K_{en} = \int dt \lim_{\Delta t \to 0} \int d\lambda \rho_d(\lambda, t) \int dQ \rho_Q(Q) \frac{\mu}{2} \left\{ \left(\frac{(\lambda_i^a(t + \Delta t) - \lambda_i^a(t))^2}{\Delta t^2} \right) + \left(\frac{(\lambda_i^a(t) - \lambda_i^a(t - \Delta t))^2}{\Delta t^2} \right) \right\}. \quad (7.20)$$

We now are ready to integrate over the Q's. The key point is that the dependence of the λ's on the Q's through a sum of a large number of independent terms, $\sum_j \frac{Q_{ij}^a Q_{ji}^a}{d_i^a - d_j^a}$, as well as the coupling of the λ's with the Q's coming from the terms in $U(\lambda, Q)$ turns the λ's into stochastic variables, described by a stochastic differential equation of the form

$$D\lambda_i^a = b_i^a(\lambda, t)dt + \Delta\lambda_i^a \qquad \Delta t > 0 \quad (7.21)$$
$$D\lambda_i^a = b_i^{*a}(\lambda, t)dt + \Delta^*\lambda_i^a \qquad \Delta t < 0 \quad (7.22)$$

with

$$\langle \Delta\lambda_i^a \Delta\lambda_j^b \rangle = \delta^{ab}\delta_{ij}\nu_\lambda dt \quad dt > 0 \quad (7.23)$$
$$\langle \Delta^*\lambda_i^a \Delta^*\lambda_j^b \rangle = -\delta^{ab}\delta_{ij}\nu_\lambda dt \quad dt < 0 \quad (7.24).$$

Here the brackets mean

$$\langle F(\lambda, Q) \rangle = \int dQ \rho_Q(Q) F(\lambda, Q). \quad (7.25)$$

We note that we can use the value of ν_λ described from the following equation

$$\nu_\lambda = \nu_d + \frac{N\nu_Q q^2}{r^2} + \ldots \approx \omega \left[\frac{dt^{3/2}}{4(d-1)^{3/2} N^{\frac{3}{4}(p-1)}} + \frac{t^{3/2}}{N^{9p/4+3/4}} \right] \quad (7.25b)$$

We have, from the Focker-Planck equations that the current velocity is

$$v_i^a(\lambda) = \frac{1}{2}\left(b_i^a + b_i^{*a}\right) \quad (7.26)$$

while the osmotic velocity is

$$u_i^a(\lambda) = \frac{1}{2}\left(b_i^a - b_i^{*a}\right) = \nu_\lambda \frac{\delta \ln \rho_\lambda}{\delta \lambda_i^a}. \quad (7.27)$$

From these we can derive

$$\lim_{\Delta t \to 0} \int dQ \rho_d(\lambda, t) \rho_Q(Q) \frac{1}{2} \left(\frac{\left(\lambda_i^a(t + \Delta t) - \lambda_i^a(t)\right)^2}{\Delta t^2} \right) = \rho_d(\lambda, t)\left[b_i^a(\lambda, t)^2 + NC\right] \quad (7.28)$$

and

$$\lim_{\Delta t \to 0} \int dQ \rho_d(\lambda, t) \rho_Q(Q) \frac{1}{2} \left(\frac{\left(\lambda_i^a(t) - \lambda_i^a(t - \Delta t)\right)^2}{\Delta t^2} \right) = \rho_d(\lambda, t)\left[b_i^{*a}(\lambda, t)^2 + NC\right] \quad (7.29)$$

where C is the infinite constant defined in the following equation

$$C = vd \lim_{\Delta t \to 0} \frac{1}{\Delta t}. \quad (7.29b)$$

To go further we need to define the effective Hamilton-Jacobi function for the eigenvalues. We define

$$S_\lambda(\lambda) = \int (dQ) \rho_Q(Q) S(\lambda, Q). \quad (7.30)$$

We now show that, to leading order in $1/N$,

$$\mu v_i^a = \frac{\delta S_\lambda(\lambda)}{\delta \lambda_a^i}. \quad (7.31)$$

Consider the probability conservation law that follows from the statistical variational principle that defines the dynamics of our matrix models, eq. (7.8).

$$\dot{\rho}(d,Q) = -\frac{1}{\mu}\left[\frac{\delta}{\delta d_{ai}}\left(\rho(d,Q)\frac{\delta S(d,Q)}{\delta d_{ai}}\right) + \frac{\delta}{\delta Q_{aij}}\left(\rho(d,Q)\frac{\delta S(d,Q)}{\delta Q_{aij}}\right)\right]. \quad (7.32)$$

But using (7.9) and (7.10) we have that

$$\dot{\rho}(d,Q) = \dot{\rho}_d(d)\rho_Q(Q). \quad (7.33)$$

We also have, by the same assumptions, since a thermal distribution is stationary and has no current velocity,

$$v^{aij}(Q) = \frac{1}{\mu}\frac{\delta S(d,Q)}{\delta Q_{aij}} = O(1/N). \quad (7.34)$$

Thus, we have, integrating over the Q's,

$$\dot{\rho}_d(d) = -\frac{1}{\mu}\frac{\delta}{\delta d_{ai}}\left(\rho_d(d)\frac{\delta}{\delta d_{ai}}\int dQ \rho_Q(Q) S(d,Q)\right) + O(1/N). \quad (7.35)$$

To leading order we can replace everywhere the dependence on d_{ai} with dependence on λ_{ai}, since the terms by which they differ are also higher order in $1/N$. Thus we have

$$\dot{\rho}_d(\lambda) = -\frac{1}{\mu}\frac{\delta}{\delta \lambda_{ai}}\left(\rho_d(\lambda)\frac{\delta S_\lambda(\lambda)}{\delta \lambda_{ai}}\right) + O(1/N). \quad (7.36)$$

But by (7.26) we must have

$$\dot{\rho}_d(\lambda) = -\frac{\delta \rho_d(\lambda) v^{ai}(\lambda)}{\delta \lambda_{ai}}. \quad (7.37)$$

This establishes eq. (7.31). With this result we have the key relation that,

$$\frac{\mu}{2}\left(b^2+b^{*2}\right)=\frac{\mu}{2}\left(v^2+u^2\right)=\left\{\frac{1}{2\mu}\left(\frac{\delta S_\lambda(\lambda)}{\delta\lambda_i^a}\right)^2+\frac{\mu v_\lambda^2}{2}\left(\frac{\delta\ln\rho_\lambda(\lambda)}{\delta\lambda_i^a}\right)^2\right\}. \quad (7.38)$$

We also define

$$E_Q=\int dQ\rho_Q(Q)\left[\frac{1}{2\mu}\left(\frac{\delta S_Q(Q)}{\delta Q_{ij}^a}\right)^2+\frac{\mu\omega^2}{2}Tr[Q,Q]^2\right] \quad (7.39)$$

and

$$\frac{\mu}{2}\Omega_d^2\sum_{aij}\left(\lambda_i^a-\lambda_j^a\right)^2=\int dQ\rho_Q(Q)U^{\mathrm{int}}(\lambda,Q). \quad (7.40)$$

We can estimate that $E_Q=TN(N-1)/4\approx N\mu\omega^2$ so this is a divergent constant in the limit. The result is

$$I[\rho_d,S,T]=\int dt\int d\lambda\left[\rho_d(\lambda,t)\dot{S}_\lambda-H^{eff}(S_\lambda,\rho_d,T)\right] \quad (7.41)$$

where, the effective Hamiltonian for the eigenvalues is

$$H^{eff}(S_\lambda,\rho_\lambda,T)=\rho_d(\lambda)\left[E_Q^{'}+\left\{\left(\frac{\frac{1}{2\mu}\delta S_\lambda(\lambda)}{\delta\lambda_i^a}\right)^2+\frac{\mu v_\lambda^2}{2}\left(\frac{\delta\ln\rho_\lambda(\lambda)}{\delta\lambda_i^a}\right)^2\right\}+\frac{\mu\Omega_d^2}{2}\sum_{aij}\left(\lambda_I^a-\lambda_j^a\right)^2\right] \quad (7.42)$$

where $E_Q^{'}=E_Q+N\mu C$ contains both infinite constants. The resulting equations of motion are

$$E_Q^{'}+\dot{S}_\lambda+\frac{1}{2\mu}\left(\frac{\delta S_d(\lambda)}{\delta\lambda_i^a}\right)^2+\frac{\mu\Omega_d^2}{2}\sum_{aij}\left(\lambda_I^a-\lambda_j^a\right)^2+U^{quantum}=0 \quad (7.43)$$

and the current conservation equation

$$\dot{\rho}_\lambda=-\frac{1}{\mu}\partial^{ai}\rho_\lambda\left(\partial_{ai}S_\lambda\right). \quad (7.44)$$

The so-called "quantum potential" is given by

$$U^{quantum}=\mu v_\lambda^2\left\{\left(\frac{\delta\ln\rho_\lambda(\lambda)}{\delta\lambda_i^a}\right)^2+\frac{1}{\rho_\lambda}\partial_{ai}\left(\rho_\lambda\partial^{ai}\ln\rho_\lambda\right)\right\}=-\mu v_\lambda^2\frac{1}{\sqrt{\rho_\lambda(\lambda)}}\nabla^2\sqrt{\rho_\lambda(\lambda)}. \quad (7.45)$$

These we recognize as the real and imaginary parts of Schroedinger equation, when we write

$$\Psi(\lambda,t)=\sqrt{\rho_\lambda}e^{S_\lambda/\hbar} \quad (7.46)$$

with

$$\hbar = \mu \nu_\lambda = \mu \omega \frac{dt^{3/2}}{4(d-1)^{3/2}}. \quad (7.47)$$

So, finally, we have in the limit $N \to \infty$,

$$i\hbar \frac{d\Psi(\lambda,t)}{dt} = \left[-\frac{\hbar^2}{2\mu} \frac{\delta^2}{\delta(\lambda_i^a)^2} + \frac{\mu\Omega_d^2}{2} \sum_{aij} (\lambda_I^a - \lambda_j^a)^2 + E_Q^{'} \right] \Psi(\lambda,t). \quad (7.48)$$

Finally, we can show that the conserved energy of the original theory splits into two pieces,

$$H = H^\Psi + E_Q^{'} \quad (7.49)$$

where

$$H^\Psi = \int d\lambda \overline{\Psi} \left[-\frac{\hbar^2}{2\mu} \frac{\delta^2}{\delta(\lambda_i^a)^2} + \frac{\mu\Omega_d^2}{2} \sum_{aij} (\lambda_I^a - \lambda_j^a)^2 \right] \psi. \quad (7.50)$$

