Non-existence of odd harmonic divisor numbers

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Abstract

Let n be a positive integer and let b be an odd harmonic divisor number. Let the prime factors of b which are different from each other be odd primes $p_1,p_2,\ldots,p_r$ and let the exponent of p_k be a positive integer q_k. If the product of the series of the prime factors is an integer a,

$$a = \prod_{k=1}^{r} (p_k{}^{q_k} + p_k{}^{q_k-1} + \cdots + 1)$$

$$b = \prod_{k=1}^{r} p_k{}^{q_k}$$

If b is a harmonic divisor number, let m be an integer,

$$m = \prod_{k=1}^{r} (q_k + 1)$$

$$an = bm$$

holds. By a research of this paper, let a_k be an integer and b_k be an odd integer and if

$$a_k = a/(p_k{}^{q_k} + \cdots + 1)$$

$$b_k = b/p_k{}^{q_k}$$

holds, when $r \geqq 3$, by a proof which uses the prime factors and the greatest common divisor (GCD) C_k included in b_k and $p_k{}^{q_k} + \cdots + 1$, we found that it becomes a contradiction when $C_k < b_k$ since a least one prime number exists only in the denominator on the left side and it does not in the denominator on the right side. When $C_k = b_k$, we found that it becomes inconsistent. We have obtained a conclusion that there are no odd harmonic divisor numbers other than 1.

Contents

1. Introduction

In mathematics, a harmonic divisor number, or Ore number (named after Øystein Ore who defined it in 1948), is a positive integer whose divisors have a harmonic mean that is an integer. For example, the harmonic divisor number 6 has the four divisors 1, 2, 3, and 6. Their harmonic mean is an integer:

$$4/(1+1/2+1/3+1/6)=2$$

(Quoted from Wikipedia)

In this paper, we prove that there are no odd harmonic divisor numbers other than 1.

2. Proof

Let n be a positive integer and let b be an odd harmonic divisor number. Let the prime factors of b which are different from each other be odd primes $p_1, p_2, \dots, p_r$ and let the exponent of p_k be a positive integer q_k. If the product of the series of the prime factors is an integer a,

$$a=\prod_{k=1}^{r}(p_k{}^{q_k}+p_k{}^{q_k-1}+\cdots+1) \quad \dots ①$$

$$b=\prod_{k=1}^{r}p_k{}^{q_k} \quad \dots ②$$

If b is a harmonic divisor number, let m be an integer,

$$m=\prod_{k=1}^{r}(q_k+1)$$

$$an=bm \quad \dots ③$$

holds. Divide m and n by the greatest common divisor and assume that they are relatively prime. Even if this calculation is performed, generality is not lost.

Let a_k be an integer and b_k be an odd integer,

$a_k = a/(p_k{}^{q_k}+\cdots+1)$

$b_k = b/p_k{}^{q_k}$

From the expression ③,

$na_k(p_k{}^{q_k}+\cdots+1) = mb_kp_k{}^{q_k} \quad \dots ④$

Ⅰ. When $r = 1$

$n(p_1^{q_1} + \cdots + 1) = (q_1 + 1)p_1^{q_1}$

Let n' be an integer and if $n = n'p_1^{q_1}$ holds,

$n'(p_1^{q_1} + \cdots + 1) = q_1 + 1$

Since $n' \geqq 1$,

$(q_1 + 1)/(p_1^{q_1} + \cdots + 1) \geqq 1$

$q_1 + 1 \geqq p_1^{q_1} + \cdots + 1 \geqq p_1^{q_1} + 1$

$q_1 \geqq p_1^{q_1}$

When $q_1 \geqq 1$ and $p_1 \geqq 3$, this inequality does not hold obviously. Therefore, odd harmonic divisor numbers do not exist when $r = 1$.

Ⅱ. When $r \geqq 2$

From the equation ④,

$na_k(p_k^{q_k+1} - 1) = mb_k p_k^{q_k}(p_k - 1)$

$\left((na_k - mb_k)p_k + mb_k\right)p_k^{q_k} = na_k$

Since $na_k/p_k^{q_k}$ is an integer, let c_k be an integer.

