The assumption of continuity of spacetime in quantum gravity

René Friedrich, Strasbourg

General relativity and quantum mechanics both have been confirmed by experiments. In order to unify them, theories of quantum gravity are becoming more and more ingenious, but without success.

The current theories of quantum gravity are based on the assumption of a continuous Lorentzian spacetime manifold. Little attention is paid to this assumption, but no theory is complete without the prior verification of the soundness of its underlying assumptions.

Based on the principles of general relativity it will be shown that the assumption may be invalidated for two reasons:

- Mathematically, the metric of Lorentzian manifolds is not properly defined, and

- Physically, the principles of relativity refer to worldlines, but not to the void between worldlines.

1. Introduction

Lorentzian spacetime models are used for the representation of the principles of special and general relativity. In 1908, it was Minkowski who provided special relativity with a geometric interpretation, and a few years later, Einstein and Grossmann developed a geometric model for the description of the principles of general relativity. In both cases it was assumed that Lorentzian spacetime was a continuous manifold, and both models worked very well, such that they were considered to be integral parts of the theory of special/ general relativity. The only issue appeared when trying to quantize Lorentzian spacetime, but nobody suspected that this problem could be due to the failure of the assumption of a continuous spacetime manifold.

However, contrary to intuition, nothing is corroborating this assumption, whatever is the approach we choose:

Mathematically, Lorentzian spacetime manifolds require a twofold (split) metric, one for timelike and one for spacelike intervals, and there is no underlying physical justification for such a split metric;

Physically, the theory of gravity of general relativity defines worldlines, but not the vacuum between worldlines.

2. The continuity of spacetime is an assumption

This assumption was expressed by Minkowski in his lecture "Space and Time" in 1908:

"To never let a yawning emptiness, let us imagine that everywhere and at any time something perceivable exists." **[1]**

Minkowski wanted to define a continuous spacetime manifold, and the quotation shows that he was aware of the fact that vacuum between worldlines was not defined by special relativity (see below **section 4**), otherwise this sentence would have been meaningless. This assumption of a continuous spacetime manifold seemed to be so obvious and so natural that since then, no serious doubts arose with respect to it[1].

However, the assumption constrained him further on in his lecture in section III to split up the metric of spacetime into two parts, in order to avoid imaginary spacelike intervals (see below **subsection 3.1**), by simply changing the sign of the square of spacelike intervals (the equation

$$" - F = x^2 + y^2 + z^2 - c^2t^2 = k^2 "$$

applying to *"all constant positive values of k^2."*)**[1]**

3. The mathematical issue: Spacelike spacetime intervals are not properly defined

3.1. Spacelike spacetime intervals are imaginary

Among the two possible sign conventions for the spacetime interval (+ - - - and - + + +), the first signature (+ - - -) is, as Penrose says, "more directly physical" **[3]** because it corresponds to proper time dτ which is the spacetime interval of the timelike worldlines of particles. Following this interpretation, we get the spacetime interval

$$ds = d\tau = \sqrt{dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2}}$$

It is obvious that this metric cannot span any real manifold: it provides real solutions only for timelike spacetime intervals. Spacelike spacetime intervals are becoming imaginary, they are not defined.

---

[1] It deserves to be mentioned, however, that even Einstein had expressed doubts with respect to continuity of spacetime, in 1916, in a letter to Dällenbach: *"But you have correctly grasped the drawback that the continuum brings. If the molecular view of matter is the correct (appropriate) one, i.e., if a part of the universe is to be represented by a finite number of moving points, then the continuum of the present theory contains too great a manifold of possibilities. I also believe that this too great is responsible for the fact that our present means of description miscarry with the quantum theory. The problem seems to me how one can formulate statements about a discontinuum without calling upon a continuum (space-time) as an aid; the latter should be banned from the theory as a supplementary construction not justified by the essence of the problem, which corresponds to nothing "real". But we still lack the mathematical structure unfortunately. How much have I already plagued myself in this way!"* **[2]**

A change of sign convention does not help: If spacelike spacetime intervals are considered as real, this implies that timelike intervals are imaginary.