Since $E_Q^{'}$ is an infinite constant the result is that H^Ψ, which is the quantum mechanical energy, is conserved as $N \to \infty$. We can then renormalize the wave-functional so that

$$\Psi_r(\lambda) = e^{iE_Q^{'}t/\hbar} \Psi(\lambda). \quad (7.51)$$

Finally, we note that as $\Omega_d^2 \approx 1/N$ the eigenvalues become free in the limit $N \to \infty$. Thus, when $N \to \infty$ the probabilities evolve according to the free Schroedinger equation

$$i\hbar \frac{d\Psi_r(\lambda,t)}{dt} = \left[-\frac{\hbar^2}{2\mu} \frac{\delta^2}{\delta(\lambda_i^a)^2} \right] \Psi_r(\lambda,t). \quad (7.52)$$

Furthermore, from eqs. (7.43) and (7.50) we can also write eq. (7.49) as:

$$H = \int d\lambda \overline{\Psi} \left[-\frac{\hbar^2}{2\mu} \frac{\delta^2}{\delta(\lambda_i^a)^2} + \frac{\mu\Omega_d^2}{2} \sum_{aij} (\lambda_I^a - \lambda_j^a)^2 \right] \psi + \mu \nu_\lambda^2 \frac{1}{\sqrt{\rho_\lambda(\lambda)}} \nabla^2 \sqrt{\rho_\lambda(\lambda)} - \dot{S}_\lambda +$$
$$- \frac{1}{2\mu} \left(\frac{\delta S_d(\lambda)}{\delta \lambda_i^a} \right)^2 - \frac{\mu\Omega_d^2}{2} \sum_{aij} (\lambda_I^a - \lambda_j^a)^2. \quad (7.53)$$

8. On some equations concerning the quantum supergravity and the role of a "free" vacuum in loop quantum gravity. [8]

Now we will consider mainly $N = 1$ supergravity. This can be formulated in chiral variables which extend the Ashtekar-Sen variables of general relativity. In this formulation, the canonical variables are the left handed $su(2)$ spin connection A_a^i and its super-partner spin-3/2 field ψ_a^A. These fit together into a connection field of the super-Lie algebra $Osp(1|2)$. We thus define the graded connection:

$$\mathrm{A}_a := A_a^i J_i + \psi_a^A Q_A \quad (8.1)$$

where a is the spatial index. If $\tilde{E}_i^a$ and π_A^a are momenta of A_a^i and ψ_a^A respectively, we can define the graded momentum as:

$$\xi^a := \tilde{E}_i^a J^i + \pi_A^a Q^A . \quad (8.2)$$

The constraints that generate local gauge transformations can then be expressed as

$$G_i = D_a \tilde{E}_i^a + \frac{i}{\sqrt{2}} \pi_A^a \psi_{aB} \tau_i^{AB} = 0 . \quad (8.3)$$

The left and right handed supersymmetry transformations are generated by,

$$L_A = D_a \pi_A^a - ig \tilde{E}_i^a \tau_{iA}^B \psi_{aB} = 0 \quad (8.4)$$

$$R^A = \varepsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \sigma_{kB}^A \left(-4iD_{[a} \psi_{b]}^B + \sqrt{2} g \varepsilon_{abc} \pi^{cB} \right) = 0 , \quad (8.5)$$

where the cosmological constant is given by $\Lambda = -g^2$. The diffeomorphism and Hamiltonian constraints can be derived by taking the Poisson brackets of (8.4) and (8.5). These may be written simply in terms of the fundamental representation of $Osp(1|2)$, which is 3 dimensional. The super-Lie algebra $Osp(1|2)$ is then generated by five 3×3 matrices $G_I (I = 1...5)$. Using them we can define

$$\mathrm{A}_a^I = \left(A_a^i , \psi_a^A \right), \quad (8.6) \qquad \xi_I^a = \left(\tilde{E}_i^a , \pi_A^a \right), \quad (8.7)$$

where $I = (i, A)$ labels the five generators of $Osp(1|2)$. Then the first two constraints can be combined into one $Osp(1|2)$ Gauss constraint:

$$D_a \xi_I^a = 0 \quad (8.8)$$

while the last one combines with the Hamiltonian constraint to give:

$$\xi^a \xi^b F_{ab} - ig^2 \varepsilon_{abc} \xi^a \xi^b \xi^c = 0 , \quad (8.9)$$

where F_{ab} is the curvature of the super connection A_a:

$$F_{ab} := d_a \mathrm{A}_b + [\mathrm{A}_a , \mathrm{A}_b]. \quad (8.10)$$

The loop representation for supergravity in the chiral representation can be constructed in terms of $Osp(1|2)$ Wilson loops. These are defined in terms of the super-trace taken in the fundamental 3 dimensional representation of $Osp(1|2)$.

$$\mathrm{T}[\gamma] = StrP \exp\left(\oint_\gamma ds \mathrm{A}_a \gamma^a \right) \equiv StrU_\gamma(\mathrm{A}) . \quad (8.11)$$

These Wilson loop states are subject to additional relations arising from intersections of loops. These are solved completely by the introduction of the spin network basis, which are complete and orthogonal. We can construct the loop-momentum variables by inserting the $Osp(1|2)$ invariant momentum ξ^a into the Wilson loops:

$$\mathrm{T}^a[\alpha](s) = Str[U_\alpha(\mathrm{A})\xi^a(\alpha(s))]. \quad (8.12)$$

It is straightforward to show that the $\mathrm{T}[\gamma]$ and $\mathrm{T}^a[\alpha](s)$ form a closed algebra under Poisson brackets, which we will call the $N=1$ super-loop algebra. We will also need to describe operators quadratic in the conjugate momenta, which in the loop representation are formed by inserting two momenta in the loop trace,

$$\mathrm{T}^{ab}[\alpha](s,t) = Str[U_\alpha(s,t)\xi^a(\alpha(t))U_\alpha(t,s)\xi^b(\alpha(s))]. \quad (8.13)$$

The higher order loop operators are similarly defined as

$$\mathrm{T}^{ab...c}[\alpha](s,t,...v) = Str[U_\alpha(s,t)\xi^a(\alpha(t))U_\alpha(t,u)\xi^b(\alpha(u))...U_\alpha(v,s)\xi^c(\alpha(s))]. \quad (8.14)$$

The supersymmetric extension of the Chern-Simons state may be formed from the Chern-Simons form of the superconnection A_a,

$$\Psi_{SCS}(\mathrm{A}_a) = \exp\left[\frac{i}{2\Lambda}\int d^3x STr\left(\mathrm{A}\wedge F - \frac{1}{3}\mathrm{A}\wedge\mathrm{A}\wedge\mathrm{A}\right)\right]. \quad (8.15)$$

This state is an exact solution to all the quantum constraints. Like the ordinary Chern-Simons state it also has a semiclassical interpretation as the ground state associated with DeSitter or Anti-DeSitter spacetime.
Now, we want to describe some equations concerning the role of a "free" vacuum in Loop Quantum Gravity.
The classical Ashtekar-Barbero variables are obtained by the transformation

$$A_a^i = \Gamma_a^i + \beta K_a^i, \quad (8.16)$$

where β is the Immirzi parameter and $\Gamma_a^i[E]$ denotes the spin connection as a function of E:

$$\Gamma_a^i[E] = \frac{1}{2}\varepsilon^{ijk}E_k^b\left[\partial_b E_a^j - \partial_a E_b^j + E_j^c E_a^l \partial_b E_c^l\right] + \frac{1}{4}\varepsilon^{ijk}\left[2E_a^j\frac{\partial_b E}{E} - E_b^j\frac{\partial_a E}{E}\right]. \quad (8.17)$$

Here, we take β to be real. The transformation leads to the following Poisson brackets

$$\{E_i^a(\underline{x}), A_b^j(\underline{y})\} = \{E_i^a(\underline{x}), \Gamma_b^j[E](\underline{y}) + \beta K_b^j(\underline{y})\} = \frac{\beta\kappa}{2}\delta_a^b\delta_i^j\delta(\underline{x}-\underline{y}), \quad (8.18)$$

or in terms of Fourier modes

$$\left\{E_i^a(\underline{k}), A_b^{j*}(\underline{k}')\right\} = \frac{\beta\kappa}{2}\delta_a^b \delta_i^j \delta_{\underline{k},\underline{k}'}. \quad (8.19)$$

The linearization of (8.16) induces a canonical transformation on the reduced variables:

$$A_{red}^{ab} = \varepsilon_{acd}\partial_c e_{db}^{red} + \beta K_{red}^{ab}. \quad (8.20)$$

One may check that A_{red}^{ab} is again symmetric, transverse and of constant trace. In the quantum theory, we introduce the new operator

$$\hat{A}_{red}^{ab} = \varepsilon_{acd}\partial_c \hat{e}_{db}^{red} + \beta \hat{K}_{red}^{ab}. \quad (8.21)$$

Up to an A-dependent phase (which we choose to be zero), eigenstates of $\hat{A}$ have the form

$$\varphi_A\left[e_i(\underline{k})\right] = \mathrm{N}\exp\left[\frac{2i}{\hbar\kappa\beta}\sum_{\underline{k}}\left(A_i^*(\underline{k})e_i(\underline{k}) - ke_1^*(\underline{k})e_2(\underline{k})\right)\right]. \quad (8.22)$$

Within the quantum theory, the canonical transformation (8.20) is implemented by a unitary map

$$\psi[e] \to e^{if[e]/\hbar}\psi[e], \qquad \hat{O} \to e^{if[e]/\hbar}\hat{O}e^{-if[e]/\hbar},$$

that turns the $\hat{A}_i$-operator into a pure functional derivative in e_i^*, i.e.

$$\hat{A}_i(\underline{k}) \to -i\hbar\frac{\kappa\beta}{2}\frac{\partial}{\partial e_i^*(\underline{k})}. \quad (8.23)$$

We see from (8.22) that the required factor is

$$e^{if[e]/\hbar} := \exp\left(\frac{2i}{\hbar\kappa\beta}\sum_{\underline{k}} ke_1^*(\underline{k})e_2(\underline{k})\right).$$

The transformed vacuum reads

$$\Psi\left[e_i(\underline{k})\right] = \mathrm{N}\exp\left[-\frac{1}{\hbar\kappa}\sum_{\underline{k}}\left(\omega(\underline{k})e_i^*(\underline{k})e_i(\underline{k}) - \frac{2i}{\beta}ke_1^*(\underline{k})e_2(\underline{k})\right)\right], \quad (8.24)$$