$\left((na_k - mb_k)p_k + mb_k\right) = na_k/p_k^{q_k} = c_k$

When $p_k \geqq 3$,

$p_k^{q_k-1} + \cdots + 1 = (p_k^{q_k} - 1)/(p_k - 1) < p_k^{q_k}/2$

From the equation ④,

$mb_k - na_k = c_k(p_k^{q_k} + \cdots + 1) - c_k p_k^{q_k} = c_k(p_k^{q_k-1} + \cdots + 1)$

$mb_k - na_k < c_k p_k^{q_k}/2 = na_k/2$

$mb_k < 3na_k/2$

$a_k/b_k > 2m/(3n) > 2/3$ …⑤

When $r = 2$,

$a_1 = p_2^{q_2} + \cdots + 1$

$b_1 = p_2^{q_2}$

$a_1/b_1 = (p_2^{q_2} + \cdots + 1)/p_2^{q_2} = (p_2^{q_2+1} - 1)/(p_2^{q_2}(p_2 - 1)) < p_2/(p_2 - 1)$

If $p_1 < p_2$, since $p_2 \geqq 5$ holds,

$a_1/b_1 < 5/4$

This inequality contradicts the inequality ⑤. Therefore, there are no odd harmonic divisor numbers when $r = 2$.

Ⅲ. When $r \geqq 3$

From the equation ④,

$na_k/b_k = mp_k{}^{q_k}/(p_k{}^{q_k} + \cdots + 1)$ …⑥

When m is divided by $p_k{}^{q_k} + \cdots + 1$, let m' be an integer,

$m' = mp_k{}^{q_k}/(p_k{}^{q_k} + \cdots + 1)$

$a_k = m'/n \times b_k$

hold.

The equation ⑥ is an equation for obtaining m'/n-multiperfect numbers. When m is divisible by $p_k{}^{q_k} + \cdots + 1$ for plural q_k with the same p_k, m must be divided by the number of one of q_k. By repeating this operation, it is possible to prevent m from being divisible by $p_k{}^{q_k} + \cdots + 1$ for all k.

A case where m cannot be divided by $p_k{}^{q_k} + \cdots + 1$ for all k is considered. At this time, the right side is not an integer. $p_k{}^{q_k} + \cdots + 1$ is the product of the prime factors p_1 to p_r excluding p_k and the prime factors of m. Let C_k be the greatest common divisor (GCD) of the denominators on both sides. When the denominator on both sides are divided by C_k, if mC_k becomes a multiple of the denominator on the right side, let s_k be an integer,

$mC_k = s_k(p_k{}^{q_k} + \cdots + 1)$

this equation is assumed to be hold, the value of the left side of the equation ⑥ is $s_k p_k{}^{q_k}/C_k$. If this value is assumed to be an integer, s_k is a multiple of C_k since p_k does not exist as the prime factor of C_k. However, this contradicts the condition that m is not divided by $p_k{}^{q_k} + \cdots + 1$. Therefore, when C_k is transposed from the denominator on the left side to the right side, the right side does not become an integer.

Let P_k be an odd integer and $P_k = b_k/C_k$ holds. When $b_k > C_k$, if the numerator on the left side is a multiple of P_k, it becomes contradiction since the left side is an integer and the right side is not. Therefore, when the left side is reduced, at least one of the prime factors p_{sk} of P_k remains in the denominator. At this time, it becomes inconsistent since the prime number p_{sk} does not exist in the denominator on the right side.

When $b_k = C_k$ and the denominators on both sides are divided by b_k, a contradiction arises from the above proof since the left side is an integer and the right side is not. Therefore, there are no odd harmonic divisor numbers when $r \geqq 3$.
From the above Ⅰ, Ⅱ, and Ⅲ, there are no odd harmonic divisor numbers other than 1.

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4. References

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