The Riemann Hypothesis Proof

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Many people are using the term “Assigned Value” or “Analytic Continuation” for divergent series
But this explanation is so lacking and can be replaced with a much easier and simpler term of explanation

For me (as I see it) when I am looking at the zeta function I dont see (or use) the term “Assigned Value” or “Analytic Continuation”
Instead I see “spirals” all around the grid

The simplest way is to first look at the Complex plane $\zeta(s) = \zeta(x+iy) = a+ib$ where $s > 1$ and the behavior of convergent points
The spiral swirls around inwards to an unique point which the series Converges - Same goes for the other way around!

When I look at the Complex plane $\zeta(s) = \zeta(x+iy) = a+ib$ where $s < 1$ and the behavior of divergent points
The spiral swirls around outwards but if you look closely you will notice that the spiral has a “center point” or an “origin”
and that “origin” is the “Assigned Value” everyone is talking about

when I first started to read about the zeta function I didn’t know what are those “Assigned Values” or “Analytic Continuation”
and how and why people are trying to give a value for divergent series And why that specific value and not something else?
I wanted an explanation other then “because the formula says so” and without going deeper into all the “Analytic Continuation stuff".

Those “origin points” did the trick!

If you are assigning a value for a series that decreases to a specific value (case #1)
Then you can assigning a value for a series that increases from a specific value (case #2)

Other then those two cases there is one more
This is when the spiral at some point start to spin around a specific value with a “fixed radius”
those cases appears at the zeta function $\zeta(s) = \zeta(x+iy) = a+ib$ when $x = 1$ and the radius will be $1/y$
meaning that this is a divergent series with a “fixed radius”

This was a small intro for the eta function spirals

Its true that the zeta function spirals have 3 cases but they are all spirals with **one arm**
Now at the eta function the spirals have **two arms** (that is because of the +/- swapping) with the same 3 cases

By the way the “fixed radius” appears at the eta function $\eta(s) = \eta(x+iy) = a+ib$ when $x = 0$

If you like to know more I am providing further details at http://myzeta.125mb.com

Removing the Riemann hypothesis from the Complex plane

$$e^{\theta} = \left(1 + \frac{\theta}{n}\right)^n$$

$$\theta = -ib \cdot \ln k$$

$$\frac{1}{k^{(a+ib)}} = \frac{1}{k^a} \cdot k^{-ib} = \frac{1}{k^a} \cdot e^{\theta} = \frac{1}{k^a} \cdot \left(1 + \frac{\theta}{n}\right)^n = \frac{1}{k^a} \cdot \left(1 - i \cdot \frac{b \cdot \ln k}{n}\right)^n$$

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

$$(x-iy)^n = x^n - i\binom{n}{1} x^{n-1} y^1 - \binom{n}{2} x^{n-2} y^2 + i\binom{n}{3} x^{n-3} y^3 + \binom{n}{4} x^{n-4} y^4 - i\binom{n}{5} x^{n-5} y^5 - \binom{n}{6} x^{n-6} y^6 + i\binom{n}{7} x^{n-7} y^7 + \binom{n}{8} x^{n-8} y^8$$

$$(x-iy)^n = \left[x^n - \binom{n}{2} x^{n-2} y^2 + \binom{n}{4} x^{n-4} y^4 - \binom{n}{6} x^{n-6} y^6 + \binom{n}{8} x^{n-8} y^8 + \ldots\right] + i\left[-\binom{n}{1} x^{n-1} y^1 + \binom{n}{3} x^{n-3} y^3 - \binom{n}{5} x^{n-5} y^5 + \binom{n}{7} x^{n-7} y^7 + \ldots\right]$$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$$(x-iy)^n = \left[x^n - \frac{n! x^{n-2} y^2}{(n-2)!2!} + \frac{n! x^{n-4} y^4}{(n-4)!4!} - \frac{n! x^{n-6} y^6}{(n-6)!6!} + \frac{n! x^{n-8} y^8}{(n-8)!8!} + \ldots\right] + i\left[-\frac{n! x^{n-1} y^1}{(n-1)!1!} + \frac{n! x^{n-3} y^3}{(n-3)!3!} - \frac{n! x^{n-5} y^5}{(n-5)!5!} + \frac{n! x^{n-7} y^7}{(n-7)!7!}\right]$$

$$x = 1, y = \frac{b \cdot \ln k}{n}$$

$$\left(1 - i \cdot \frac{b \cdot \ln k}{n}\right)^n = \left[1 - \frac{n!\left(\frac{b \cdot \ln k}{n}\right)^2}{(n-2)!2!} + \frac{n!\left(\frac{b \cdot \ln k}{n}\right)^4}{(n-4)!4!} - \frac{n!\left(\frac{b \cdot \ln k}{n}\right)^6}{(n-6)!6!} + \frac{n!\left(\frac{b \cdot \ln k}{n}\right)^8}{(n-8)!8!} + \ldots\right] + i\left[-\frac{n!\left(\frac{b \cdot \ln k}{n}\right)^1}{(n-1)!1!} + \frac{n!\left(\frac{b \cdot \ln k}{n}\right)^3}{(n-3)!3!} - \frac{n!\left(\frac{b \cdot \ln k}{n}\right)^5}{(n-5)!5!} + \frac{n!\left(\frac{b \cdot \ln k}{n}\right)^7}{(n-7)!7!}\right]$$

$$\frac{1}{k^a}\cdot\left(1-i\cdot\frac{b\cdot\ln k}{n}\right)^n=\frac{1}{k^a}\cdot\left[1-\frac{n!\left(\frac{b\cdot\ln k}{n}\right)^2}{(n-2)!2!}+\frac{n!\left(\frac{b\cdot\ln k}{n}\right)^4}{(n-4)!4!}-\frac{n!\left(\frac{b\cdot\ln k}{n}\right)^6}{(n-6)!6!}+\frac{n!\left(\frac{b\cdot\ln k}{n}\right)^8}{(n-8)!8!}+\ldots\right]+i\cdot\frac{1}{k^a}\cdot\left[-\frac{n!\left(\frac{b\cdot\ln k}{n}\right)^1}{(n-1)!1!}+\frac{n!\left(\frac{b\cdot\ln k}{n}\right)^3}{(n-3)!3!}-\frac{n!\left(\frac{b\cdot\ln k}{n}\right)^5}{(n-5)!5!}+\frac{n!\left(\frac{b\cdot\ln k}{n}\right)^7}{(n-7)!7!}\right]$$

$$\frac{1}{k^{(a+ib)}}=\frac{1}{k^a}\cdot\left(1-i\cdot\frac{b\cdot\ln k}{n}\right)^n=\frac{1}{k^a}\cdot\left[1-\frac{n!(b\cdot\ln k)^2}{n^2(n-2)!2!}+\frac{n!(b\cdot\ln k)^4}{n^4(n-4)!4!}-\frac{n!(b\cdot\ln k)^6}{n^6(n-6)!6!}+\frac{n!(b\cdot\ln k)^8}{n^8(n-8)!8!}+\ldots\right]+i\cdot\frac{1}{k^a}\cdot\left[-\frac{n!(b\cdot\ln k)^1}{n^1(n-1)!1!}+\frac{n!(b\cdot\ln k)^3}{n^3(n-3)!3!}-\frac{n!(b\cdot\ln k)^5}{n^5(n-5)!5!}+\frac{n!(b\cdot\ln k)^7}{n^7(n-7)!7!}\right]$$

$$\boxed{\lim_{n\to\infty}\frac{n!(b\cdot\ln k)^m}{n^m(n-m)!m!}=\frac{(b\cdot\ln k)^m}{m!}}$$

$$\frac{1}{k^{(a+ib)}}=\frac{1}{k^a}\cdot\left[1-\frac{(b\cdot\ln k)^2}{2!}+\frac{(b\cdot\ln k)^4}{4!}-\frac{(b\cdot\ln k)^6}{6!}+\frac{(b\cdot\ln k)^8}{8!}+\ldots\right]+i\cdot\frac{1}{k^a}\cdot\left[-\frac{(b\cdot\ln k)^1}{1!}+\frac{(b\cdot\ln k)^3}{3!}-\frac{(b\cdot\ln k)^5}{5!}+\frac{(b\cdot\ln k)^7}{7!}\right]$$

$$\cos(x)=\frac{1}{0!}-\frac{(x)^2}{2!}+\frac{(x)^4}{4!}-\frac{(x)^6}{6!}+\frac{(x)^8}{8!}-\ldots \qquad \sin(x)=\frac{(x)^1}{1!}-\frac{(x)^3}{3!}+\frac{(x)^5}{5!}-\frac{(x)^7}{7!}-\ldots$$

$$\boxed{\frac{1}{k^{(a+ib)}}=\frac{\cos(b\cdot\ln k)}{k^a}-i\cdot\frac{\sin(b\cdot\ln k)}{k^a}}$$

$$+\frac{1}{1^{(a+ib)}}=\left[+\frac{\cos(b\cdot\ln 1)}{1^a}\right]+i\cdot\left[-\frac{\sin(b\cdot\ln 1)}{1^a}\right]$$

$$-\frac{1}{2^{(a+ib)}}=\left[-\frac{\cos(b\cdot\ln 2)}{2^a}\right]+i\cdot\left[+\frac{\sin(b\cdot\ln 2)}{2^a}\right]$$

$$+\frac{1}{3^{(a+ib)}}=\left[+\frac{\cos(b\cdot\ln 3)}{3^a}\right]+i\cdot\left[-\frac{\sin(b\cdot\ln 3)}{3^a}\right]$$

$$-\frac{1}{4^{(a+ib)}}=\left[-\frac{\cos(b\cdot\ln 4)}{4^a}\right]+i\cdot\left[+\frac{\sin(b\cdot\ln 4)}{4^a}\right]$$

$$\eta(a+ib)=\frac{1}{1^{(a+ib)}}-\frac{1}{2^{(a+ib)}}+\frac{1}{3^{(a+ib)}}-\frac{1}{4^{(a+ib)}}+\ldots.=\left[\frac{\cos(b\ln 1)}{1^a}-\frac{\cos(b\ln 2)}{2^a}+\frac{\cos(b\ln 3)}{3^a}-\frac{\cos(b\ln 4)}{4^a}+\ldots.\right]+i\cdot\left[-\frac{\sin(b\ln 1)}{1^a}+\frac{\sin(b\ln 2)}{2^a}-\frac{\sin(b\ln 3)}{3^a}+\frac{\sin(b\ln 4)}{4^a}+\ldots.\right]$$

another way (and much more easier way) to look at this is:

$$\eta(a+ib)=\left[\frac{1}{1^a}\cdot\cos(-b\ln 1)-\frac{1}{2^a}\cdot\cos(-b\ln 2)+\frac{1}{3^a}\cdot\cos(-b\ln 3)-\frac{1}{4^a}\cdot\cos(-b\ln 4)+....\right]+\left[\frac{1}{1^a}\cdot\sin(-b\ln 1)-\frac{1}{2^a}\cdot\sin(-b\ln 2)+\frac{1}{3^a}\cdot\sin(-b\ln 3)-\frac{1}{4^a}\cdot\sin(-b\ln 4)+....\right]\cdot i$$

$$\vec{V}_k=\frac{1}{1^k} \qquad \theta_k=-b\ln k$$

$$\boxed{\eta(a+ib)=\left[\vec{V}_1\cdot\cos(\theta_1)-\vec{V}_2\cdot\cos(\theta_2)+\vec{V}_3\cdot\cos(\theta_3)-\vec{V}_4\cdot\cos(\theta_4)+....\right]+\left[\vec{V}_1\cdot\sin(\theta_1)-\vec{V}_2\cdot\sin(\theta_2)+\vec{V}_3\cdot\sin(\theta_3)-\vec{V}_4\cdot\sin(\theta_4)+....\right]\cdot i}$$

moving on the xAxis $\vec{V}_1\cdot\cos(\theta_1)-\vec{V}_2\cdot\cos(\theta_2)+\vec{V}_3\cdot\cos(\theta_3)-\vec{V}_4\cdot\cos(\theta_4)+....$

moving on the yAxis $\vec{V}_1\cdot\sin(\theta_1)-\vec{V}_2\cdot\sin(\theta_2)+\vec{V}_3\cdot\sin(\theta_3)-\vec{V}_4\cdot\sin(\theta_4)+....$

when xAxis=0 and yAxis=0 you are at the number zero!

this alternative series extends the zeta function from Re(s) > 1 to the larger domain Re(s) > 0

meaning that all the points of $\eta(a+ib)$ when Re(s) > 0 are converging!