In terms of reduced Fourier components, it takes the form

$$\Psi\left[e_{ab}^{red}(\underline{k})\right] = \mathrm{N}\exp\left[-\frac{1}{\hbar\kappa}\sum_{\underline{k}}\left(\omega(\underline{k})e_{ab}^{red*}(\underline{k})e_{ab}^{red}(\underline{k}) + \frac{1}{\beta}e_{ab}^{red*}(\underline{k})\varepsilon_{acd}k_c e_{db}^{red}(\underline{k})\right)\right]. \quad (8.25)$$

By doing a Gaussian integration, we can transform (8.24) to the A-representation:

$$\Psi\left[A_i(\underline{k})\right]=\mathrm{N}\exp\left\{-\frac{1}{\hbar\kappa}\left[\frac{1}{\beta^2\omega_0}A_i^2(0)+\frac{3/4}{1+\beta^2}\sum_{k>0}\frac{1}{k}\left(A_i^*(\underline{k})A_i(\underline{k})+\frac{2i}{\beta}A_1^*(\underline{k})A_2(\underline{k})\right)\right]\right\}. \quad (8.26)$$

Hence, the canonically transformed and regularized vacuum becomes

$$\Psi\left[e_{ab}^{red}(\underline{k})\right]=\mathrm{N}\exp\left[-\frac{1}{\hbar\kappa}\sum_{k\leq\Lambda}\left(\omega(\underline{k})e_{ab}^{red*}(\underline{k})e_{ab}^{red}(\underline{k})+\frac{1}{\beta}e_{ab}^{red*}(\underline{k})\varepsilon_{acd}k_c e_{db}^{red}(\underline{k})\right)\right]. \quad (8.27)$$

In the regularized scheme, we consider the position space field $e_{ab}^{red}(\underline{x})$ as a function

$$e_{ab}^{red}(\underline{x}):=\frac{1}{\sqrt{V}}\sum_{k\leq\Lambda}e^{i\underline{k}\cdot\underline{x}}e_{ab}^{red}(\underline{k}) \quad (8.28)$$

of the Fourier modes $e_{ab}^{red}(\underline{k})$. Hence, we can write the state functional also as

$$\Psi\left[e_{ab}^{red}(\underline{k})\right]=\mathrm{N}\exp\left[-\frac{1}{\hbar\kappa}\left(\int d^3x\int d^3y W_\Lambda(\underline{x},\underline{y})e_{ab}^{red}(\underline{x})e_{ab}^{red}(\underline{y})-\frac{i}{\beta}\int d^3x e_{ab}^{red}(\underline{x})\varepsilon_{acd}\partial_c e_{db}^{red}(\underline{x})\right)\right], \quad (8.29)$$

where the kernel W_Λ is defined by

$$W_\Lambda(\underline{x},\underline{y})=\frac{1}{V}\sum_{k\leq\Lambda}e^{i\underline{k}\cdot(\underline{x}-\underline{y})}\omega(\underline{k}). \quad (8.30)$$

Now, we simplify the state (8.29) by dropping the β-dependent phase factor:

$$\Psi\left[e_{ab}^{red}(\underline{k})\right]=\mathrm{N}\exp\left[-\frac{1}{\hbar\kappa}\int d^3x\int d^3y W_\Lambda(\underline{x},\underline{y})e_{ab}^{red}(\underline{x})e_{ab}^{red}(\underline{y})\right]. \quad (8.31)$$

Furthermore, we extend the functional (8.31) to the full configuration space. The most simple possibility would be to use the projection map

$$e_{ab}^{red}(\underline{k})=P_{ab}^{cd}(\underline{k})e_{cd}(\underline{k}), \quad (8.32)$$

and define the extended state by the pull-back, i.e.

$$\Psi_{ext}\left[E_i^a(\underline{k})\right]:=\mathrm{N}\exp\left[-\frac{1}{\hbar\kappa}\int d^3x\int d^3y W_\Lambda(\underline{x},\underline{y})(Pe)_{ab}(\underline{x})(Pe)_{ab}(\underline{y})\right]. \quad (8.33)$$

Now, we could drop the projectors in (8.33) and define the state as

$$\Psi_{ext}\left[E_i^a(\underline{k})\right]:=\mathrm{N}\exp\left[-\frac{1}{\hbar\kappa}\int d^3x\int d^3y W_\Lambda(\underline{x},\underline{y})e_{ab}(\underline{x})e_{ab}(\underline{y})\right]. \quad (8.34)$$

We replace the triad fluctuations $e_{ab}(\underline{x})$ in (8.34) by the fluctuation of the densitized inverse metric $\tilde{g}^{ab}(\underline{x})$:

$$e_{ab}(\underline{x}) \to \frac{1}{2}\left(\tilde{g}^{ab}(\underline{x}) - \delta^{ab}\right) = \frac{1}{2}\left(E_i^a(\underline{x})E_i^b(\underline{x}) - \delta^{ab}\right). \quad (8.35)$$

The triad fields are 1-densitites, so $\tilde{g}^{ab}(\underline{x})$ has density weight 2. Since E_i^a contains only modes up to $k = \Lambda$, we can write $\tilde{g}^{ab}(\underline{x})$ also in a smeared form

$$\tilde{g}^{ab}_\Lambda := \int d^3x' \, E_i^a(\underline{x})\delta_\Lambda(\underline{x} - \underline{x}')E_i^b(\underline{x}'), \quad (8.36)$$

where the smearing is done with the regularized delta function

$$\delta_\Lambda(\underline{x}) := \frac{1}{V}\sum_{k \le \Lambda} e^{i\underline{k}\cdot\underline{x}} . \quad (8.37)$$

The new state Ψ_{ext} is defined as

$$\Psi_{ext}\left[E_i^a(\underline{k})\right] := \mathrm{N}\exp\left[-\frac{1}{4\hbar\kappa}\int d^3x\int d^3y W_\Lambda(\underline{x},\underline{y})\left(\int d^3x' \, E_i^a(\underline{x})\delta_\Lambda(\underline{x} - \underline{x}')E_i^b(\underline{x}') - \delta^{ab}\right)\right.$$
$$\left.\times\left(\int d^3y' \, E_i^a(\underline{y})\delta_\Lambda(\underline{y} - \underline{y}')E_i^b(\underline{y}') - \delta^{ab}\right)\right]. \quad (8.38)$$

This state is almost gauge-invariant, but not completely, due to the smearing at the cutoff scale. By a Gaussian integration, we transform the state (8.38) to the A-representation:

$$\Psi_{ext}\left[A_a^i(\underline{k})\right] = \mathrm{N}\int DE \exp\left(-\frac{2i}{\hbar\kappa\beta}\sum_{k\le\Lambda}A_i^{a*}(\underline{k})E_i^a(\underline{k})\right)\times\exp\left[-\frac{1}{4\hbar\kappa}\int d^3x\int d^3y W_\Lambda(\underline{x},\underline{y})\right.$$
$$\left.\left(\int d^3x' \, E_i^a(\underline{x})\delta_\Lambda(\underline{x} - \underline{x}')E_i^b(\underline{x}') - \delta^{ab}\right)\times\left(\int d^3y' \, E_i^a(\underline{y})\delta_\Lambda(\underline{y} - \underline{y}')E_i^b(\underline{y}') - \delta^{ab}\right)\right]. \quad (8.39)$$

Using that

$$\int DE \exp\left(-\frac{2i}{\hbar\kappa\beta}\sum_{k\le\Lambda}A_i^{a*}(\underline{k})E_{ir}^a(\underline{k})\right) = \left(\prod_{k\le\Lambda, k^1>0}\prod_{i=1}^{3}\prod_{a=1}^{3}\prod_{r=0}^{1}\int_{-\infty}^{\infty} dE_{ir}^a(\underline{k})\right)\exp\left(-\frac{2i}{\hbar\kappa\beta}\sum_{k\le\Lambda}A_{ir}^a(\underline{k})E_{ir}^a(\underline{k})\right)$$

is the delta functional on the connection, and that the operator $\hat{E}_i^{a1}$ acts like

$$i\hbar\frac{\kappa\beta}{2}\frac{\partial}{\partial A_a^i(\underline{k})},$$

we can write the entire expression (8.39) as

$$\Psi_{ext}\left[A_a^i(\underline{k})\right] = \mathrm{N}\exp\left[-\frac{1}{4\hbar\kappa}\int d^3x\int d^3y W_\Lambda(\underline{x},\underline{y})\left(\int d^3x' \, \hat{E}_i^a(\underline{x})\delta_\Lambda(\underline{x} - \underline{x}')\hat{E}_i^b(\underline{x}') - \delta^{ab}\right)\right.$$
$$\left.\times\left(\int d^3y' \, \hat{E}_i^a(\underline{y})\delta_\Lambda(\underline{y} - \underline{y}')\hat{E}_i^b(\underline{y}') - \delta^{ab}\right)\right]\delta(A), \quad (8.40)$$

where

$$\hat{E}_i^a(\underline{x}) = \frac{1}{\sqrt{V}}\sum_{k\le\Lambda} e^{i\underline{k}\cdot\underline{x}}\hat{E}_i^a(\underline{k}). \quad (8.41)$$

This form of the state is similar to Thiemann's general complexifier form for coherent states.

