Non-existence of odd harmonic divisor numbers

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April 28th, 2020

Abstract

Let b be an odd harmonic divisor number. Let the prime factors of b which are different from each other be odd primes $p_1, p_2, \ldots, p_r$ and let the exponent of p_k be a positive integer q_k. If the product of the series of the prime factors is an integer a,

$$a = \prod_{k=1}^{r} (p_k{}^{q_k} + p_k{}^{q_k-1} + \cdots + 1)$$

$$b = \prod_{k=1}^{r} p_k{}^{q_k}$$

If b is a harmonic divisor number, let m and n be positive integers,

$$m = \prod_{k=1}^{r} (q_k + 1)$$

$$an = bm$$

holds. By a research of this paper, let a_k be an integer and b_k be an odd integer and if

$$a_k = a/(p_k{}^{q_k} + \cdots + 1)$$

$$b_k = b/p_k{}^{q_k}$$

holds, when $r \geqq 3$, by a proof which uses the prime factors and the greatest common divisor (GCD) C_k included in b_k and $p_k{}^{q_k} + \cdots + 1$, we found that it becomes a contradiction when $C_k < b_k$ since a least one prime number exists only in the denominator on the left side and it does not in the denominator on the right side. When $C_k = b_k$, we found that it becomes inconsistent. We have obtained a conclusion that there are no odd harmonic divisor numbers other than 1.

Contents

1. Introduction

In mathematics, a harmonic divisor number, or Ore number (named after Øystein Ore who defined it in 1948), is a positive integer whose divisors have a harmonic mean that is an integer. For example, the harmonic divisor number 6 has the four divisors 1, 2, 3, and 6. Their harmonic mean is an integer:

$$4/(1+1/2+1/3+1/6)=2$$

(Quoted from Wikipedia)

In this paper, we prove that there are no odd harmonic divisor numbers other than 1.

2. Proof

Let b be an odd harmonic divisor number. Let the prime factors of b which are different from each other be odd primes $p_1, p_2, \ldots, p_r$ and let the exponent of p_k be a positive integer q_k. If the product of the series of the prime factors is an integer a,

$$a=\prod_{k=1}^{r}(p_k{}^{q_k}+p_k{}^{q_k-1}+\cdots+1) \quad \ldots①$$

$$b=\prod_{k=1}^{r}p_k{}^{q_k} \quad \ldots②$$

If b is a harmonic divisor number, let m and n be positive integers,

$$m=\prod_{k=1}^{r}(q_k+1)$$

$$an=bm \quad \ldots③$$

holds. Divide m and n by the greatest common divisor and assume that they are relatively prime. Even if this calculation is performed, generality is not lost.

Let $\mathrm{a_k}$ be an integer and $\mathrm{b_k}$ be an odd integer,

$\mathrm{a_k}=\mathrm{a}/(\mathrm{p_k}^{\mathrm{q_k}}+\cdots+1)$

$\mathrm{b_k}=\mathrm{b}/\mathrm{p_k}^{\mathrm{q_k}}$

From the expression ③,

$\mathrm{na_k}(\mathrm{p_k}^{\mathrm{q_k}}+\cdots+1)=\mathrm{mb_k}\mathrm{p_k}^{\mathrm{q_k}} \quad \ldots④$

Ⅰ. When $r = 1$

$n(p_1^{q_1} + \cdots + 1) = (q_1 + 1)p_1^{q_1}$

Let n' be an integer and if $n = n'p_1^{q_1}$ holds,

$n'(p_1^{q_1} + \cdots + 1) = q_1 + 1$

Since $n' \geqq 1$,

$(q_1 + 1)/(p_1^{q_1} + \cdots + 1) \geqq 1$

$q_1 + 1 \geqq p_1^{q_1} + \cdots + 1 \geqq p_1^{q_1} + 1$

$q_1 \geqq p_1^{q_1}$

When $q_1 \geqq 1$ and $p_1 \geqq 3$, this inequality does not hold obviously. Therefore, odd harmonic divisor numbers do not exist when $r = 1$.