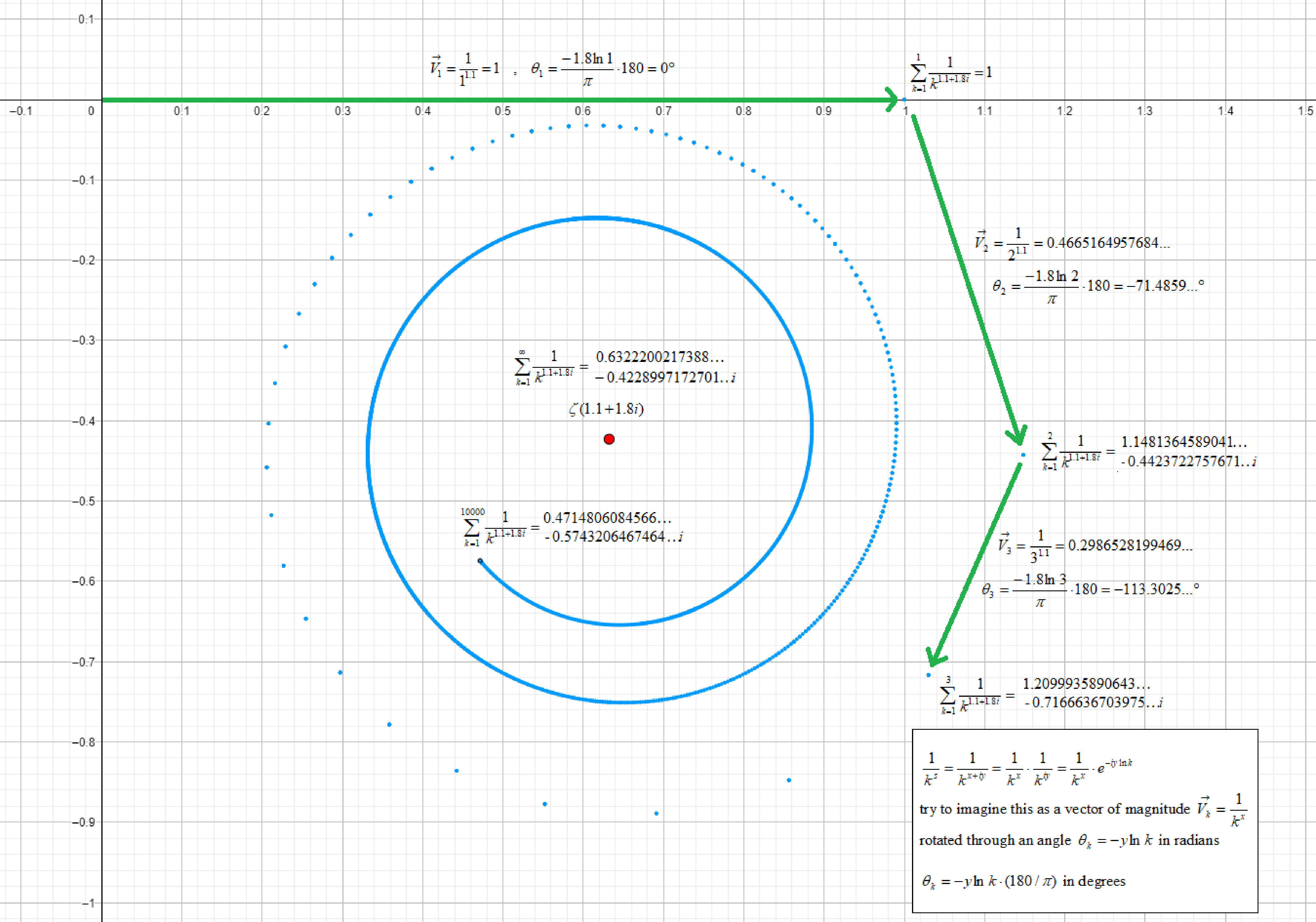
Meaning that you **<u>don't</u>** need to use the complex plane to solve or understand the the famous Riemann hypothesis

because $(1-2^{1-s})\xi(s)=\eta(s)$ when one side is zero the other one is zero as well
the only difference between the two is that one is not a "real zero" only "Analytic Continuation" value of the center of the spiral of a divergent series
while the other is a "real zero" the spiral converges to the value zero

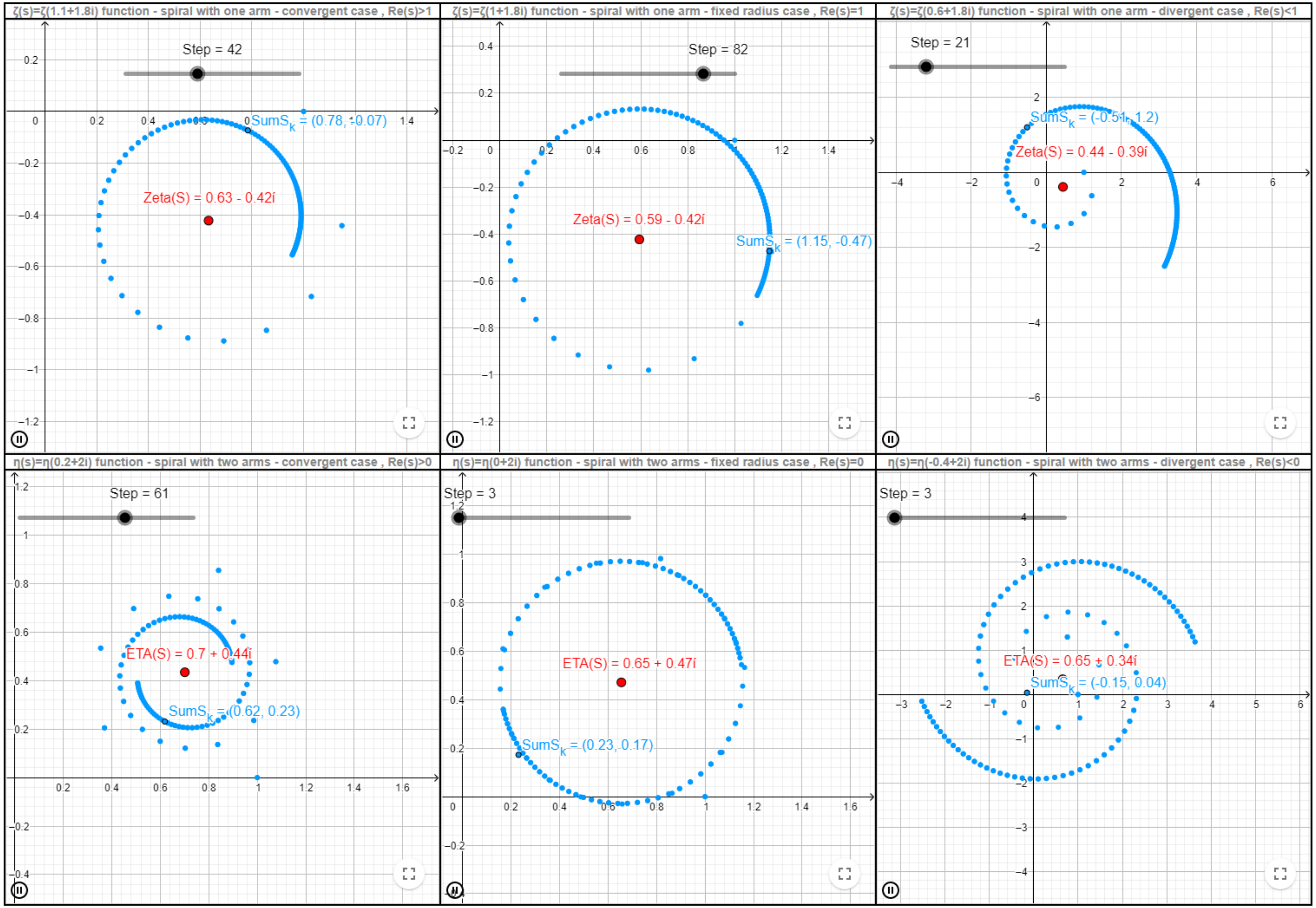
The Riemann hypothesis equivalent to:

$$\boxed{0=\frac{\cos(b\ln 1)}{1^a}-\frac{\cos(b\ln 2)}{2^a}+\frac{\cos(b\ln 3)}{3^a}-\frac{\cos(b\ln 4)}{4^a}+....} \text{ and } \boxed{0=\frac{\sin(b\ln 1)}{1^a}-\frac{\sin(b\ln 2)}{2^a}+\frac{\sin(b\ln 3)}{3^a}-\frac{\sin(b\ln 4)}{4^a}+....}$$

where a and b are real numbers and the only solution for $0<a<1$ is when $a=\frac{1}{2}$



$\vec{V}_1 = \frac{1}{1^{1.1}} = 1$, $\theta_1 = \frac{-1.8 \ln 1}{\pi} \cdot 180 = 0°$
$\sum_{k=1}^{1} \frac{1}{k^{1.1+1.8i}} = 1$
$\vec{V}_2 = \frac{1}{2^{1.1}} = 0.4665164957684...$
$\theta_2 = \frac{-1.8 \ln 2}{\pi} \cdot 180 = -71.4859...°$
$\sum_{k=1}^{\infty} \frac{1}{k^{1.1+1.8i}} = 0.6322200217388... - 0.4228997172701...i$
$\zeta(1.1+1.8i)$
$\sum_{k=1}^{2} \frac{1}{k^{1.1+1.8i}} = 1.1481364589041... - 0.4423722757671...i$
$\sum_{k=1}^{10000} \frac{1}{k^{1.1+1.8i}} = 0.4714806084566... - 0.5743206467464...i$
$\vec{V}_3 = \frac{1}{3^{1.1}} = 0.2986528199469...$
$\theta_3 = \frac{-1.8 \ln 3}{\pi} \cdot 180 = -113.3025...°$
$\sum_{k=1}^{3} \frac{1}{k^{1.1+1.8i}} = 1.2099935890643... - 0.7166636703975...i$
$\frac{1}{k^s} = \frac{1}{k^{x+iy}} = \frac{1}{k^x} \cdot \frac{1}{k^{iy}} = \frac{1}{k^x} \cdot e^{-iy \ln k}$
try to imagine this as a vector of magnitude $\vec{V}_k = \frac{1}{k^x}$
rotated through an angle $\theta_k = -y \ln k$ in radians
$\theta_k = -y \ln k \cdot (180/\pi)$ in degrees