As shown in **section 2**, the issue was already seen by Minkowski, but he overrode the problem, simply by defining a real metric for the imaginary intervals. The result was a sort of patchwork of two complementary metrics, each metric applying where the other metric gave imaginary results.

Mathematically, it is possible to redefine the metric of a certain zone in order to obtain a continuous manifold, in the same way as we may define that 2 x 2 = 5 or that $(-1)^{0,5} = +1$. However, the problem is that this redefinition is not corresponding to any physical reality.

In spite of the absence of physical justification, the twofold metric of Lorentzian manifolds is generally accepted. As an example, Misner/ Thorne/ Wheeler (p. 305) defined the metric by the means of the equation

$$\Delta s^2 = -\Delta\tau^2 = g_{\mu\nu} x^\alpha \Delta x_\mu \Delta x_\nu \quad \textbf{[4]},$$

implying that the spacetime interval and proper time are two opposed metrics, because accordingly, the equation

$$\Delta\tau^2 = -\Delta s^2 = -g_{\mu\nu} x^\alpha \Delta x_\mu \Delta x_\nu$$

must be a second, independent metric. Again, such distinct treatment of timelike and spacelike intervals does not correspond to any physical reality.

3.2. No "nice" topology

Topology does not provide corroboration of the assumption of continuous spacetime. Quite the contrary, it has been noticed that there is no "nice" topology for Lorentzian spacetimes **[5][6]**. One possible topology is among others the 1+3 standard topology which is treating space and time separately, corroborating rather the assumption of a threedimensional space manifold than the one of a fourdimensional spacetime manifold **[7-9]**.

3.3. From the Euclidean manifold to Lorentzian spacetime

Spacetime has a Lorentzian metric. However, it must be kept in mind that spacetime coordinates have Euclidean geometry, and that without Euclidean geometry there could be no light cone: As the Lorentzian spacetime interval of lightlike phenomena is zero, a light cone would just be reduced to one single point.

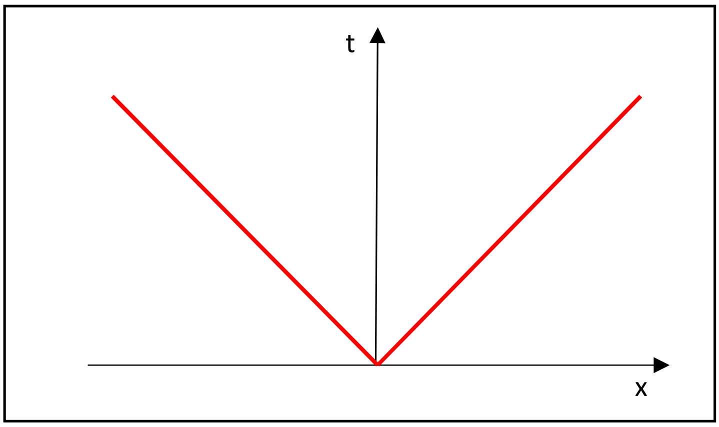

Fig. 1: Spacetime representations with light cones are always Euclidean because if they were Lorentzian, the light cones would be reduced to zero

Representations of spacetime such as the diagram in **fig. 1** are similar to representations of Newtonian space and time: They are always continuous Euclidean manifolds, and if they are flat and twodimensional they may even have the form of a sheet of paper. With a meter stick, we may measure space intervals and time periods, measurements along one axis (time axis or space axis) correspond to observation. In contrast, for mixed intervals of space and time, the Euclidean metric provides meaningless results, and the Lorentzian interval must be calculated. In Euclidean manifolds, the Lorentzian metric is not measurable, it may only be calculated. Newtonian spacetime becomes Lorentzian by this hidden metric. However, when we apply a Lorentzian metric to a spacetime manifold, the continuity gets lost because spacelike spacetime intervals are imaginary and not defined (see above **subsection 3.1**).