Ⅱ. When $r \geqq 2$

From the equation ④,

$na_k(p_k^{q_k+1} - 1) = mb_k p_k^{q_k}(p_k - 1)$

$\left((na_k - mb_k)p_k + mb_k\right)p_k^{q_k} = na_k$

Since $na_k/p_k^{q_k}$ is an integer, let c_k be an integer.

$\left((na_k - mb_k)p_k + mb_k\right) = na_k/p_k^{q_k} = c_k$

When $p_k \geqq 3$,

$p_k^{q_k-1} + \cdots + 1 = (p_k^{q_k} - 1)/(p_k - 1) < p_k^{q_k}/2$

From the equation ④,

$mb_k - na_k = c_k(p_k^{q_k} + \cdots + 1) - c_k p_k^{q_k} = c_k(p_k^{q_k-1} + \cdots + 1)$

$mb_k - na_k < c_k p_k^{q_k}/2 = na_k/2$

$mb_k < 3na_k/2$

$a_k/b_k > 2m/(3n) > 2/3$ …⑤

When $r = 2$,

$a_1 = p_2^{q_2} + \cdots + 1$

$b_1 = p_2^{q_2}$

$a_1/b_1 = (p_2^{q_2} + \cdots + 1)/p_2^{q_2} = (p_2^{q_2+1} - 1)/(p_2^{q_2}(p_2 - 1)) < p_2/(p_2 - 1)$

If $p_1 < p_2$, since $p_2 \geqq 5$ holds,

$a_1/b_1 < 5/4$

This inequality contradicts the inequality ⑤. Therefore, there are no odd harmonic divisor numbers when $r = 2$.

Ⅲ. When $r \geqq 3$

From the equation ④,

$na_k/b_k = mp_k{}^{q_k}/(p_k{}^{q_k} + \cdots + 1)$ …⑥

When m is divided by $p_k{}^{q_k} + \cdots + 1$, let m' be an integer,

$m' = mp_k{}^{q_k}/(p_k{}^{q_k} + \cdots + 1)$

$a_k = m'/n \times b_k$

hold.

The equation ⑥ is an equation for obtaining m'/n-multiperfect numbers. When m is divisible by $p_k{}^{q_k} + \cdots + 1$ for plural q_k with the same p_k, m must be divided by the number of one of q_k. By repeating this operation, it is possible to prevent m from being divisible by $p_k{}^{q_k} + \cdots + 1$ for all k.

A case where m cannot be divided by $p_k{}^{q_k} + \cdots + 1$ for all k is considered. At this time, the right side is not an integer. $p_k{}^{q_k} + \cdots + 1$ is the product of the prime factors p_1 to p_r excluding p_k and the prime factors of m. Let C_k be the greatest common divisor (GCD) of the denominators on both sides. When the denominator on both sides are divided by C_k, if mC_k becomes a multiple of the denominator on the right side, let s_k be an integer,

$mC_k = s_k(p_k{}^{q_k} + \cdots + 1)$

this equation is assumed to be hold, the value of the left side of the equation ⑥ is $s_k p_k{}^{q_k}/C_k$. If this value is assumed to be an integer, s_k is a multiple of C_k since p_k does not exist as the prime factor of C_k. However, this contradicts the condition that m is not divided by $p_k{}^{q_k} + \cdots + 1$. Therefore, when C_k is transposed from the denominator on the left side to the right side, the right side does not become an integer.

Let P_k be an odd integer and $P_k = b_k/C_k$ holds. When $b_k > C_k$, if the numerator on the left side is a multiple of P_k, it becomes contradiction since the left side is an integer and the right side is not. Therefore, when the left side is reduced, at least one of the prime factors p_{sk} of P_k remains in the denominator. At this time, it becomes inconsistent since the prime number p_{sk} does not exist in the denominator on the right side.

When $b_k = C_k$ and the denominators on both sides are divided by b_k, a contradiction arises from the above proof since the left side is an integer and the right side is not. Therefore, there are no odd harmonic divisor numbers when $r \geqq 3$.
From the above Ⅰ, Ⅱ, and Ⅲ, there are no odd harmonic divisor numbers other than 1.

3. Acknowledgement

We would like to thank the family members who sustained the research environment and the mathematicians who reviewed this research.

4. References

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