ζ(s)=ζ(1.1+1.8i) function - spiral with one arm - convergent case , Re(s)>1
Step = 42
SumS_k = (0.78, -0.07)
Zeta(S) = 0.63 - 0.42i
ζ(s)=ζ(1+1.8i) function - spiral with one arm - fixed radius case , Re(s)=1
Step = 82
Zeta(S) = 0.59 - 0.42i
SumS_k = (1.15, -0.47)
ζ(s)=ζ(0.6+1.8i) function - spiral with one arm - divergent case , Re(s)<1
Step = 21
SumS_k = (-0.51, 1.2)
Zeta(S) = 0.44 - 0.39i
η(s)=η(0.2+2i) function - spiral with two arms - convergent case , Re(s)>0
Step = 61
ETA(S) = 0.7 + 0.44i
SumS_k = (0.62, 0.23)
η(s)=η(0+2i) function - spiral with two arms - fixed radius case , Re(s)=0
Step = 3
ETA(S) = 0.65 + 0.47i
SumS_k = (0.23, 0.17)
η(s)=η(-0.4+2i) function - spiral with two arms - divergent case , Re(s)<0
Step = 3
ETA(S) = 0.65 + 0.34i
SumS_k = (-0.15, 0.04)

$$\eta(a+ib)=\left[\vec{V_1}\cdot\cos(\theta_1)-\vec{V_2}\cdot\cos(\theta_2)+\vec{V_3}\cdot\cos(\theta_3)-\vec{V_4}\cdot\cos(\theta_4)+\dots\right]+\left[\vec{V_1}\cdot\sin(\theta_1)-\vec{V_2}\cdot\sin(\theta_2)+\vec{V_3}\cdot\sin(\theta_3)-\vec{V_4}\cdot\sin(\theta_4)+\dots\right]\cdot i$$

a = 0.5 (slider from -5 to 5)

b = 15 (slider from 0 to 15)

$s = a + b\,i$ → $0.5 + 15i$

n = 20000 (slider from 1 to 2.0×10^4)

$$SumS_k = Sum\left(\frac{1}{(-1)^{k+1}\,k^s}, k, 1, n\right)$$

→ (1.0900094407307, 1.0915070434392)

Input...

moving on the xAxis $\vec{V_1}\cdot\cos(\theta_1)-\vec{V_2}\cdot\cos(\theta_2)+\vec{V_3}\cdot\cos(\theta_3)-\vec{V_4}\cdot\cos(\theta_4)+\dots$

moving on the yAxis $\vec{V_1}\cdot\sin(\theta_1)-\vec{V_2}\cdot\sin(\theta_2)+\vec{V_3}\cdot\sin(\theta_3)-\vec{V_4}\cdot\sin(\theta_4)+\dots$

when xAxis=0 and yAxis=0 you are at the number zero!

this alternative series extends the zeta function from Re(s) > 1 to the larger domain Re(s) > 0

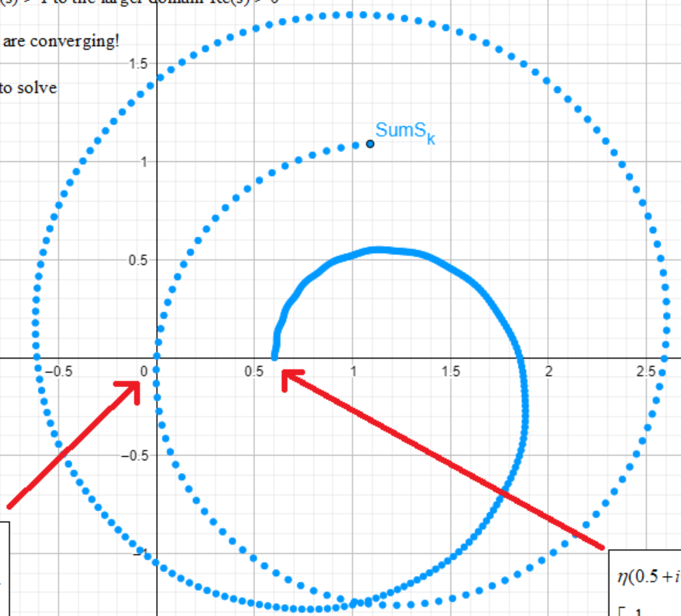
meaning that all the points of $\eta(a+ib)$ when Re(s) > 0 are converging!

Meaning that you **don't** need to use the complex plane to solve the the famous Riemann hypothesis or understand it

The Riemann hypothesis equivalent to:

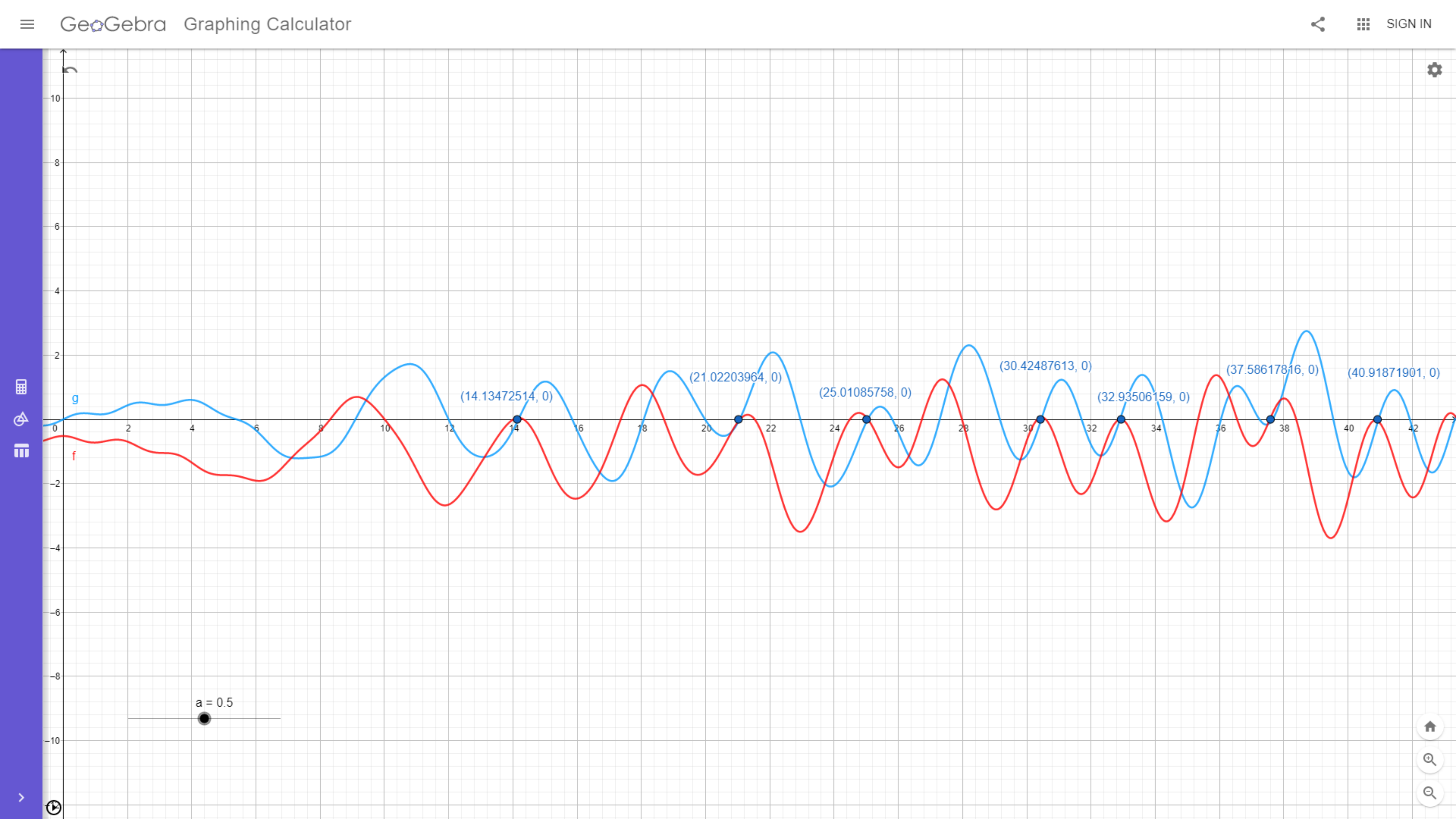
$$0=\frac{\cos(b\ln 1)}{1^a}-\frac{\cos(b\ln 2)}{2^a}+\frac{\cos(b\ln 3)}{3^a}-\frac{\cos(b\ln 4)}{4^a}+\dots \quad \text{and} \quad 0=\frac{\sin(b\ln 1)}{1^a}-\frac{\sin(b\ln 2)}{2^a}+\frac{\sin(b\ln 3)}{3^a}-\frac{\sin(b\ln 4)}{4^a}+\dots$$

where a and b are real numbers and the only solution for $0<a<1$ is when $a=\frac{1}{2}$



$$\eta(0.5+i14.134725)=$$
$$\left[\frac{1}{1^{0.5}}\cdot\cos(\ln\frac{1}{1^{14.134725}})-\frac{1}{2^{0.5}}\cdot\cos(\ln\frac{1}{2^{14.134725}})+\frac{1}{3^{0.5}}\cdot\cos(\ln\frac{1}{3^{14.134725}})-\frac{1}{4^{0.5}}\cdot\cos(\ln\frac{1}{4^{14.134725}})+\dots\right]+$$
$$\left[\frac{1}{1^{0.5}}\cdot\sin(\ln\frac{1}{1^{14.134725}})-\frac{1}{2^{0.5}}\cdot\sin(\ln\frac{1}{2^{14.134725}})+\frac{1}{3^{0.5}}\cdot\sin(\ln\frac{1}{3^{14.134725}})-\frac{1}{4^{0.5}}\cdot\sin(\ln\frac{1}{4^{14.134725}})+\dots\right]\cdot i$$

$$\eta(0.5+i0)=\frac{1}{1^{0.5}}-\frac{1}{2^{0.5}}+\frac{1}{3^{0.5}}-\frac{1}{4^{0.5}}+\dots=0.604898643421630370247\dots=$$
$$\left[\frac{1}{1^{0.5}}\cdot\cos(\ln\frac{1}{1^0})-\frac{1}{2^{0.5}}\cdot\cos(\ln\frac{1}{2^0})+\frac{1}{3^{0.5}}\cdot\cos(\ln\frac{1}{3^0})-\frac{1}{4^{0.5}}\cdot\cos(\ln\frac{1}{4^0})+\dots\right]+$$
$$\left[\frac{1}{1^{0.5}}\cdot\sin(\ln\frac{1}{1^0})-\frac{1}{2^{0.5}}\cdot\sin(\ln\frac{1}{2^0})+\frac{1}{3^{0.5}}\cdot\sin(\ln\frac{1}{3^0})-\frac{1}{4^{0.5}}\cdot\sin(\ln\frac{1}{4^0})+\dots\right]\cdot i$$


GeoGebra Graphing Calculator
SIGN IN
g
f
(14.13472514, 0)
(21.02203964, 0)
(25.01085758, 0)
(30.42487613, 0)
(32.93506159, 0)
(37.58617816, 0)
(40.91871901, 0)
a = 0.5

$$f(x)=\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 4)}{4^a}+....$$

$$g(x)=\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 4)}{4^a}+....$$

I am going to make a new function $h(x)$ that will include but cases and will be 0 only when both functions $f(x)$ and $g(x)$ are 0 as well