4. The physical issue: The vacuum between worldlines is not defined

4.1. The two postulates of special relativity

The two postulates of special relativity:

1. *The laws of physics are the same in all inertial reference frames.*
2. *Speed of light is measured with the same value c in all inertial reference frames.*

Manifestly, both postulates are talking about inertial reference frames (and also about lightlike phenomena) with their respective worldlines, but not about the vacuum between worldlines. Vacuum is described by quantum physics and possibly by cosmology in the form of dark energy, but it is neither defined by special relativity, nor by the theory of gravity of general relativity. Example: the Schwarzschild metric introduces the warping of the worldlines by gravity, but vacuum between worldlines is not described.

That means that general relativity itself is contradicting the assumption of a continuous spacetime manifold which should include vacuum points.

4.2. Vacuum points have no time evolution

What happens with vacuum points in special relativity?
For this question we consider a Minkowski diagram with two lines of simultaneity.

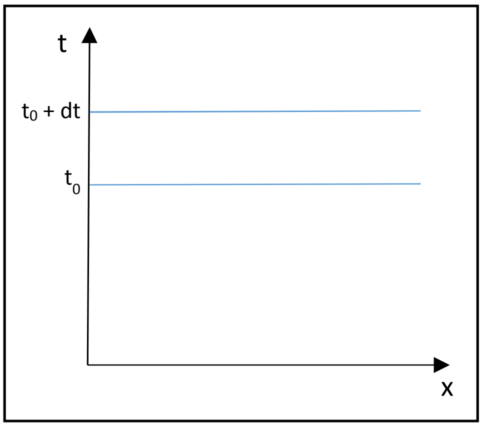

Fig. 2: Two lines of simultaneity seem to be continuous

The two lines of simultaneity seem to be perfectly continuous.

Now we introduce two particle worldlines. The worldline of each particle is determined by its position and its velocity.

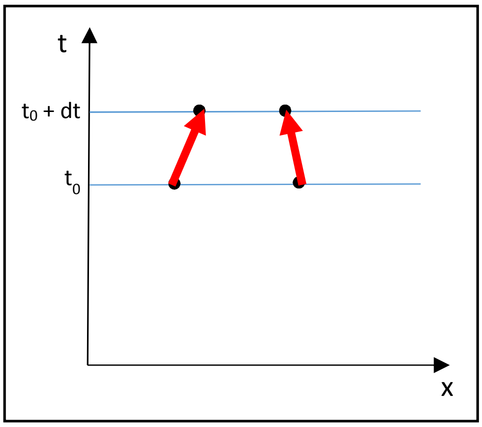

Fig. 3: Two particle worldlines

But what about a vacuum point between both particles? A vacuum point has no defined velocity. Contrarily to Euclidean spacetimes such as Newtonian spacetime, it does not travel simply upwards through time, because this would imply a preferred observer. The result: There is no point on the upper line $t_0 + dt$ which corresponds to the vacuum point on the lower line t_0. Vacuum has no time evolution.

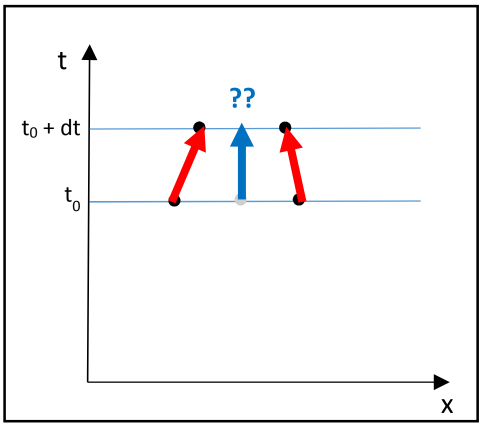

Fig. 4: No point on the upper line corresponds to the vacuum point on the lower line

That means: Every vacuum point is just an independent point with time and space coordinates, without any additional information. Or, conversely: The only "reason to be" of a vacuum point between worldlines is the coordinate system which has been drawn for the description of the worldlines. The vacuum point is simply representing the void between worldlines, without physical relevance.

For lightlike phenomena which are propagating with velocity c (such as electromagnetic or gravitational fields), the problem is a different one: One could presume that lightlike phenomena are continuous and everywhere, even in the vacuum between particles. But the problem is here that many lightlike phenomena go through the same point such that there is no unambiguously defined point on the upper line which corresponds to the point on the lower line.