Now, we have the following action

$$\int d^3x' \hat{E}_i^a(\underline{x})\delta_\Lambda(\underline{x}-\underline{x}')\hat{E}_i^b(\underline{x}')\widetilde{S} = \widetilde{g}_{\widetilde{S}}^{ab}(\underline{x})\widetilde{S}\,, \quad (8.42)$$

and the delta functional

$$\delta_{T^*}(\overline{A}) = \sum_{\widetilde{S}\subset T_\Lambda^*} \widetilde{S}(0)\widetilde{S}^*(\overline{A}). \quad (8.43)$$

With the action (8.42) and the replacement of the delta functional by (8.43), we obtain the state

$$\Psi = \mathrm{N}\sum_{\widetilde{S}\subset T_\Lambda^*} \widetilde{S}(0)\exp\left[-\frac{1}{4\hbar\kappa}\int d^3x\int d^3y W_\Lambda(\underline{x}-\underline{y})\widetilde{h}_{\widetilde{S}}^{ab}(\underline{x})\widetilde{h}_{\widetilde{S}}^{ab}(\underline{y})\right]\widetilde{S}^*. \quad (8.44)$$

Now we make the transition from (8.44) to a gauge-invariant state in H_0. Gauge-averaging simply yields

$$\Psi_0 = \mathrm{N}\sum_{S\subset T_\Lambda^*} S(0)\exp\left[-\frac{1}{4\hbar\kappa}\int d^3x\int d^3y W_\Lambda(\underline{x}-\underline{y})\widetilde{h}_S^{ab}(\underline{x})\widetilde{h}_S^{ab}(\underline{y})\right]S^*, \quad (8.45)$$

where the sum ranges over all gauge-invariant spin networks S on T_Λ^*. We introduce the coefficient

$$\Psi_0(S) := S(0)\exp\left[-\frac{1}{4\hbar\kappa}\int d^3x\int d^3y W_\Lambda(\underline{x}-\underline{y})\widetilde{h}_S^{ab}(\underline{x})\widetilde{h}_S^{ab}(\underline{y})\right], \quad (8.46)$$

and write this more compactly as

$$\Psi_0 = \mathrm{N}\sum_{S\subset T_\Lambda^*}\Psi_0(S)S^*. \quad (8.47)$$

One can think of $\Psi_0(S)$ as the wave-function of Ψ_0 in the S-representation.
From the equation (8.45), including a phase factor, if we were using a hypercubic lattice, we would simply get the following wave-function:

$$\Psi_0(S) = S(0)\exp\left[-\frac{1}{4\hbar\kappa}\int d^3x\int d^3y W_\Lambda(\underline{x}-\underline{y})\widetilde{h}_S^{ab}(\underline{x})\widetilde{h}_S^{ab}(\underline{y}) + \frac{i}{4\hbar\kappa\beta}\sum_v \widetilde{h}_S^{ab}(v)\varepsilon^{acd}\nabla^c\widetilde{h}_S^{ab}(v)\right]. \quad (8.48)$$

With regard the graviton states, we take the Schrodinger representation of linearized extended ADM gravity. For $\underline{k}\neq 0$, we define creation and annihilation operators such that

$$\left[a_i(\underline{k}), a_j^\dagger(\underline{k}')\right] = \delta_{ij}\delta_{\underline{k},\underline{k}'}. \quad (8.49)$$

With regard the one-graviton state with polarization i and momentum $\underline{k}$, we can write this also with tensors:

$$\Psi_{i,\underline{k}}\left[e_{ab}^{red}\left(\underline{k}\right)\right]=2\sqrt{\frac{k}{\hbar\kappa}}\varepsilon_i^{ab}\left(\underline{k}\right)e_{ab}^{red*}\left(\underline{k}\right)\Psi\left[e_{ab}^{red}\left(\underline{k}\right)\right]. \quad (8.50)$$

The canonical transformation to Ashtekar-Barbero variables adds the phase factor

$$\exp\left(\frac{2i}{\hbar\kappa\beta}\sum_{\underline{k}}ke_1^*\left(\underline{k}\right)e_2\left(\underline{k}\right)\right). \quad (8.51)$$

Next we extend the functional from the reduced to the full configuration space. This gives us

$$\Psi_{i,\underline{k}}\left[E_i^a\left(\underline{k}\right)\right]=2\sqrt{\frac{k}{\hbar\kappa}}\varepsilon_i^{la}\left(\underline{k}\right)\frac{1}{\sqrt{V}}\int d^3xe^{i\underline{k}\cdot\underline{x}}e_{la}\left(\underline{x}\right)\Psi\left[E_i^a\left(\underline{k}\right)\right]. \quad (8.52)$$

We replace $e_{la}\left(\underline{x}\right)$ by $\frac{1}{2}\left(\tilde{g}_\Lambda^{la}\left(\underline{x}\right)-\delta^{la}\right)$ and arrive at

$$\Psi_{i,\underline{k}}\left[E_l^a\left(\underline{k}\right)\right]=\sqrt{\frac{k}{\hbar\kappa}}\varepsilon_i^{la}\left(\underline{k}\right)\frac{1}{\sqrt{V}}\int d^3xe^{i\underline{k}\cdot\underline{x}}\left(\tilde{g}_\Lambda^{la}\left(\underline{x}\right)-\delta^{la}\right)\Psi\left[E_l^a\left(\underline{k}\right)\right]. \quad (8.53)$$

We bring this into the complexifier form, make the transition to H and finally apply the gauge projector. The result is the state

$$\Psi_{i,\underline{k}}=\mathrm{N}\sum_{S\subset\mathrm{T}_\Lambda^*}S(0)\sqrt{\frac{k}{\hbar\kappa}}\varepsilon_i^{la}\left(\underline{k}\right)\tilde{h}_S^{la*}\left(\underline{k}\right)\times\exp\left[-\frac{1}{4\hbar\kappa}\int d^3x\int d^3yW_\Lambda\left(\underline{x}-\underline{y}\right)\tilde{h}_S^{ab}\left(\underline{x}\right)\tilde{h}_S^{ab}\left(\underline{y}\right)+p_t\right]S^* \quad (8.54)$$

in the gauge-invariant Hilbert space H_0 (here p_t is the phase term). We define an associated wave-function

$$\Psi_{i,\underline{k}}(S):=S(0)\sqrt{\frac{k}{\hbar\kappa}}\varepsilon_i^{la}\left(\underline{k}\right)\tilde{h}_S^{la*}\left(\underline{k}\right)\times\exp\left[-\frac{1}{4\hbar\kappa}\int d^3x\int d^3yW_\Lambda\left(\underline{x}-\underline{y}\right)\tilde{h}_S^{ab}\left(\underline{y}\right)+p_t\right], \quad (8.55)$$

and write the state as

$$\Psi_{i,\underline{k}}=\mathrm{N}\sum_{S\subset\mathrm{T}_\Lambda^*}\Psi_{i,\underline{k}}(S)S^*. \quad (8.56)$$

In the same way, we construct multiply excited states. Denote the polarizations and momenta of the gravitons by $i_1,\underline{k}_1;\ldots;i_N,\underline{k}_N$. Then, the N-graviton state in H_0 becomes

$$\Psi_{i_1,\underline{k}_1;\ldots;i_N,\underline{k}_N}=\mathrm{N}_{i_1,\underline{k}_1;\ldots;i_N,\underline{k}_N}\sum_{S\subset\mathrm{T}_\Lambda^*}\Psi_{i_1,\underline{k}_1;\ldots;i_N,\underline{k}_N}(S)S^*, \quad (8.57)$$

where

$$\Psi_{i_1,\underline{k}_1;\ldots;i_N,\underline{k}_N}(S):=S(0)\left(\prod_{n=1}^N\sqrt{\frac{k_n}{\hbar\kappa}}\varepsilon_{i_n}^{la}\left(\underline{k}_n\right)\tilde{h}_S^{la*}\left(\underline{k}_n\right)\right)\times\exp\left[-\frac{1}{4\hbar\kappa}\int d^3x\int d^3yW_\Lambda\left(\underline{x}-\underline{y}\right)\tilde{h}_S^{ab}\left(\underline{x}\right)\tilde{h}_S^{ab}\left(\underline{y}\right)+p_t\right].$$

(8.58)

The normalization factor $N_{i_1,\underline{k}_1;\ldots;i_N,\underline{k}_N}$ depends on the excitation number of each mode.
The basic idea of this approach is to start from the free Fock vacuum of linearized gravity and construct from it a state Ψ_0 that could play the role of the "free" vacuum in loop quantum gravity. The state we get is a superposition

$$\Psi_0 = \mathrm{N}\sum_{S\subset \mathrm{T}_\Lambda^*}\Psi_0(S)S^* . \quad (8.59)$$

The sum ranges over all spin networks S whose graph lies on the dual complex T_Λ^* of the triangulation. The coefficients $\Psi_0(S)$ are given by

$$\Psi_0(S) = S(0)\exp\left[-\frac{1}{4\hbar\kappa}\int d^3x\int d^3y W_\Lambda(\underline{x}-\underline{y})\tilde{h}_S^{ab}(\underline{x})\tilde{h}_S^{ab}(\underline{y}) + p_t\right]. \quad (8.60)$$

9. On some equations concerning the unifying role of equivariant cohomology in the Topological Field Theories. [9]

Now, we describe the Yang-Mills equations and action. Let Σ_T be a spacetime of dimension n, $G_{\mu\nu}$ a metric on it. $P \to \Sigma_T$ a principal G bundle. Let $\mathrm{A}(P)$ be the space of all connections on P. This space is infinite-dimensional, and is the space of gauge fields in nonabelian gauge theory. We would like to write an action on $\mathrm{A}(P)$ which is gauge invariant.
To get an action consider $*F$. This is an $(n-2)-$form with values in the Lie algebra. Let "Tr" be an invariant form on the Lie algebra – for example the ordinary trace in the fundamental representation for $SU(N)$. The gauge-invariant action is

$$I_{YM}[A] = \frac{1}{4e^2}\int_{\Sigma_T} Tr\tilde{F}\wedge *F = \frac{1}{4e^2}\int_{\Sigma_T} d^D x\sqrt{\det G_{\mu\nu}}G^{\mu\lambda}G^{\nu\rho}TrF_{\mu\nu}F_{\lambda\rho} . \quad (9.1)$$

The equations of motion and Bianchi identities are:

$$D_A F = dF + [A,F] = 0 \quad \text{Bianchi identity} \qquad D_A * F = 0 \quad \text{Equations of motion.} \quad (9.2)$$

In local coordinates (9.2) is:

$$D_{[\mu}F_{\nu\lambda]} = 0 \qquad D^\mu F_{\mu\nu} = 0 . \quad (9.3)$$

Consider quantizing the theory on a cylinder with periodic spatial coordinate x of period L. With the gauge choice $A_0 = 0$, we may characterize the Hilbert space as follows. The constraint obtained from varying A_0 in the Yang-Mills action (9.1) is

$$D_1 F_{10} = 0 . \quad (9.4)$$

In canonical quantization we must impose the constraint (9.4) on wavefunctions $\Psi[A_1^a(x)]$. Let T_a be an orthonormal (ON) basis of $\underline{g}$, with structure constants $[T_a,T_b] = f_{ab}^c T_c$. Then the Gauss law constraint becomes:

$$\nabla \cdot E^a \Psi = \left(\partial_1 \frac{\delta}{\delta A_1^a(x)} + f_{bc}^a A_1^b(x) \frac{\delta}{\delta A_1^c(x)} \right) \Psi = 0, \quad (9.5)$$

which is solved by wavefunctionals of the form:

$$\Psi\left[A_1^a(x)\right] = \Psi\left[P \exp\left[\int_0^L dx A_1\right]\right]. \quad (9.6)$$