The simplest way is to have $h(x)=f(x)f(x)+g(x)g(x)$ where $h(x)\geq 0$ always and $h(x)=0$ only when you have those non-trival zeros

$$f(x)=\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 4)}{4^a}+....$$

$$f(x)f(x)=?$$

$$\begin{aligned}
&+\frac{\cos(x\ln 1)}{1^a}\cdot\left(\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 4)}{4^a}+...\right)\\
&-\frac{\cos(x\ln 2)}{2^a}\cdot\left(\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 4)}{4^a}+...\right)\\
&+\frac{\cos(x\ln 3)}{3^a}\cdot\left(\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 4)}{4^a}+...\right)\\
&-\frac{\cos(x\ln 4)}{4^a}\cdot\left(\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 4)}{4^a}+...\right)
\end{aligned}$$

$$\begin{aligned}
&+\frac{\cos(x\ln 1)}{1^a}\cdot\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 1)}{1^a}\cdot\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 1)}{1^a}\cdot\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 1)}{1^a}\cdot\frac{\cos(x\ln 4)}{4^a}+...\\
&-\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 1)}{1^a}+\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 2)}{2^a}-\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 3)}{3^a}+\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 4)}{4^a}-...\\
&+\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 4)}{4^a}+...\\
&-\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 1)}{1^a}+\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 2)}{2^a}-\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 3)}{3^a}+\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 4)}{4^a}-...
\end{aligned}$$

$$\begin{aligned}
&+\frac{\cos(x\ln 1)}{1^a}\cdot\frac{\cos(x\ln 1)}{1^a}\\
&-2\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 1)}{1^a}+\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 2)}{2^a}\\
&+2\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 1)}{1^a}-2\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 3)}{3^a}\\
&-2\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 1)}{1^a}+2\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 2)}{2^a}-2\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 3)}{3^a}+\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 4)}{4^a}
\end{aligned}$$

$$g(x)=\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 4)}{4^a}+....$$

$$g(x)g(x)=?$$

$$\begin{aligned}
&+\frac{\sin(x\ln 1)}{1^a}\cdot\left(\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 4)}{4^a}+...\right)\\
&-\frac{\sin(x\ln 2)}{2^a}\cdot\left(\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 4)}{4^a}+...\right)\\
&+\frac{\sin(x\ln 3)}{3^a}\cdot\left(\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 4)}{4^a}+...\right)\\
&-\frac{\sin(x\ln 4)}{4^a}\cdot\left(\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 4)}{4^a}+...\right)
\end{aligned}$$

$$\begin{aligned}
&+\frac{\sin(x\ln 1)}{1^a}\cdot\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 1)}{1^a}\cdot\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 1)}{1^a}\cdot\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 1)}{1^a}\cdot\frac{\sin(x\ln 4)}{4^a}+...\\
&-\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 1)}{1^a}+\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 2)}{2^a}-\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 3)}{3^a}+\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 4)}{4^a}-...\\
&+\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 4)}{4^a}+...\\
&-\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 1)}{1^a}+\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 2)}{2^a}-\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 3)}{3^a}+\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 4)}{4^a}-...
\end{aligned}$$

$$\begin{aligned}
&+\frac{\sin(x\ln 1)}{1^a}\cdot\frac{\sin(x\ln 1)}{1^a}\\
&-2\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 1)}{1^a}+\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 2)}{2^a}\\
&+2\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 1)}{1^a}-2\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 3)}{3^a}\\
&-2\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 1)}{1^a}+2\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 2)}{2^a}-2\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 3)}{3^a}+\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 4)}{4^a}
\end{aligned}$$

$h(x) = f(x)f(x) + g(x)g(x) = ?$

$$+\frac{\cos(x\ln 1)}{1^a}\cdot\frac{\cos(x\ln 1)}{1^a}$$
$$-2\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 1)}{1^a}+\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 2)}{2^a}$$
$$+2\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 1)}{1^a}-2\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 3)}{3^a}$$
$$-2\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 1)}{1^a}+2\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 2)}{2^a}-2\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 3)}{3^a}+\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 4)}{4^a}$$

$$+\frac{\sin(x\ln 1)}{1^a}\cdot\frac{\sin(x\ln 1)}{1^a}$$
$$-2\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 1)}{1^a}+\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 2)}{2^a}$$
$$+2\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 1)}{1^a}-2\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 3)}{3^a}$$
$$-2\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 1)}{1^a}+2\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 2)}{2^a}-2\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 3)}{3^a}+\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 4)}{4^a}$$

$$\cos(a-b)=\cos(a)\cos(b)+\sin(a)\sin(b)$$

$$+\frac{\cos(x\ln 1-x\ln 1)}{1^a\cdot 1^a}$$
$$-2\frac{\cos(x\ln 2-x\ln 1)}{2^a\cdot 1^a}+\frac{\cos(x\ln 2-x\ln 2)}{2^a\cdot 2^a}$$
$$+2\frac{\cos(x\ln 3-x\ln 1)}{3^a\cdot 1^a}-2\frac{\cos(x\ln 3-x\ln 2)}{3^a\cdot 2^a}+\frac{\cos(x\ln 3-x\ln 3)}{3^a\cdot 3^a}$$
$$-2\frac{\cos(x\ln 4-x\ln 1)}{4^a\cdot 1^a}+2\frac{\cos(x\ln 4-x\ln 2)}{4^a\cdot 2^a}-2\frac{\cos(x\ln 4-x\ln 3)}{4^a\cdot 3^a}+\frac{\cos(x\ln 4-x\ln 4)}{4^a\cdot 4^a}$$

$$
\begin{array}{|l|}\hline
+\frac{\cos(x\ln 1/1)}{1^a\cdot 1^a} \\
-2\frac{\cos(x\ln 2/1)}{2^a\cdot 1^a}+\frac{\cos(x\ln 2/2)}{2^a\cdot 2^a} \\
+2\frac{\cos(x\ln 3/1)}{3^a\cdot 1^a}-2\frac{\cos(x\ln 3/2)}{3^a\cdot 2^a}+\frac{\cos(x\ln 3/3)}{3^a\cdot 3^a} \\
-2\frac{\cos(x\ln 4/1)}{4^a\cdot 1^a}+2\frac{\cos(x\ln 4/2)}{4^a\cdot 2^a}-2\frac{\cos(x\ln 4/3)}{4^a\cdot 3^a}+\frac{\cos(x\ln 4/4)}{4^a\cdot 4^a} \\ \hline
\end{array}
$$

$$
\begin{array}{|l|}\hline
+\frac{1}{1^a\cdot 1^a} \\
-2\frac{\cos(x\ln 2/1)}{2^a\cdot 1^a}+\frac{1}{2^a\cdot 2^a} \\
+2\frac{\cos(x\ln 3/1)}{3^a\cdot 1^a}-2\frac{\cos(x\ln 3/2)}{3^a\cdot 2^a}+\frac{1}{3^a\cdot 3^a} \\
-2\frac{\cos(x\ln 4/1)}{4^a\cdot 1^a}+2\frac{\cos(x\ln 4/2)}{4^a\cdot 2^a}-2\frac{\cos(x\ln 4/3)}{4^a\cdot 3^a}+\frac{1}{4^a\cdot 4^a} \\ \hline
\end{array}
$$

$$
\begin{array}{|l|}\hline
+\frac{1}{1^a\cdot 1^a} \\
+\frac{1}{2^a\cdot 2^a}-2\frac{\cos(x\ln 2/1)}{2^a\cdot 1^a} \\
+\frac{1}{3^a\cdot 3^a}+2\frac{\cos(x\ln 3/1)}{3^a\cdot 1^a}-2\frac{\cos(x\ln 3/2)}{3^a\cdot 2^a} \\
+\frac{1}{4^a\cdot 4^a}-2\frac{\cos(x\ln 4/1)}{4^a\cdot 1^a}+2\frac{\cos(x\ln 4/2)}{4^a\cdot 2^a}-2\frac{\cos(x\ln 4/3)}{4^a\cdot 3^a} \\ \hline
\end{array}
$$

$$
\zeta(2a)+
\begin{array}{|l|}\hline
-2\frac{\cos(x\ln 2/1)}{2^a\cdot 1^a} \\
+2\frac{\cos(x\ln 3/1)}{3^a\cdot 1^a}-2\frac{\cos(x\ln 3/2)}{3^a\cdot 2^a} \\
-2\frac{\cos(x\ln 4/1)}{4^a\cdot 1^a}+2\frac{\cos(x\ln 4/2)}{4^a\cdot 2^a}-2\frac{\cos(x\ln 4/3)}{4^a\cdot 3^a} \\
+2\frac{\cos(x\ln 5/1)}{5^a\cdot 1^a}-2\frac{\cos(x\ln 5/2)}{5^a\cdot 2^a}+2\frac{\cos(x\ln 5/3)}{5^a\cdot 3^a}-2\frac{\cos(x\ln 5/4)}{5^a\cdot 4^a} \\ \hline
\end{array}
$$

$$\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\frac{2\cos(x\ln(n/k))}{(nk)^a}=\begin{array}{|l|}\hline -2\frac{\cos(x\ln 2/1)}{2^a\cdot 1^a}\\ +2\frac{\cos(x\ln 3/1)}{3^a\cdot 1^a}-2\frac{\cos(x\ln 3/2)}{3^a\cdot 2^a}\\ -2\frac{\cos(x\ln 4/1)}{4^a\cdot 1^a}+2\frac{\cos(x\ln 4/2)}{4^a\cdot 2^a}-2\frac{\cos(x\ln 4/3)}{4^a\cdot 3^a}\\ +2\frac{\cos(x\ln 5/1)}{5^a\cdot 1^a}-2\frac{\cos(x\ln 5/2)}{5^a\cdot 2^a}+2\frac{\cos(x\ln 5/3)}{5^a\cdot 3^a}-2\frac{\cos(x\ln 5/4)}{5^a\cdot 4^a}\\ -\dots\\ +\dots\\ -\dots\\ \hline\end{array}$$

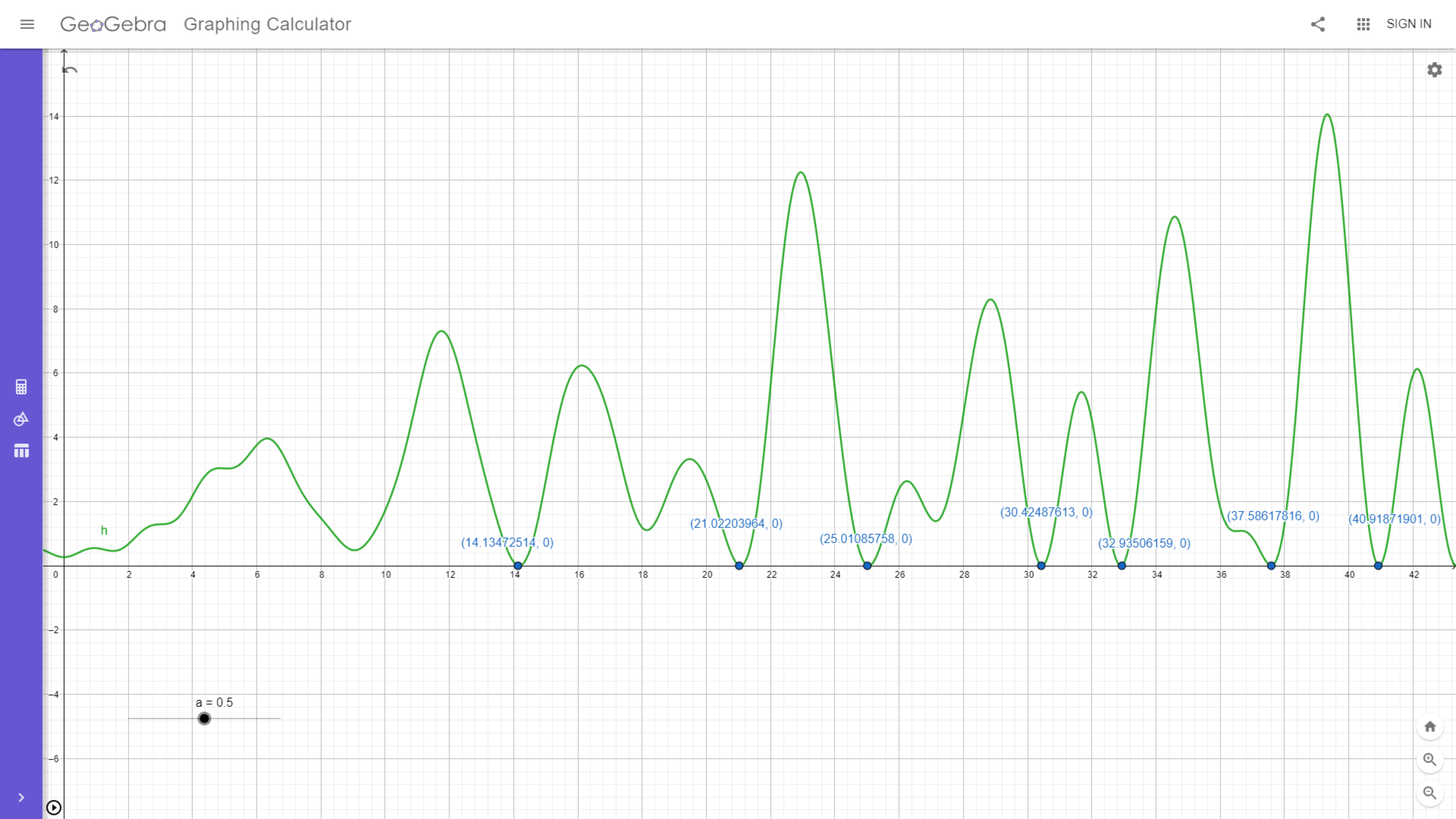
$$h(x)=f(x)f(x)+g(x)g(x)=\zeta(2a)+\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\frac{2\cos(x\ln(n/k))}{(nk)^a}$$

$$\boxed{0\le h(x)=\zeta(2a)+\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\frac{2\cos(x\ln(n/k))}{(nk)^a}}$$