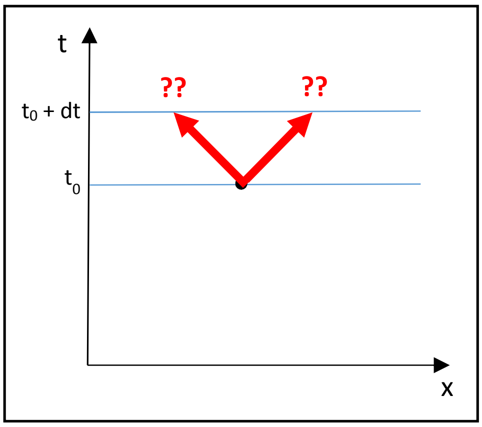

Fig. 5: No unambiguous solution for lightlike phenomena

5. The causal set theory

The causal set theory comes very close to the solution which is proposed here, but instead of searching for the solution inside of general relativity, the causal set theory is searching unnecessarily for a more fundamental concept beyond general relativity.

The causal set theory is based on a continuous Lorentzian manifold, but it says that spacetime is fundamentally discrete. Its aim is the recovery of a continuous Lorentzian manifold from a causal order, which by nature excludes spacelike spacetime intervals. The causal set theory proposes a solution with a discrete spacetime consisting of discrete spacetime points (events), one of its main concerns is Lorentz invariance (in lieu of all, see **[10-11]**).

The causal set theory is trying to build a framework of causal order beyond Lorentzian spacetime, unfortunately without being aware of the fact that such Lorentz-invariant causal framework exists already within general relativity in the form of worldlines: Timelike worldlines of particles and lightlike worldlines of fields are transmitting perfectly 100 % of the causality of events of the universe, and we have the possibility to parameterize each of these worldlines by its respective proper time in order to make it Lorentz-invariant. One specificity of such solipsistic worldlines which are parameterized by their respective proper time is that they cannot be represented together within common spacetime coordinates, but no common spacetime and no spacetime coordinates are required at this fundamental level of the representation of worldlines before time dilation. In a second step, an observer is measuring these worldlines within his own coordinates, substituting their respective proper time parameter with his own coordinate time parameter. This is what we call "observation".

6. Conclusion

We examined the assumption of continuity of spacetime under various aspects, and the result was negative whatever was the approach. The driving force of the assumption might have been Lorentz symmetry, but we know that space and time are different things, and that their similarity is strictly limited to Lorentz symmetry. Today, the interest is focused on quantum gravity, and doubts have already been expressed with respect to the current model of the spacetime manifold. The invalidation of the assumption of continuity of spacetime could permit progresses in this field.

7. References

[1] Hermann Minkowski: Raum und Zeit (1908), in: Space and Time, Minkowski's Papers on Relativity, Minkowski Institute Press 2012, p. 111

[2] Albert Einstein, letter to Walter Dällenbach, Nov. 1916, cited and translated by J. Stachel: Einstein and the quantum: Fifty years of struggle. In Robert G. Colodny (ed.): From Quarks to Quasars, Philosophical Problems of Modern Physics (1986), p. 379

[3] Roger Penrose: The Road to Reality, 2004, § 18.1

[4] Charles Misner, Kip Thorne, John Archibald Wheeler: Gravitation, 1973

[5] R. Göbel: Zeeman topologies on space-times of general relativity theory, Comm. Math. Phys. 1976, p. 289

[6] Renee Hoekzema: On the Topology of Lorentzian manifolds, 2011

[7] E.C. Zeeman: Causality Implies the Lorentz Group, 1963, Journal of Mathematical Physics 1964, p. 490

[8] Steven Hawking: Singularities and the geometry of spacetime, The European Physical Journal H 2014

[9] Steven Hawking, A.R. King and P.J. McCarthy: A new topology for curved space-time which incorporates the causal, differential and conformal structures, Journal of Mathematical Physics, 1976 p. 174

[10] Rafael D. Sorkin: Causal Sets: Discrete Gravity, arxiv, 01/09/2003

[11] Astrid Eichhorn: Steps towards Lorentzian quantum gravity with causal sets, arxiv, 01/02/2019