Demanding invariance under x-independent gauge transformations shows that Ψ only depends on the conjugacy class of $U = P\exp\int_0^L dx A_1$. Hence, we conclude that: the Hilbert space of states is the space of L^2-class functions on G. The inner product will be

$$\langle f_1 | f_2 \rangle = \int_G dU f_1^*(U) f_2(U)$$

where dU is the Haar measure normalized to give volume one.
A form $U \in W(\underline{g}_S) \otimes \Omega(V)_{RD}$ will be called a universal Thom form (in the Weil model) if it satisfies: (i) U is basic; (ii) $QU = 0$, where $Q = d_W + d$; (iii) $\int_V U = 1$.
In order to write a manifestly closed expression for U we enlarge the equivariant cohomology complex to:

$$W(\underline{g}_S) \otimes \Omega^\bullet(V) \otimes \Omega^\bullet(\Pi V^*) \quad (9.7)$$

and consider the following differential

$$Q_W = d_W \otimes 1 \otimes 1 + 1 \otimes d \otimes 1 + 1 \otimes 1 \otimes \delta \quad (9.8)$$

d_W is the Weil differential, while δ is the de Rham differential in ΠV^*. Explicitly:

$$\delta \begin{pmatrix} \rho_a \\ \pi_a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_a \\ \pi_a \end{pmatrix}.$$

The grading, or ghost numbers of ρ and π are -1 and 0, respectively. Consider the "gauge fermion"

$$\Psi = -i\langle \rho, x \rangle + \frac{1}{4}(\rho, \theta\rho)_{V^*} - \frac{1}{4}(\rho, \pi)_{V^*} \in W(\underline{g}_S) \otimes \Omega^\bullet(V) \otimes \Omega^\bullet(\Pi V^*) \quad (9.9)$$

which in orthonormal coordinates reads:

$$\Psi = -\rho_a \left(i x^a - \frac{1}{4}\theta^{ab}\rho_b + \frac{1}{4}\pi_a \right). \quad (9.10)$$

Expanding the action and doing the Gaussian integral on π leads to the third representation:

$$U = \int_{V^* \times \Pi V^*} \prod_{a=1}^{2m} \frac{d\pi_a}{\sqrt{2\pi}} \frac{d\rho_a}{\sqrt{2\pi}} e^{Q_W(\Psi)}. \quad (9.11)$$

The advantage of this representation is that

$$\int Q_W(\ldots) = (d + d_W)\int(\ldots) \quad (9.12)$$

which follows from the simple observation that

$$\delta = \pi_a \frac{\partial}{\partial \rho_a} \quad (9.13)$$

and hence $\int \delta(\ldots) = 0$ by properties of the Berezin integral over ρ. Since the integrand is Q_W-closed, it immediately follows from (9.12) that U is closed in $W(\underline{g}_S) \otimes \Omega^\bullet(V)$. Thus, we have finally proven that U satisfies criteria (i), (ii) and (iii) and hence U is a universal Thom form.

A universal Thom form $U_C \in \left(S(\underline{g}_s^{\,*}) \otimes \Omega(V)\right)^G$ can also be constructed in the Cartan model of equivariant cohomology. $U_{C,t}$ is obtained by a differential on the complex $S(\underline{g}_S^{\,*}) \otimes \Omega^\bullet(V) \otimes \Omega^\bullet(\Pi V^*)$ defined by:

$$Q_C x = (d - \iota_\phi)x, \qquad Q_C \begin{pmatrix} \rho_a \\ \pi_a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -L_\phi & 0 \end{pmatrix} \begin{pmatrix} \rho_a \\ \pi_a \end{pmatrix}. \quad (9.14)$$

We then may take the much simpler gauge fermion:

$$U_{C,t} = \frac{1}{(2\pi)^{2m}} \int_{V^* \times \Pi V^*} d\pi d\rho e^{Q_C\left[\rho_a\left(-ix^a - t\pi_a\right)\right]} = \left(\frac{1}{4\pi t}\right)^m \int_{\Pi V^*} d\rho \exp\left[-\frac{1}{4t}(x,x)_V + i\langle \rho, dx\rangle + t(\rho, \phi\rho)_{V^*}\right]. \quad (9.15)$$

It is important that we note that the Cartan model is used in topological gauge theories and string theories.

One application of the nonabelian localization theorem is a formula for the partition function of YM_2. The key observation is that we can write the YM_2 partition function as

$$Z_{phys} = \frac{1}{volG} \int_{LieG \times \hat{A}} d\phi \hat{\mu}_A \exp\left\{-\left[\frac{i}{4\pi^2}\int_\Sigma Tr\left(\phi F - \frac{1}{2}\psi \wedge \psi\right)\right] - \left[\varepsilon \int_\Sigma \mu \frac{1}{8\pi^2} Tr\phi^2(P)\right]\right\} \quad (9.16)$$

where ε is related to the gauge coupling by

$$e^2 = 2\pi^2 \varepsilon \quad (9.17)$$

and $\hat{\mu}_A = dAd\psi$ is the usual superspace measure. (9.16) coincides with the partition function of the theory because the integral over ψ is just such as to give the symplectic volume element on A:

$$\int_{\hat{A}} \hat{\mu}_A \exp\left[\frac{i}{4\pi^2}\int_\Sigma Tr\left(\frac{1}{2}\psi \wedge \psi\right)\right] \leftrightarrow \frac{\omega^n}{n!}. \quad (9.18)$$

The remaining action then coincides with the following equation

$$I = -\frac{1}{2}\int iTr(\phi F) + \frac{1}{2}e^2 \mu Tr\phi^2 \ .$$

(9.16) is a special case of integration of equivariant differential forms for $\Omega_G(\mathrm{A})$.
The Supersymmetric Quantum Mechanics (SQM) is an example of a topological field theory.
Let X be a Riemannian manifold with metric $G_{\mu\nu}$. The degrees of freedom of supersymmetric quantum mechanics on X, $SQM(X)$, consist of a coordinate $\phi^\mu(t)$ on X and fermionic coordinates $\Psi^\mu(t)$ and $\overline{\Psi}^\mu(t)$, which together may be thought of as components of a superfield:

$$\Phi^\mu = \phi^\mu + \overline{\theta}\Psi^\mu + \theta\overline{\Psi}^\mu + \overline{\theta}\theta F^\mu \ . \quad (9.19)$$

Here F^μ is an auxiliary field. The action of $SQM(X)$ is given by:

$$I_{SQM} = \int dt \left(G_{\mu\nu} s^\mu s^\nu - i\overline{\Psi}_\mu D_t \Psi^\mu - \frac{1}{4} R^\mu{}_{\nu\rho\sigma} \overline{\Psi}_\mu G^{\nu\nu'} \overline{\Psi}_{\nu'} \Psi^\rho \Psi^\sigma \right) \quad (9.20)$$

where

$$s^\mu = \dot{\phi}^\mu + G^{\mu\nu}\partial_\nu W \ , \quad D_t\Psi^\mu = \nabla_t \Psi^\mu + G^{\mu\nu}\left(\nabla_\nu \nabla_\lambda W\right)\Psi^\lambda \ , \quad \nabla_t \Psi^\mu = \frac{d}{dt}\Psi^\mu + \Gamma^\mu_{\nu\rho}\dot{\phi}^\nu \Psi^\rho \quad (9.21)$$

and W is a real-valued function on X.
The theory is supersymmetric and has a standard superspace construction. If we make the field redefinition

$$\overline{F}_\mu = 2iG_{\mu\nu}\left(F^\nu + \dot{\phi}^\nu\right) + \partial_\lambda G_{\mu\nu}\Psi^\lambda \overline{\Psi}^\nu \quad (9.22)$$

the usual supersymmetry transformations take the simple form:

$$Q\phi^\mu = \Psi^\mu \ , \quad Q\overline{\Psi}_\mu = \overline{F}_\mu \ , \quad Q\Psi^\mu = 0 \ , \quad Q\overline{F}_\mu = 0 \ . \quad (9.23)$$

Evidently $Q^2 = 0$. SQM provides a simple example of a topological field theory. The nilpotent fermionic symmetry Q can be interpreted as a BRST[1] operator. Moreover, (9.20), is derived from a Q-exact action:

$$I_{SQM} = \int dt \left\{ Q, \overline{\Psi}_\mu \left(is^\mu + \frac{1}{4} G^{\lambda\mu} \overline{\Psi}_\kappa \Gamma^\kappa_{\lambda\nu} \Psi^\nu + \frac{1}{4} G^{\mu\nu} \overline{F}_\nu \right) \right\} . \quad (9.24)$$

After integrating out $\overline{F}_\mu$, to get

[1] We remember that the BRST invariance, is a nilpotent symmetry of Faddeev-Popov gauge-fixed theories, which encodes the information container in the original gauge symmetry. Furthermore, we remember that the Faddeev-Popov determinant, is the Jacobian determinant arising from the reduction of a gauge-invariant functional integral to an integral over a gauge slice. While, the Faddeev-Popov ghosts are the wrong-statistics quantum fields used to give a functional integral representation of the Faddeev-Popov determinant.

$$\overline{F}_\mu = -2\left(iG_{\mu\nu}s^\nu - \frac{1}{2}\Gamma^\nu_{\lambda\mu}\Psi^\lambda\overline{\Psi}_\nu \right) \quad (9.25)$$

we recover (9.20). Topological invariance implies that the partition function

$$Z_{SQM} = \int e^{-I_{SQM}} \quad (9.26)$$

is a topological invariant, i.e. independent of any einbein one puts on the one-dimensional space, and, if X is compact, is independent of W.
We will apply the MQ formalism to the bundle $E = T\ LX$. The Riemannian geometry of LX is almost identical to that of X, with some extra delta functions entering expressions when written in terms of local coordinates. The Levi-Civita connection, ∇, on LX is just the pullback connection from X. It acts on a vector field

$$V = \oint dt V^\mu(\phi,t)\frac{\partial}{\partial\phi^\mu(t)}$$

to produce

$$\nabla V = \oint dt_1 dt_2 \left[\frac{\delta V^\mu(\phi,t_1)}{\delta\phi^\nu(t_2)} + \Gamma^\mu_{\nu\lambda}(\phi(t_2))V^\lambda(\phi,t_1)\delta(t_1-t_2) \right] \frac{\partial}{\partial\phi^\mu(t_1)} \otimes \tilde{d}\phi^\nu(t_2)$$

where $\left\{ \frac{\partial}{\partial\phi^\mu(t)} \right\}$ and $\left\{ \tilde{d}\phi^\mu(t) \right\}$ are to be viewed as bases of $T\ LX$ and T^*LX, respectively.
Having specified our connection we choose a section of E to be:

$$[s(\phi)]^\mu(t) = \dot{\phi}^\mu(t) + G^{\mu\nu}\partial_\nu W(t). \quad (9.27)$$