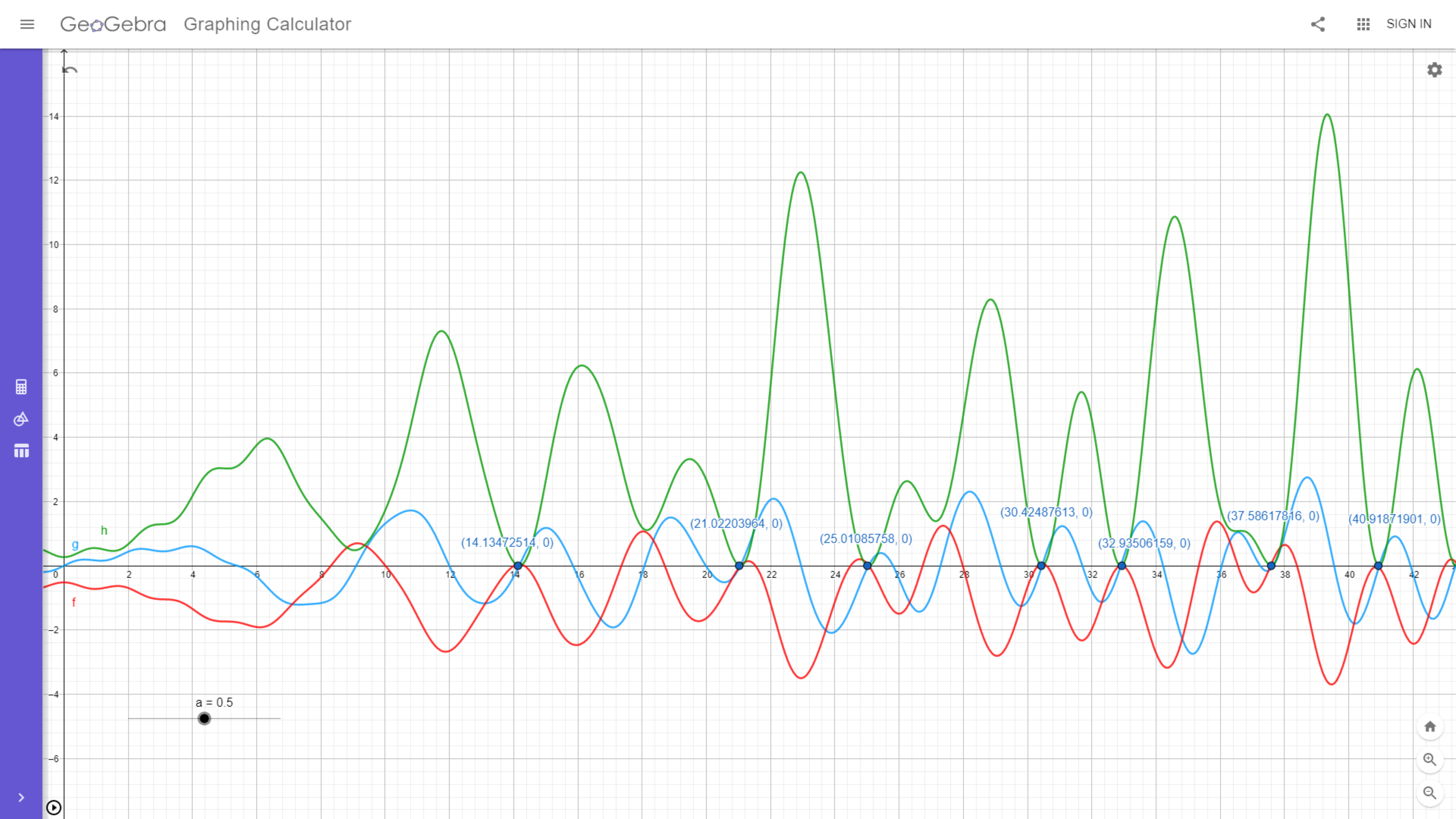
$$-\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\frac{2\cos(x\ln(n/k))}{(nk)^a}\le\zeta(2a)$$

$$\boxed{q(x)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x\ln(n/k))}{(nk)^a}\le\zeta(2a)}$$

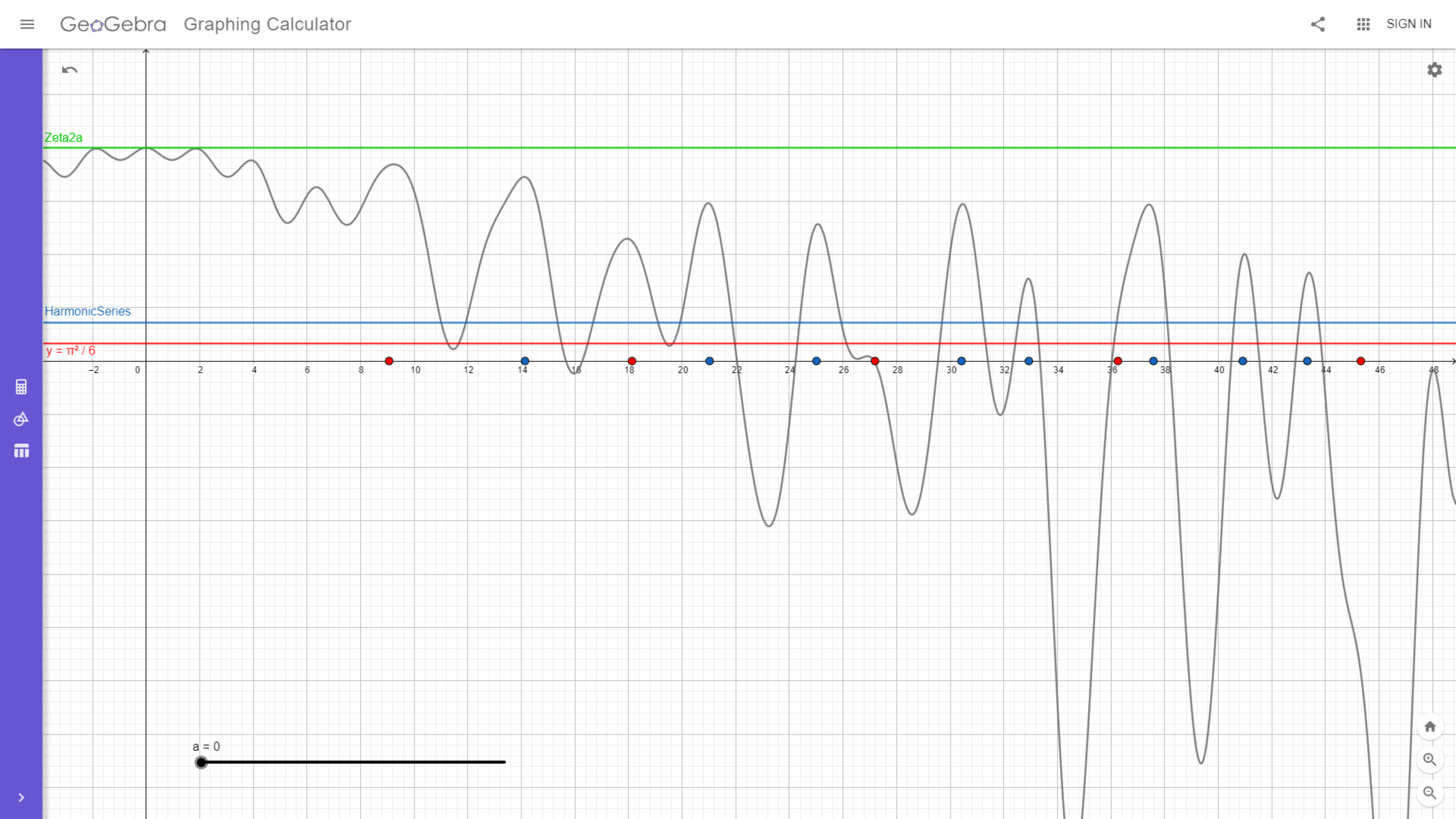
When $q(x)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x\ln(n/k))}{(nk)^a}=\zeta(2a)$ then x is a non trivial zero (because $h(x)=0$)

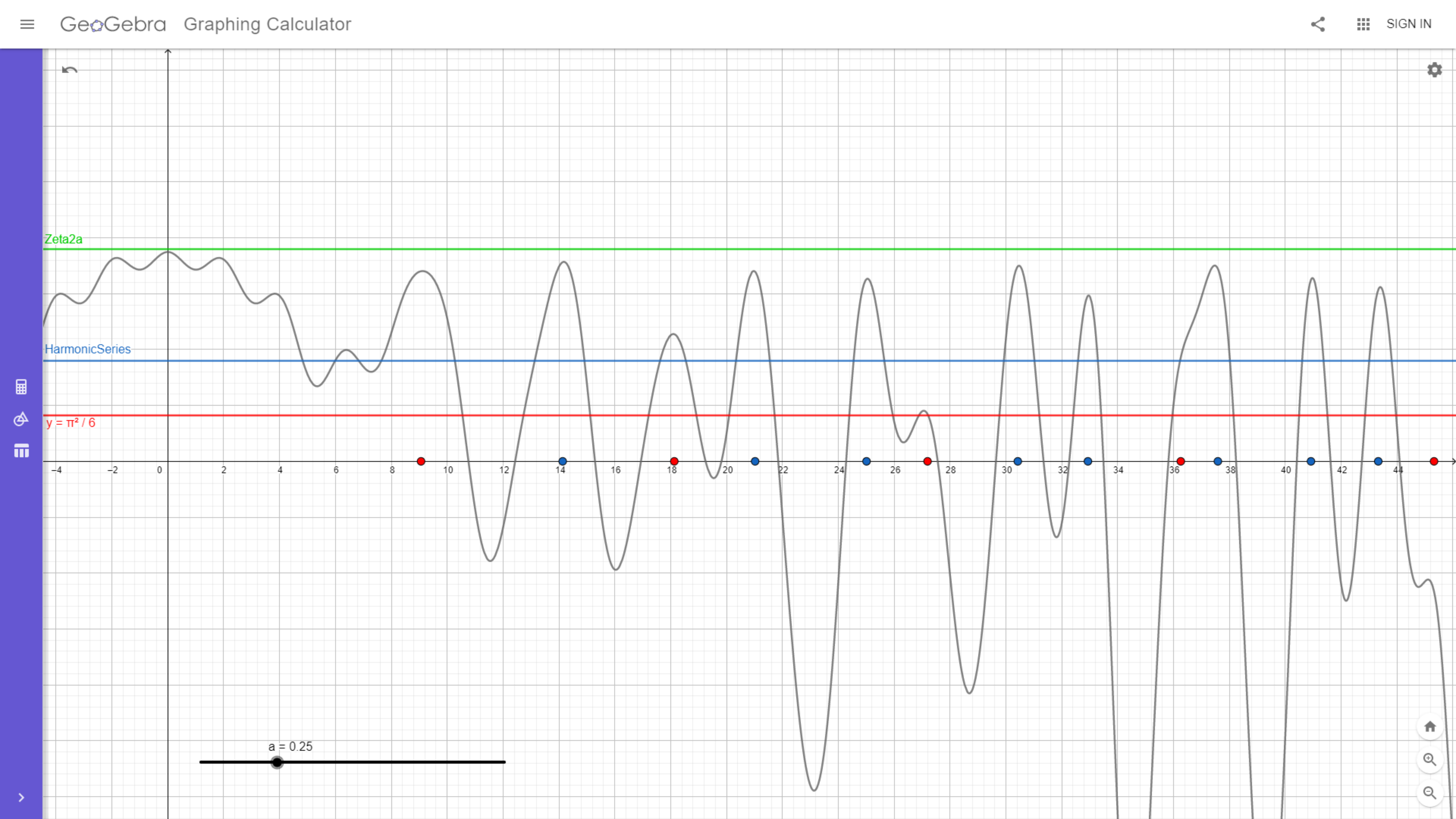


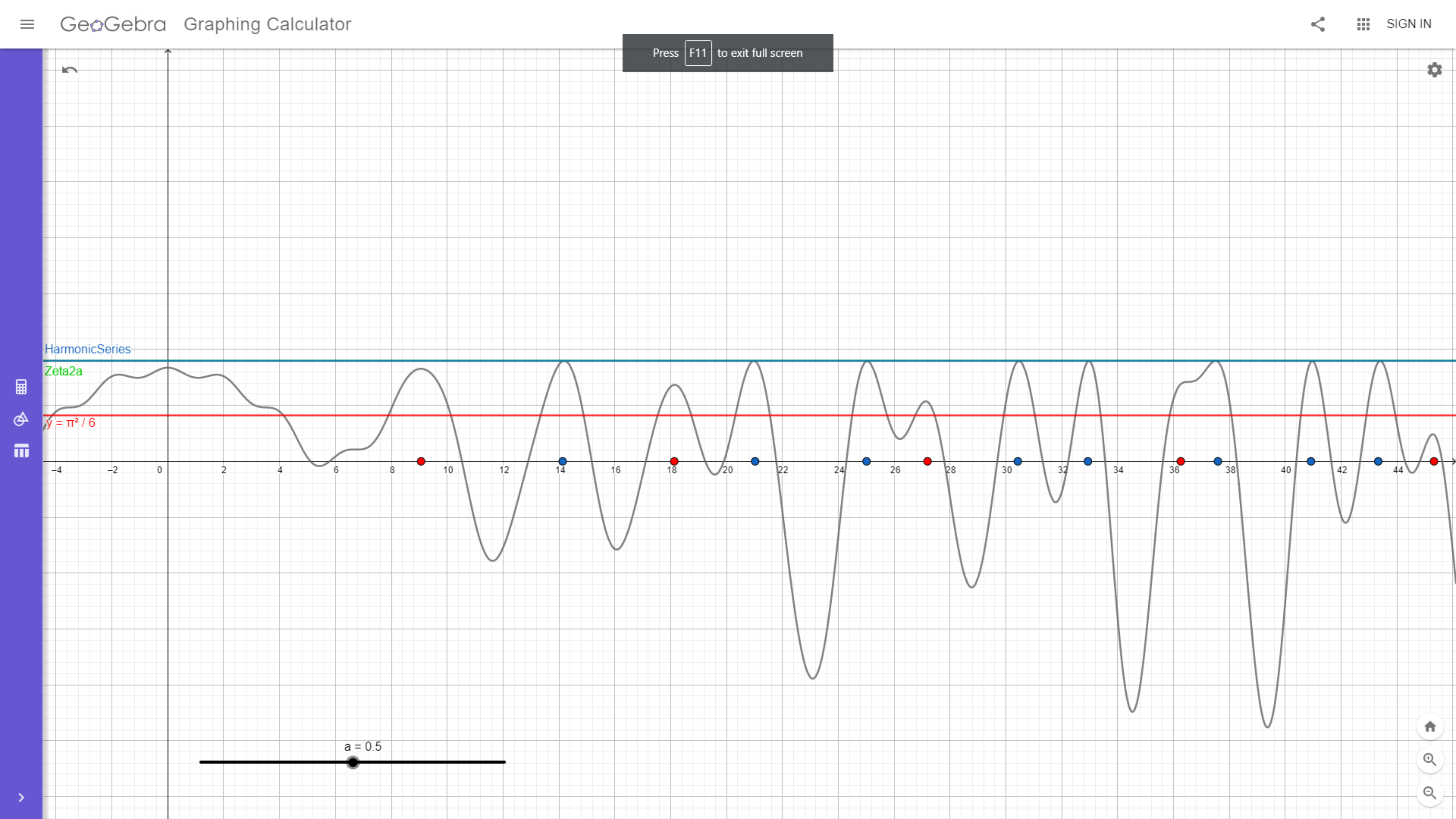
GeoGebra
Graphing Calculator
SIGN IN
h
(14.13472514, 0)
(21.02203964, 0)
(25.01085758, 0)
(30.42487613, 0)
(32.93506159, 0)
(37.58617816, 0)
(40.91871901, 0)
a = 0.5

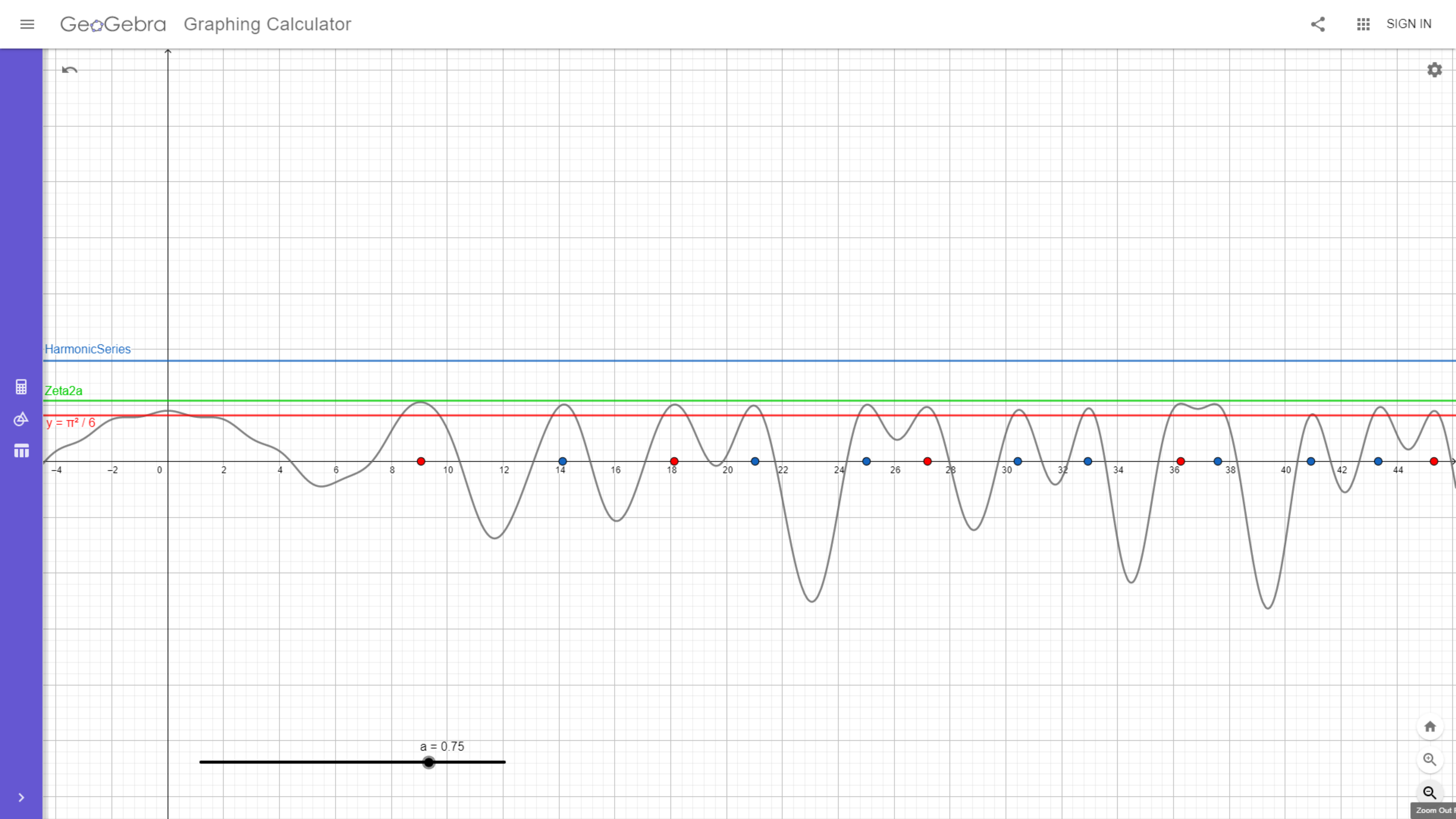


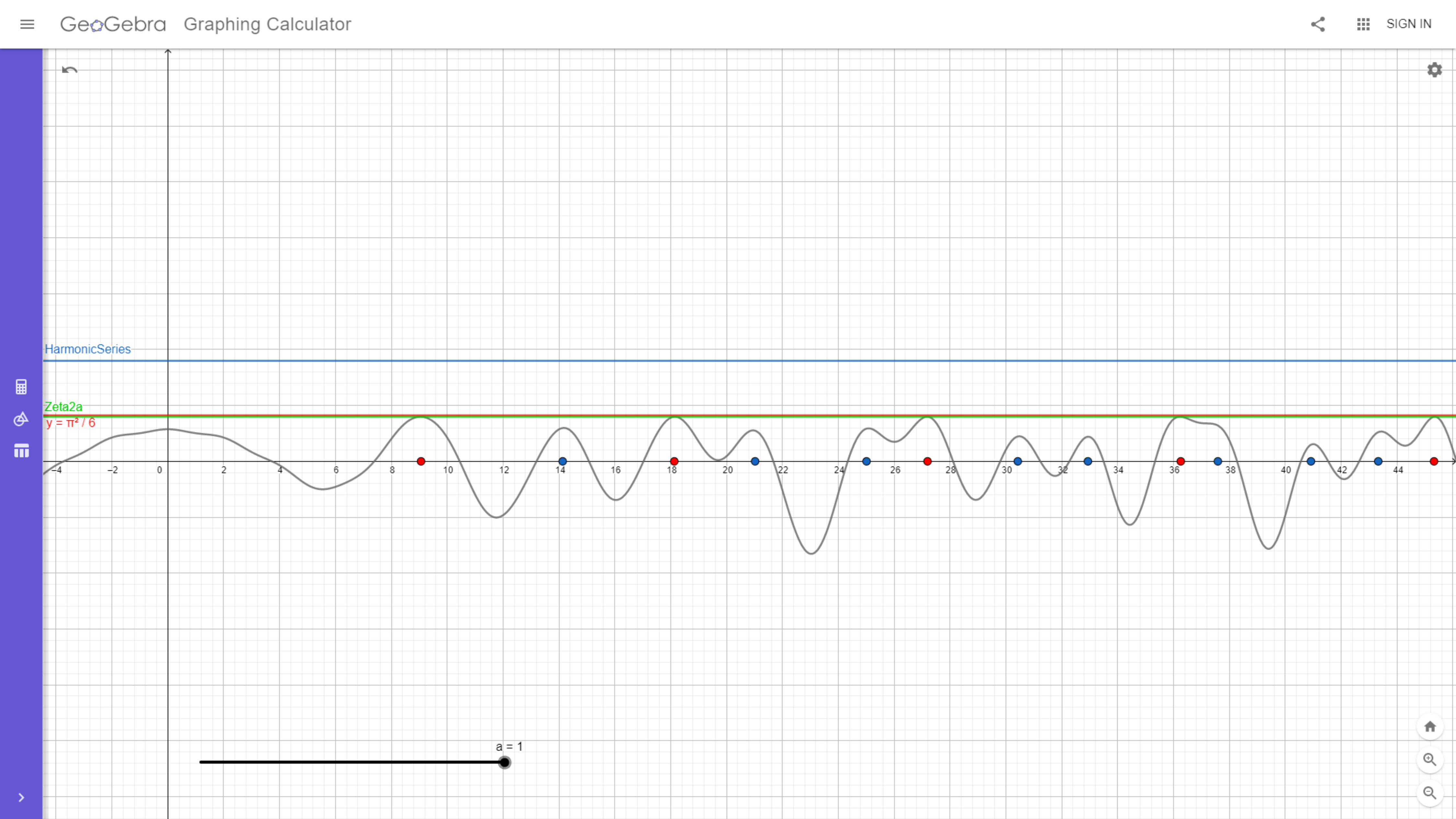
GeoGebra Graphing Calculator
SIGN IN
h
g
f
(14.13472514, 0)
(21.02203964, 0)
(25.01085758, 0)
(30.42487613, 0)
(32.93506159, 0)
(37.58617816, 0)
(40.91871901, 0)
a = 0.5