An easy calculation shows that

$$\nabla s = \oint dt \left[\nabla_t \Psi^\mu + \left(\nabla^\mu \nabla_\lambda W \right) \Psi^\lambda \right] \frac{\partial}{\partial\phi^\mu(t)}. \quad (9.28)$$

Identifying coordinates for the dual bundle ΠE^* as $\rho \to \overline{\Psi}_\mu(t)$, we see that $\langle \rho, \nabla s \rangle$ corresponds to fermion bilinear terms in (9.20). Thus the SQM action (9.20) coincides exactly with the MQ formula. Moreover, the expression (9.24) for the action coincides with the gauge fermion (9.10). The differential is:

$$Q = \oint dt \left[\Psi^\mu(t)\frac{\partial}{\partial\phi^\mu(t)} + \overline{F}_\mu(t)\frac{\partial}{\partial\overline{\Psi}_\mu(t)} \right].$$

The index of the Dirac operator is obtained by reducing the supersymmetry to $N = \frac{1}{2}$, by imposing the condition

$$\Psi^\mu_+ = \Psi^\mu_- = 2^{-1/2}\tilde{\Psi}^\mu.$$

From the Bianchi identity it follows that the curvature term in I disappears, leaving

$$I_{SQM}^{N=\frac{1}{2}} = \int dt \left\{ -\frac{1}{2} G_{\mu\nu} \dot{\phi}^{\mu} \dot{\phi}^{\nu} + \frac{1}{2} G_{\mu\nu} \tilde{\Psi}^{\mu} D_t \tilde{\Psi}^{\nu} \right\}.$$

There is one remaining supersymmetry and the supercharge Q for this model is related to the Dirac operator, $i\mathrm{D}$, on the target space. Evaluating the path integral of this theory one finds

$$\operatorname{ind}\ i\mathrm{D} = \int_M \hat{A}(M) \quad (9.29)$$

where $\hat{A}(M)$ is the *A-roof genus*

$$\hat{A}(M) = \prod_{a=1}^{\frac{1}{2}\dim M} \frac{\frac{1}{2} x_a}{\sinh \frac{1}{2} x_a} = 1 - \frac{1}{24} p_1(M) + \frac{1}{5760} \left(7 p_1^2 - 4 p_2\right)(M) + \ldots \quad (9.30)$$

Hence, the eq. (9.29) can be rewritten also

$$\operatorname{ind}\ i\mathrm{D} = \int_M \prod_{a=1}^{\frac{1}{2}\dim M} \frac{\frac{1}{2} x_a}{\sinh \frac{1}{2} x_a} = 1 - \frac{1}{24} p_1(M) + \frac{1}{5760} \left(7 p_1^2 - 4 p_2\right)(M) + \ldots \quad (9.30b)$$

Here x_a label the eigenvalues of the skew-diagonalized form $\frac{1}{2\pi} R_{ab}$ and $p_i(M)$ are the Pontryagin classes. Taking the tensor product of the Dirac complex with a vector bundle, E, one obtains the twisted Dirac complex, whose index may be computed from the following SQM action

$$I = \int dt \left\{ G_{\mu\nu} \left[\frac{1}{2} \dot{\phi}^{\mu} \dot{\phi}^{\nu} - \frac{i}{2} \tilde{\Psi}^{\mu} D_t \tilde{\Psi}^{\nu} \right] + i C_i^* \left(\dot{C}_i - A_{\mu}^a(\phi) \dot{\phi}^{\mu} T_{ij}^a C_j \right) - \frac{i}{2} \tilde{\Psi}^{\mu} \tilde{\Psi}^{\nu} F_{\mu\nu}^a C_i^* T_{ij}^a C_j \right\}$$

where A_{μ} is the connection on the associated bundle (viewed here as an external gauge field) and the C_i form a Clifford algebra in the representation of G generated by T_{ij}^a.
Evaluating the partition function, one finds

$$\operatorname{ind} i\mathrm{D}_A = \int_M ch(F) \wedge \hat{A}(M) \quad (9.31)$$

where $ch(F)$ is the Chern character

$$ch(F) = Tr \exp \frac{i}{2\pi} F = \operatorname{rank} E + c_1(F) + \frac{1}{2}\left(c_1^2 - 2c_2\right)(F) + \ldots \quad (9.32)$$

and the $c_1(F)$ are the Chern classes of F. Hence the eq. (9.31) can be rewritten

$$\operatorname{ind} i\mathrm{D}_A = \int_M Tr \exp \frac{i}{2\pi} F \wedge \prod_{a=1}^{\frac{1}{2}\dim M} \frac{\frac{1}{2} x_a}{\sinh \frac{1}{2} x_a} = 1 - \frac{1}{24} p_1(M) + \frac{1}{5760} \left(7 p_1^2 - 4 p_2\right)(M) + \ldots \quad (9.32b)$$

The Cartan representation of the MQ form in (9.15) becomes:

$$\Psi_L = \frac{1}{e^2}\int_M d^4x\sqrt{g}Tr\chi(iF_+ - tH) \quad (9.33)$$

and

$$Q_C\Psi_L = \frac{1}{e^2}\int d^4x\sqrt{g}\{TrH(iF_+ - tH) - Tr\chi(i(D_A\psi)_+ + t[\phi,\chi])\}. \quad (9.34)$$

Integrating out H gives

$$H = \frac{i}{2t}F_+, \quad (9.35)$$

yielding the Lagrangian:

$$\frac{1}{e^2}\int d^4x\sqrt{g}\left\{-\frac{1}{4t}TrF_+^2 - iTr\chi^{\mu\nu}(D_A\psi)^+_{\mu\nu} + t\chi_{\mu\nu}[\phi,\chi]^{\mu\nu})\right\}. \quad (9.36)$$

We now apply to the present case the following form

$$\Phi_p(P \to M) = \left(\frac{1}{2\pi i}\right)^{\dim G}\int_{\underline{\hat{g}}} d\lambda d\eta e^{Q_C\Psi_p}, \quad \Psi_p = i(\lambda, C^{\mathrm{l}})_{\underline{g}} \in \Omega^1(P).$$

Here, C^{l} is a one-form with values in the Lie algebra $Lie(G)$. From the following expression

$$D_A^*\tau = (D_A^\mu\tau_\mu) = g^{\mu\nu}(\partial_\nu\tau_\mu + [A_\nu, \tau_\mu] + \Gamma^\lambda_{\nu\mu}\tau_\lambda) = 0,$$

we see that it may be identified with

$$C^{\mathrm{l}} \to -*D_A*\psi = -(D^\mu\psi_\mu)^a \in \Omega^1(\mathrm{A}; Lie(G)) \quad (9.37)$$

so that

$$\Psi_P = -\frac{i}{e^2}(\lambda, C^{\mathrm{l}}) = \frac{i}{e^2}\int_M Tr\lambda D_A*\psi = -\frac{i}{e^2}\int_M d^4x\sqrt{g}Tr(\lambda*D_A*\psi) \quad (9.38)$$

and hence

$$Q_C\Psi_P = -\frac{i}{e^2}\int_M Tr(\eta D_A*\psi + \lambda\{\psi,*\psi\} + \lambda D_A*D_A\phi). \quad (9.39)$$

Combining the actions for projection and localization we recover Witten's celebrated Lagrangian for Donaldson theory

$$I_D = \frac{1}{e^2}\int d^4x\sqrt{g}\left[iTr(\eta D_A^\mu\psi_\mu + \lambda\{\psi_\mu,\psi^\mu\} + \lambda D_A*D_A\phi) + \left(-\frac{1}{4t}TrF_+^2 - Tr\chi_{\mu\nu}[i(D_A\psi)_+^{\mu\nu} + t[\phi,\chi^{\mu\nu}]]\right)\right]. \quad (9.40)$$

In the paper "Two Dimensional Gauge Theories Revisited" (Witten, 1992), Witten shows that the cohomological theory with the following gauge fermion

$$\lambda = t\int Tr\psi * Df = t\int \mu Tr\psi^{\alpha} D_{\alpha} f$$

is equivalent to a theory with gauge fermion

$$\Psi = \Psi_D + \frac{1}{te^2}\int_{\Sigma} Tr\chi\lambda . \quad (9.41)$$

He then shows that the theory with gauge fermion (9.41) is equivalent to $D = 2$ Donaldson theory with the standard gauge fermion (9.38) + (9.33), up to terms of order $\approx \mathrm{O}\left(e^{-c/te^2}\right)$ for $t \to 0^+$. The difference comes about because the second term in (9.41) introduces new Q-fixed points.
The result is that the generator of intersection numbers on the moduli space of flat connections is related to the physical partition function by

$$\frac{1}{Vol(G)}\int DA \exp\left[\frac{2\pi^2}{\varepsilon}\int_{\Sigma} d\mu Trf^2\right] = e^{\alpha_1(2-2p)}\sum_R (\dim R)^{2-2p} e^{-e^2 a(C_2(R)+\alpha_2)} =$$
$$= \left\langle e^{\omega + \varepsilon a \mathrm{O}^{(0)}} \right\rangle_{\mathrm{M}_N} + \mathrm{O}\left(e^{-c/(\varepsilon a)}\right), \quad (9.42)$$

where M_N is the moduli space of flat connections on Σ for $G = SU(N)$.

10. On some possible mathematical connections. [10] [11]

Now, in this Section, we describe the various possible and interesting mathematical connections that we have obtained, principally with the Hartle-Hawking wave-function but also with some sectors of Number Theory (Riemann zeta function and Ramanujan's modular equations) and with the fundamental equation of Palumbo-Nardelli model.