GeoGebra Graphing Calculator
SIGN IN
Zeta2a
HarmonicSeries
y = π² / 6
-2
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40
42
44
46
48
a = 0


GeoGebra Graphing Calculator
SIGN IN
Zeta2a
HarmonicSeries
y = π² / 6
–4
–2
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40
42
44
a = 0.25


GeoGebra Graphing Calculator
SIGN IN
Press F11 to exit full screen
HarmonicSeries
Zeta2a
y = π² / 6
–4
–2
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40
42
44
a = 0.5


GeoGebra Graphing Calculator
SIGN IN
HarmonicSeries
Zeta2a
y = π² / 6
−4 −2 0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44
a = 0.75


GeoGebra Graphing Calculator
SIGN IN
HarmonicSeries
Zeta2a
y = π² / 6
−4 −2 0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44
a = 1

(For $1 < a$ there are no non trivial zeros this is a known fact so I am not showing why)

i used eta function summation to get $h(x)$ and because $\boxed{\left(1-\frac{2}{2^s}\right)\zeta(s) = \eta(s)}$ then for $b = \frac{2\pi t}{\ln 2}$ when $a = 1$ we get

$$q(b) = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(b\ln(n/k))}{(nk)^a} = \zeta(2)$$

$$\boxed{\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos\left(2\pi t\frac{\ln(n/k)}{\ln 2}\right)}{(nk)^a} = \frac{\pi^2}{6}}$$

<u>side note:</u>

if $\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(b\ln(n/k))}{(nk)^1} = \zeta(1)$ then there were zeros on the $\zeta(1)$ line

but $\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(b\ln(n/k))}{(nk)^1} = \zeta(2) = \frac{\pi^2}{6} < \zeta(1)$ so no zeros on the $\zeta(1)$ line ☺

$$\boxed{q(x) = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x\ln(n/k))}{(nk)^a} \le \zeta(2a)}$$

the range of q(x) is only for 1<=a so I will extand the range below 1 by using eta function instead of zeta function:

$$\boxed{q(x) = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x\ln(n/k))}{(nk)^a} \le \frac{\eta(2a)}{\left(1-\frac{2}{2^{2a}}\right)}}$$

Critical Strip

When $q(x)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x\ln(n/k))}{(nk)^a}=\frac{\eta(2a)}{\left(1-\frac{2}{2^{2a}}\right)}$ then x is a non trivial zero (because $h(x)=0$)

$\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x_0\ln(n/k))}{(nk)^a}=\frac{\eta(2a)}{\left(1-\frac{2}{2^{2a}}\right)}$ this is point $a\pm ix_0$ when $\zeta(a\pm ix_0)=0$

Case #1

for the range $0.5<a<1$ the function $\eta(2a)/\left(1-\frac{2}{2^{2a}}\right)$ is converging

meaning the function $q(x)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x\ln(n/k))}{(nk)^a}$ has a sup value of $\eta(2a)/\left(1-\frac{2}{2^{2a}}\right)<\zeta(1)$ which is a fixed value (a real number!)

and because of that the function (theoretically) can have values of x that will result $q(x)=0$

(I am going to eliminate those theoretical zeros by using case #2 later on)

Case #2

for the range $0<a<0.5$ the function $\eta(2a)/\left(1-\frac{2}{2^{2a}}\right)$ is diverging

meaning the function $q(x)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x\ln(n/k))}{(nk)^a}$ has no (fixed) sup value!

The sup value should have been $\eta(2a)/\left(1-\frac{2}{2^{2a}}\right)$ but this is not a fixed value

and because of that the function changing all the time as n gets biger and biger
making the values of x to changed on the cos function summation

side note:

I am not using analytic continuation at all! So this means the range for case #2 can go from $0<a<0.5$ to $a<0.5$ because I don't have any analytic continuation zeroes
The sup value of the function is diverging for $a<0.5$ all the time ☺

GeoGebra Graphing Calculator SIGN IN

a(x) = (-1)^(1 + 1) / 1^x + (-1)^(2 +

b(x) = (-1)^(1 + 1) / 1^x + (-1)^(2 +

c(x) = (-1)^(1 + 1) / 1^(2x) + (-1)^(2

d(x) = (-1)^(1 + 1) / 1^(2x) + (-1)^(2

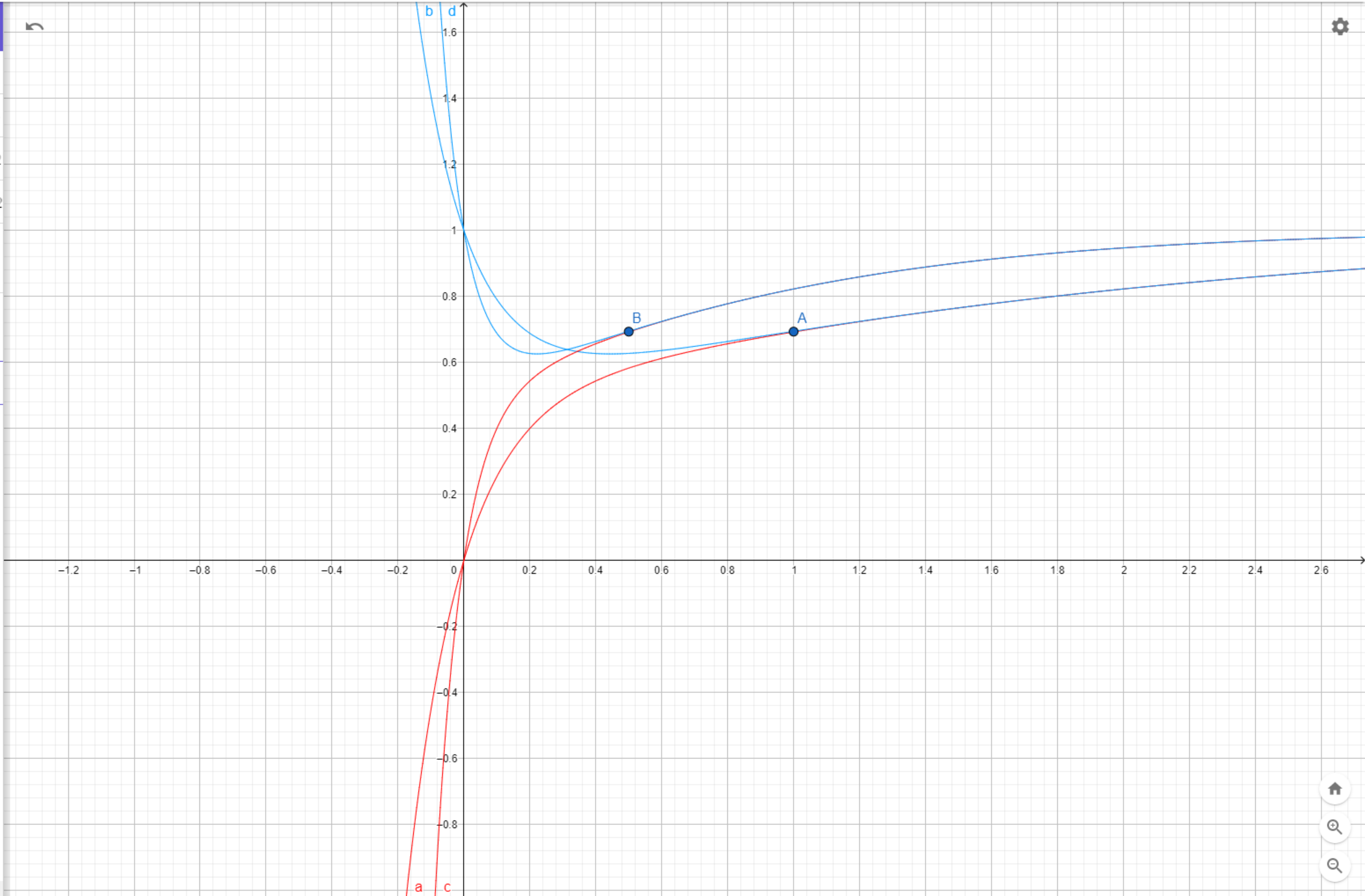
A = (1, ln(2))

→ (1, 0.6931471805599)

B = (0.5, ln(2))

→ (0.5, 0.6931471805599)

+



GeoGebra Graphing Calculator

SIGN IN

a(x) = (-1)^(1 + 1) / 1^x + (-1)^(2 +

b(x) = (-1)^(1 + 1) / 1^x + (-1)^(2 +

c(x) = (-1)^(1 + 1) / 1^(2x) + (-1)^(2

d(x) = (-1)^(1 + 1) / 1^(2x) + (-1)^(2

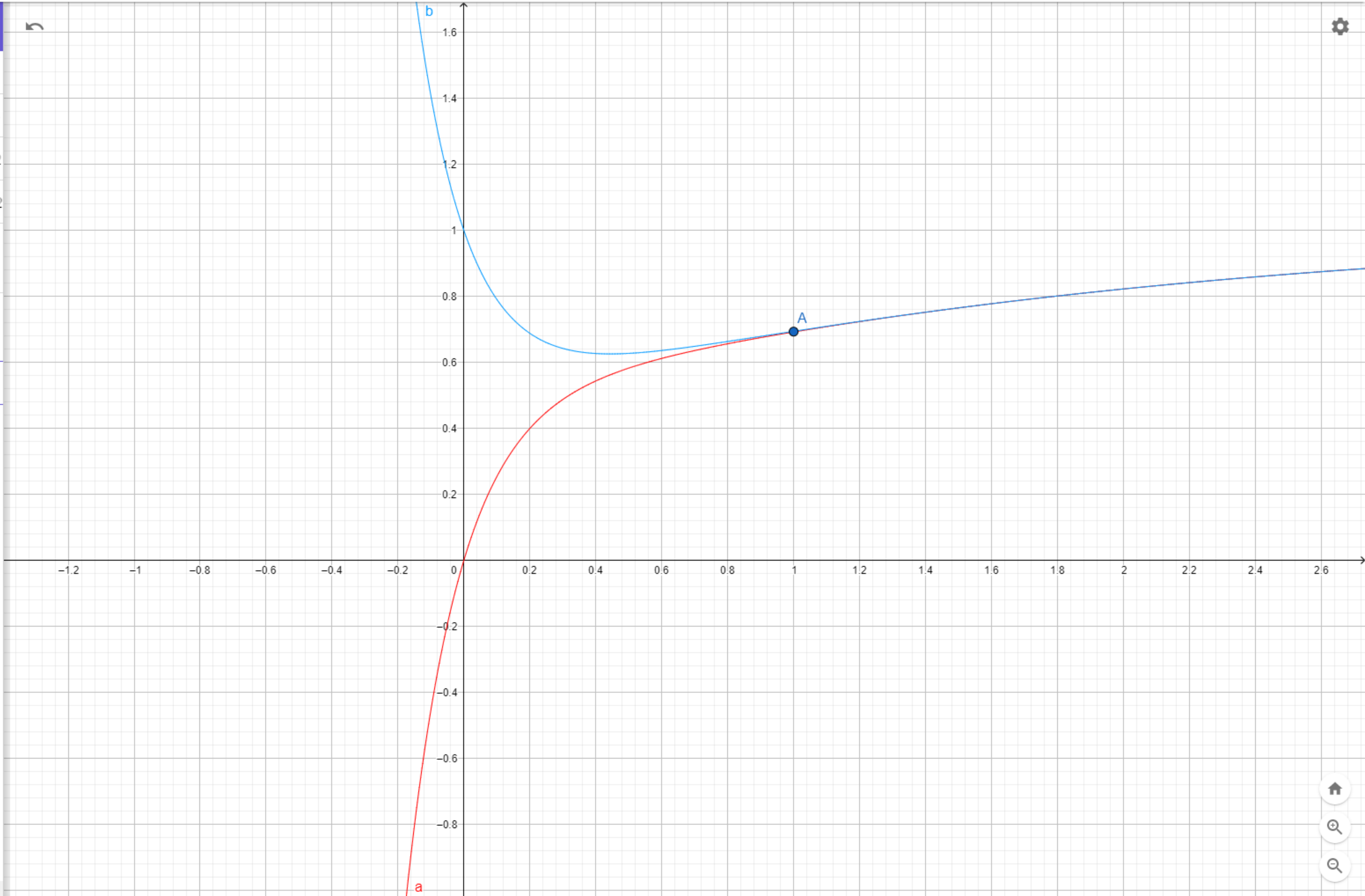
A = (1, ln(2))

→ (1, 0.6931471805599)

B = (0.5, ln(2))

→ (0.5, 0.6931471805599)

Input...



GeoGebra Graphing Calculator SIGN IN

a(x) = (-1)^(1 + 1) / 1^x + (-1)^(2 +

b(x) = (-1)^(1 + 1) / 1^x + (-1)^(2 +

c(x) = (-1)^(1 + 1) / 1^(2x) + (-1)^(2

d(x) = (-1)^(1 + 1) / 1^(2x) + (-1)^(2

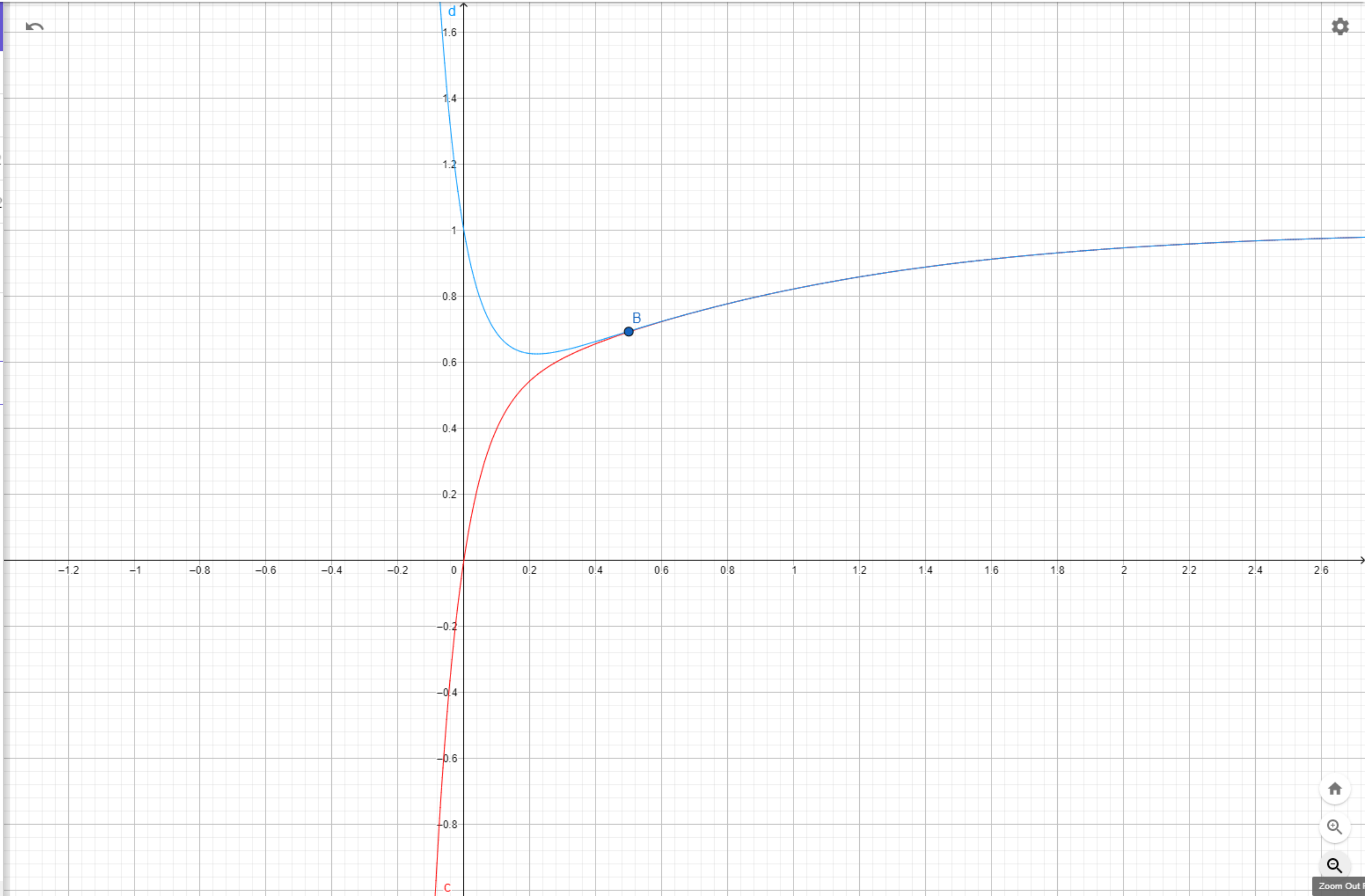
A = (1, ln(2))

→ (1, 0.6931471805599)

B = (0.5, ln(2))

→ (0.5, 0.6931471805599)

+ Input...



GeoGebra Graphing Calculator

SIGN IN

a(x) = (-1)^(1 + 1) / 1^x + (-1)^(2 +

b(x) = (-1)^(1 + 1) / 1^x + (-1)^(2 +

c(x) = (-1)^(1 + 1) / 1^(2x) + (-1)^(2

d(x) = (-1)^(1 + 1) / 1^(2x) + (-1)^(2

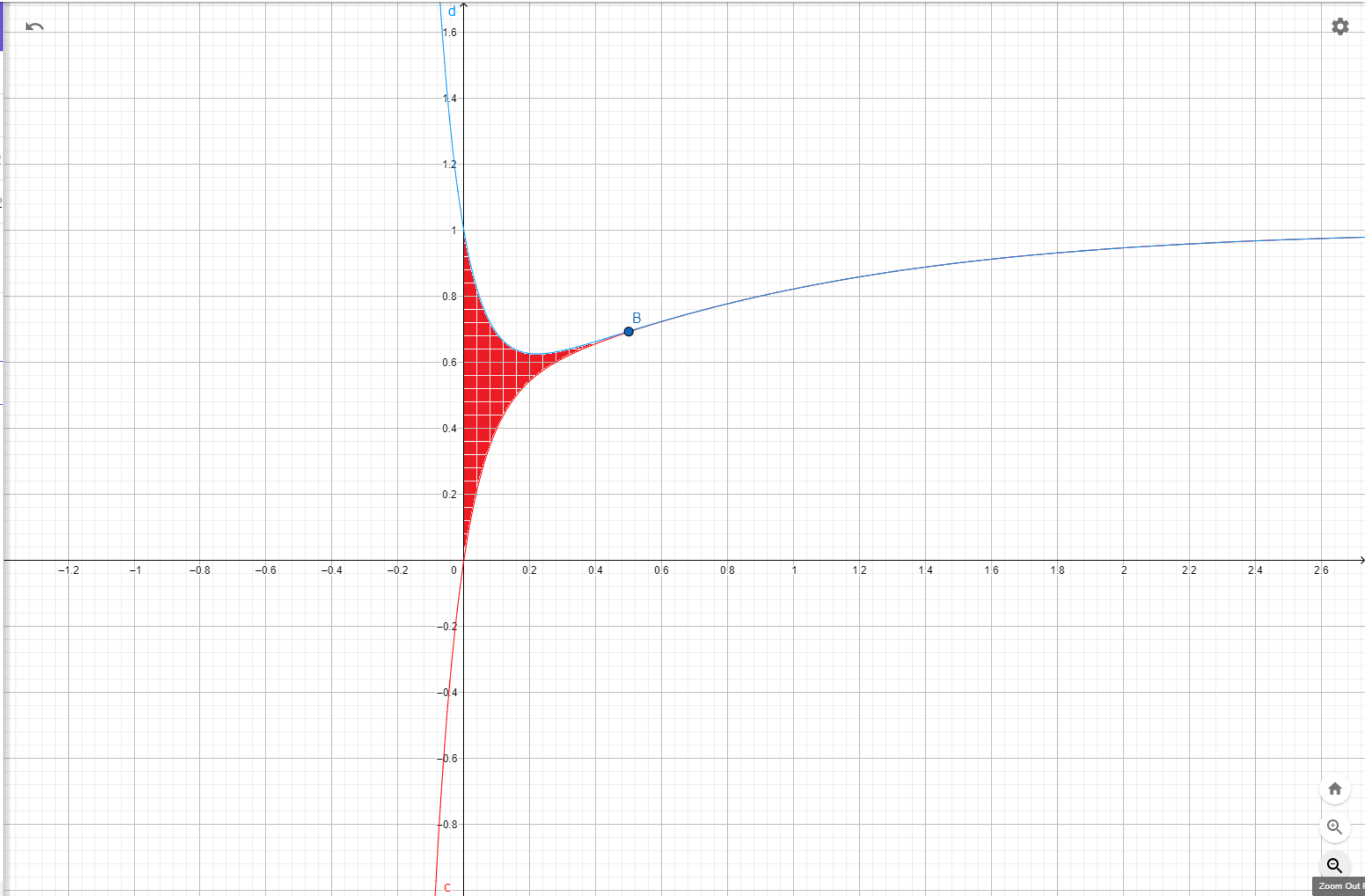
A = (1, ln(2))

→ (1, 0.6931471805599)

B = (0.5, ln(2))

→ (0.5, 0.6931471805599)

\+ Input...



(This time I am using something that already been proven) (Functional equation)

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

Because in the range $0 < a < 0.5$ the function has no zeros that means that in the range $0.5 < a < 1$ there are no zeros as well! (Functional equation)

Case #3

when a=0.5 the function $\frac{\eta(2a)}{\left(1-\frac{2}{2^{2a}}\right)} = \zeta(1)$ is devergent **but** the eta value is fixed

making the devergent part only to reflect the going to infinity part in the formula

$$\lim_{M\to\infty}\sum_{n=2}^{M}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x\ln(n/k))}{(nk)^a} \le \frac{1}{1^{2a}} + \frac{1}{2^{2a}} + \frac{1}{3^{2a}} + ... + \frac{1}{M^{2a}} = \frac{\eta(2a)}{\left(1-\frac{2}{2^{2a}}\right)}$$ (the dividing by 0 part is for illustration purposes only)

$$\lim_{M\to\infty}\sum_{n=2}^{M}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x_0\ln(n/k))}{(nk)^{1/2}} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{M} = \zeta(1)$$