We remember that the corresponding real Einstein-Hilbert action of the two minisuperspace models of de Sitter type, with D = 4 and D = 3 space-time dimensions, is:

$$S = \frac{1}{16\pi G}\int_M d^D x\sqrt{-g}(R - 2\Lambda) + \frac{1}{8\pi G}\int_{\partial M} d^{D-1}x\sqrt{h}K . \quad (10.1)$$

With regard the de Sitter model in D = 3 dimensions, the p-adic Hartle-Hawking wave function is

$$\Psi_p(a) = \int_{|N|_p \le 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}} \chi_p\left(-\frac{N}{2} + \frac{\sqrt{\lambda}\coth\left(N\sqrt{\lambda}\right)}{2}a^2\right), \quad (10.2)$$

while, with regard the de Sitter model in D = 4 dimensions, it is possible to obtain the p-adic Hartle-Hawking wave function by the following equations:

$$\Psi_p(q)=\int_{|T|_p\le 1} dT\,\frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2T^3}{24}+(\lambda q-2)\frac{T}{4}+\frac{q^2}{8T}\right), \quad (10.3)$$

$$\psi_p(q)=\int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left[-\frac{\lambda^2T^3}{24}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right]. \quad (10.4)$$

Now, we recall that Ramanujan have shown that the definite integral

$$\phi_w(t)=\int_0^\infty \frac{\cos \pi tx}{\cosh \pi x}e^{-\pi wx^2}dx,$$

can be evaluated in finite terms if w is any rational multiple of i. Furthermore, this integral can be evaluated not only for these values but also for many other values of t and w. Now we have

$$\phi_w(t)=2\int_0^\infty\int_0^\infty \frac{\cos 2\pi xz}{\cosh \pi z}\cos \pi txe^{-\pi wx^2}dxdz=\frac{e^{-\frac{\pi t^2 w'}{4}}}{\sqrt{w}}\int_0^\infty \frac{\cosh \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx, \quad (10.4a)$$

here w' stands for $1/w$. It follows that

$$\phi_w(t)=\frac{1}{\sqrt{w}}e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw'). \quad (10.4b)$$

Now, it is possible to obtain the π value utilizing the following expression

$$\phi_w(t)=\frac{e^{-\frac{\pi t^2 w'}{4}}}{\sqrt{w}}\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx=\frac{1}{\sqrt{w}}e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw'), \quad (10.5)$$

$$e^{-\frac{\pi t^2 w'}{4}}\frac{1}{\sqrt{w}}\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx=\phi_w(t),$$

$$e^{-\frac{\pi t^2 w'}{4}}=\frac{\phi_w(t)}{\frac{1}{\sqrt{w}}\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}=\frac{\frac{1}{\sqrt{w}}e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}{\frac{1}{\sqrt{w}}\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx},$$

$$e^{-\frac{\pi t^2 w'}{4}}=\frac{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}; \quad \log\frac{\pi t^2 w'}{4}=\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')};$$

$$\pi=4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{1}{t^2 w'}. \quad (10.6)$$

With regard the number 24, from the following Ramanujan's modular equation

$$\pi = \frac{24}{\sqrt{142}} \log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right],$$

for the eq. (10.6), we have that

$$\frac{24}{\sqrt{142}} \log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right] = 4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{1}{t^2 w'};$$

$$24 = \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (10.7)$$

When a string moves in space-time by splitting and recombining, a large number of mathematical identities must be satisfied. These are the identities of Ramanujan's modular function (Ramanujan' modular equations).
The Ramanujan function, has 24 "modes" that correspond to the physical vibrations of a bosonic string. When the Ramanujan function is generalized, 24 is replaced by 8 (8 + 2 = 10), hence, has 8 "modes" that correspond to the physical vibrations of a superstring.
With regard the fundamental equation of Palumbo-Nardelli model, we have that:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr\left(G_{\mu\nu} G_{\rho\sigma}\right) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi}\left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_v\left(\left|F_2\right|^2\right)\right], \quad (10.8)$$

Hence, the following connections with the equations (10.3), (10.4), (10.7) and (10.8):

$$\int_{|T|_p \le 1} dT \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p\left(-\frac{\lambda^2 T^3}{24} + (\lambda q - 2)\frac{T}{4} + \frac{q^2}{8T}\right) \Rightarrow$$

$$\Rightarrow \int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left[-\frac{\lambda^2T^3}{24}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right]\Rightarrow$$

$$\Rightarrow \int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left\{-\lambda^2T^3\frac{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right\}\Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (10.9)$$

With regard the **Section 1**, if we take the eqs. (1.16) and (1.19), we note that are possible the following connections:

$$^4R(t)=4\Lambda\left[1-\frac{\theta^2\Lambda^2}{1152\left(1+\frac{\theta^2\Lambda^2}{576}\right)ch^2\left(\sqrt{\Lambda/3}t\right)}\right]\Rightarrow\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\Rightarrow$$
$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (10.10)$$

(Note that $576=24\times24$ and $1152=2\times24\times24$)

$$\psi'^{\pm}_\theta(a''=a(1))\approx\exp\left\{\pm\frac{1}{3\alpha}\left(1+\frac{\theta^2\alpha^2}{64}\right)^{3/2}\left[1-\alpha a''^2\left(1+\frac{\theta^2\alpha^2}{64}\right)^{-1}\right]^{3/2}\right\}\Rightarrow$$
$$\int_{|T|_p\le1}dT\frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2T^3}{24}+(\lambda q-2)\frac{T}{4}+\frac{q^2}{8T}\right)\Rightarrow$$

$$\Rightarrow \int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left[-\frac{\lambda^2T^3}{24}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right]\Rightarrow$$

$$\Rightarrow \int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left\{-\lambda^2T^3\frac{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right\}\Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (10.11)$$

With regard the eq. (1.13b), we have the following connection with some expressions concerning the Riemann zeta function:

$$\phi=\phi_C+\frac{1}{4}\ln\left|\frac{1-\frac{\theta}{4a_C^4}\left(p_{\phi_C}-a_Cp_{a_C}\right)}{1-\frac{\theta}{4a_C^4}\left(p_{\phi_C}+a_Cp_{a_C}\right)}\right|\Rightarrow\frac{1}{4}\ln\left|\frac{1-\frac{\theta}{4a_C^4}\left(p_{\phi_C}-a_Cp_{a_C}\right)}{1-\frac{\theta}{4a_C^4}\left(p_{\phi_C}+a_Cp_{a_C}\right)}\right|=\phi-\phi_C\Rightarrow$$

$$\Rightarrow(1+\varepsilon')T\log T=\int_0^T f(t)dt\Rightarrow$$

$$\Rightarrow\left\{\int_T^{T+H}\left(\int_{0.5}^\infty x^{\frac{1}{2}-u}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}}\right|du\right)^{4k}dt\right\}^{1/2}<<$$

$$<<(\log x)^{-4k+1}\int_T^{T+H}\int_{0.5}^\infty x^{\frac{1}{2}-u}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}}\right|^{4k}dtdu=$$

$$=(\log x)^{4k+1}\int_{0.5}^\infty x^{\frac{1}{2}-u}\left(\int_T^{T+H}\left|\sum_{p<x^3}\frac{\Lambda_x(p)\log(xp)}{p^{u+it}\log^2 x}\right|^{4k}dt\right)du<<H(\log x)^{4k}. \quad (10.12)$$

With regard the **Section 2**, for the eq. (2.1), we have the following connection with the eqs. (10.3), (10.7) and (10.8):

$$S=-\frac{1}{2\lambda_s}\int d^4x\sqrt{-g}e^{-\phi}\left(R+\partial_\mu\phi\partial^\mu\phi+V\right)\Rightarrow\int_{|T|_p\le 1}dT\frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2T^3}{24}+(\lambda q-2)\frac{T}{4}+\frac{q^2}{8T}\right)\Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr\left(G_{\mu\nu}G_{\rho\sigma}\right) f(\phi) - \frac{1}{2} g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x(-G)^{1/2} e^{-2\Phi}\left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_v\left(\left|F_2\right|^2\right)\right], \quad (10.13)$$

while, for the eq. (2.11), we have the following connection with the eq. (1.2) and the solutions (1.5) of the **Section 1**:

$$\frac{1}{2\lambda_s} \star \left[\frac{\partial^2}{\partial \bar{\phi}^2} - \frac{\partial^2}{\partial \beta^2} + \lambda_s^2 V\left(\bar{\phi}, \beta\right) e^{-2\bar{\phi}}\right] \star \psi(\phi, \beta) = 0 \Rightarrow$$

$$\Rightarrow \left[4\left(\partial_{x_C}^2 - \partial_{y_C}^2\right) + \frac{i\theta}{2}\left(\alpha\partial_{y_C} - \beta\partial_{x_C}\right) + \alpha x_C + \beta y_C - 1\right]\psi(x_C, y_C) = 0 \Rightarrow$$

$$\Rightarrow \psi_{\theta NB}\left(x_C'', y_C''\right) = \int dN \; \frac{1}{8\pi N} \exp\left\{-\frac{1}{4}\left\{\frac{-x_C''^2 + y_C''^2}{2N} + \frac{N^3}{6}\left(\alpha^2 - \beta^2\right) - \right.\right.$$

$$\left.\left. - N\left[2 - \alpha x_C'' - \beta y_C'' + \frac{\theta^2}{32}\left(\alpha^2 - \beta^2\right)\right]\right\}\right\} \times \exp\left[\frac{i\theta}{16}\left(\beta x_C'' + \alpha y_C''\right)\right] = \exp\left[\frac{i\theta\alpha}{16}\left(x_C'' + y_C''\right)\right]\psi_{NB}\left(x_C'', y_C''\right).$$

(10.13b)

With regard the **Section 3**, for the eq. (3.17), we have the following connection with the eq. (1.2):

$$\left[-P_{\Omega_c}^2 + P_{\beta_c}^2 - 48\exp\left(-2\sqrt{3}\Omega_c\right)\right] \star \Psi(\Omega_c, \beta_c) = 0 \Rightarrow$$

$$\Rightarrow \left[4\left(\partial_{x_C}^2 - \partial_{y_C}^2\right) + \frac{i\theta}{2}\left(\alpha\partial_{y_C} - \beta\partial_{x_C}\right) + \alpha x_C + \beta y_C - 1\right]\psi(x_C, y_C) = 0. \quad (10.14)$$

For the eq. (3.26), we have the following connection with the eqs. (10.3), (10.7) and with the fundamental equation of Palumbo-Nardelli model (10.8):

$$\Psi(\Omega_c, \beta_c) = \sum_\nu C_\nu e^{i\nu\sqrt{3}\beta_c} K_{i\nu}\left\{4\exp\left[-\sqrt{3}\left(\Omega_c - \frac{\sqrt{3}}{2}\nu\theta\right)\right]\right\} = R \cdot e^{iS} \Rightarrow$$

$$\Rightarrow \int_{|T|_p \le 1} dT \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p\left(-\frac{\lambda^2 T^3}{24} + (\lambda q - 2)\frac{T}{4} + \frac{q^2}{8T}\right) \Rightarrow \frac{4\left[anti\log \dfrac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu\left(|F_2|^2\right)\right]. \quad (10.15)$$

With regard the **Section 4**, the eq. (4.20) can be connected with the eqs. (10.3), (10.7) and (10.8), obtaining:

$$\frac{N}{48\pi^2} \int d^4x\sqrt{g}\left[-\frac{3}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{1}{120}\left(11 R^* R^* + 12 R;_\mu{}^\mu\right) + \frac{g^2}{N} F_{\mu\nu}^i F^{\mu\nu i}\right] \Rightarrow$$

$$\Rightarrow \int_{|T|_p \le 1} dT \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p\left(-\frac{\lambda^2 T^3}{24} + (\lambda q - 2)\frac{T}{4} + \frac{q^2}{8T}\right) \Rightarrow \frac{4\left[anti\log \dfrac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu\left(|F_2|^2\right)\right]. \quad (10.16)$$

The eq. (4.33), can be connected with the (10.8), obtaining:

$$\int d^4x\sqrt{g}\left[\frac{1}{2\kappa_0^2} R - \mu_0^2 (H^* H) + a_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + b_0 R^2 + c_0{}^* R^* R + d_0 R;_\mu{}^\mu + e_0 + \frac{1}{4} G_{\mu\nu}^i G^{\mu\nu i} + \right.$$

$$\left. + \frac{1}{4} F_{\mu\nu}^\alpha F^{\mu\nu\alpha} + \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + |D_\mu H|^2 - \xi_0 R |H|^2 + \lambda_0 (H^* H)^2\right] \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu\left(|F_2|^2\right)\right]. \quad (10.17)$$

With regard the **Section 5**, the eq. (5.12) can be connected with the eq. (1.2), obtaining:

$$\varepsilon_{abc}\frac{\delta}{\delta A_m{}^a(x)}\frac{\delta}{\delta A_n{}^b(x)}\left[F_{mnc}(A(x))-\frac{i\hbar\Lambda}{6}\varepsilon_{mnp}\frac{\delta}{\delta A_p{}^c(x)}\right]\Psi_\Lambda[A]=0\Rightarrow$$

$$\Rightarrow\left[4\left(\partial_{x_C}^2-\partial_{y_C}^2\right)+\frac{i\theta}{2}\left(\alpha\partial_{y_C}-\beta\partial_{x_C}\right)+\alpha x_C+\beta y_C-1\right]\psi(x_C,y_C)=0. \quad (10.18)$$

The eq. (5.13), can be connected with the eq. (10.2), obtaining:

$$\Psi_\Lambda[A]=\exp\left(\frac{i}{\hbar\Lambda}\int_\Sigma d^3xA\wedge dA+iA\wedge A\wedge A(A)\right)\Rightarrow$$

$$\Rightarrow\int_{|N|_p\le 1}dN\frac{\lambda_p(-2N)}{|N|_p^{1/2}}\chi_p\left(-\frac{N}{2}+\frac{\sqrt{\lambda}\coth\left(N\sqrt{\lambda}\right)}{2}a^2\right). \quad (10.19)$$

With regard the **Section 6**, for the eq. (6.2), we have the following possible connection with the eq. (10.2):

$$\int dt\int_\Sigma d^3x\left[\frac{1}{8\pi G\gamma}E_i^aL_tA_a^i-\left(\Lambda^iG_i+N^aC_a+NC\right)\right]\Rightarrow$$

$$\Rightarrow\int_{|N|_p\le 1}dN\frac{\lambda_p(-2N)}{|N|_p^{1/2}}\chi_p\left(-\frac{N}{2}+\frac{\sqrt{\lambda}\coth\left(N\sqrt{\lambda}\right)}{2}a^2\right). \quad (10.20)$$

With regard the **Section 7**, the eq. (7.53) can be connected with the eqs. (10.3), (10.7) and (10.8) obtaining:

$$\int d\lambda\overline{\Psi}\left[-\frac{\hbar^2}{2\mu}\frac{\delta^2}{\delta\left(\lambda_i^a\right)^2}+\frac{\mu\Omega_d^2}{2}\sum_{aij}\left(\lambda_I^a-\lambda_j^a\right)^2\right]\psi+\mu v_\lambda^2\frac{1}{\sqrt{\rho_\lambda(\lambda)}}\nabla^2\sqrt{\rho_\lambda(\lambda)}-\dot{S}_\lambda+$$

$$-\frac{1}{2\mu}\left(\frac{\delta S_d(\lambda)}{\delta\lambda_i^a}\right)^2-\frac{\mu\Omega_d^2}{2}\sum_{aij}\left(\lambda_I^a-\lambda_j^a\right)^2\Rightarrow$$

$$\Rightarrow\int_{|T|_p\le 1}dT\frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2T^3}{24}+(\lambda q-2)\frac{T}{4}+\frac{q^2}{8T}\right)\Rightarrow\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\Rightarrow$$

$$\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (10.21)$$

With regard the **Section 8**, the eqs. (8.15), (8.31), (8.38), (8.40), (8.46), (8.48), (8.58) and (8.60), can be connected with the eq. (10.2). Hence, we obtain, for example:

$$\Psi_{SCS}(\mathrm{A}_a)=\exp\left[\frac{i}{2\Lambda}\int d^3xSTr\left(\mathrm{A}\wedge F-\frac{1}{3}\mathrm{A}\wedge\mathrm{A}\wedge\mathrm{A}\right)\right]\Rightarrow$$

$$\Rightarrow \int_{|N|_p \leq 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}} \chi_p \left(-\frac{N}{2} + \frac{\sqrt{\lambda}\coth\left(N\sqrt{\lambda}\right)}{2} a^2 \right), \quad (10.22)$$

$$\Psi_0(S) := S(0)\exp\left[-\frac{1}{4\hbar\kappa} \int d^3x \int d^3y W_\Lambda\left(\underline{x} - \underline{y}\right) \widetilde{h}_S^{ab}\left(\underline{x}\right) \widetilde{h}_S^{ab}\left(\underline{y}\right) \right] \Rightarrow$$

$$\Rightarrow \int_{|N|_p \leq 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}} \chi_p \left(-\frac{N}{2} + \frac{\sqrt{\lambda}\coth\left(N\sqrt{\lambda}\right)}{2} a^2 \right), \quad (10.23)$$

$$\Psi_0(S) = S(0)\exp\left[-\frac{1}{4\hbar\kappa} \int d^3x \int d^3y W_\Lambda\left(\underline{x} - \underline{y}\right) \widetilde{h}_S^{ab}\left(\underline{x}\right) \widetilde{h}_S^{ab}\left(\underline{y}\right) + p_t \right] \Rightarrow$$

$$\Rightarrow \int_{|N|_p \leq 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}} \chi_p \left(-\frac{N}{2} + \frac{\sqrt{\lambda}\coth\left(N\sqrt{\lambda}\right)}{2} a^2 \right). \quad (10.24)$$

In conclusion, with regard the **Section 9**, the eqs. (9.30) and (9.30b) can be related with the eqs. (10.7) and (10.8), obtaining the following connections:

$$\hat{A}(M) = \prod_{a=1}^{\frac{1}{2}\dim M} \frac{\frac{1}{2}x_a}{\sinh\frac{1}{2}x_a} = 1 - \frac{1}{24}p_1(M) + \frac{1}{5760}\left(7p_1^2 - 4p_2\right)(M) + \ldots \Rightarrow$$

$$\Rightarrow \text{ind } i\mathrm{D} = \int_M \prod_{a=1}^{\frac{1}{2}\dim M} \frac{\frac{1}{2}x_a}{\sinh\frac{1}{2}x_a} = 1 - \frac{1}{24}p_1(M) + \frac{1}{5760}\left(7p_1^2 - 4p_2\right)(M) + \ldots \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4}w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4}\right)} \right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x(-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\widetilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu\left(\left|F_2\right|^2\right) \right]. \quad (10.25)$$

While, the eqs. (9.33)-(9.34), (9.38)-(9.39) and (9.40), can be connected with the eqs. (10.3), (10.7) and (10.8), obtaining the following interesting connections:

$$\Psi_L = \frac{1}{e^2}\int_M d^4x\sqrt{g}Tr\chi\left(iF_+ - tH\right) \Rightarrow \; Q_C\Psi_L = \frac{1}{e^2}\int d^4x\sqrt{g}\left\{ TrH\left(iF_+ - tH\right) - Tr\chi\left(i\left(D_A\psi\right)_+ + t[\phi,\chi]\right)\right\} \Rightarrow$$

$$\Rightarrow \int_{|T|_p \le 1} dT \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p\left(-\frac{\lambda^2 T^3}{24} + (\lambda q - 2)\frac{T}{4} + \frac{q^2}{8T}\right) \Rightarrow \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr\left(G_{\mu\nu} G_{\rho\sigma}\right) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi}\left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_v\left(\left|F_2\right|^2\right)\right], \quad (10.26)$$

$$\Psi_P = -\frac{i}{e^2}\left(\lambda, C^{|}\right) = \frac{i}{e^2}\int_M Tr\lambda D_A * \psi = -\frac{i}{e^2}\int_M d^4x\sqrt{g} Tr\left(\lambda * D_A * \psi\right) \Rightarrow$$

$$\Rightarrow Q_C \Psi_P = -\frac{i}{e^2}\int_M Tr\left(\eta D_A * \psi + \lambda\{\psi, *\psi\} + \lambda D_A * D_A \phi\right) \Rightarrow$$

$$\Rightarrow I_D = \frac{1}{e^2}\int d^4x\sqrt{g}\left[iTr\left(\eta D_A^\mu \psi_\mu + \lambda\left\{\psi_\mu, \psi^\mu\right\} + \lambda D_A * D_A \phi\right) + \left(-\frac{1}{4t} TrF_+^2 - Tr\chi_{\mu\nu}\left[i\left(D_A \psi\right)_+^{\mu\nu} + t\left[\phi, \chi^{\mu\nu}\right]\right]\right)\right] \Rightarrow$$

$$\Rightarrow \int_{|T|_p \le 1} dT \frac{\lambda_p(-8T)}{|4T|_p^{1/2}} \chi_p\left(-\frac{\lambda^2 T^3}{24} + (\lambda q - 2)\frac{T}{4} + \frac{q^2}{8T}\right) \Rightarrow \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr\left(G_{\mu\nu} G_{\rho\sigma}\right) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi}\left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (10.27)$$

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