Comments on: A new additive decomposition of velocity gradient, by B. Sun [Phys. Fluids 31, 061702 (2019)]

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Abstract

Comments on "A new additive decomposition of velocity gradient [Phys. Fluids 31, 061702 (2019)]" is presented.

The Cauchy-Stokes decomposition of the velocity gradient tensor into a symmetric strain rate tensor $\mathbf{D}$ and an anti-symmetric spin tensor $\mathbf{W}$ is well-known (Aris, 1962; Tennekes and Lumley, 1972; Kundu and Cohen, 2002).

$$\nabla\mathbf{u} = \mathbf{D} + \mathbf{W} \tag{1}$$

The spin tensor $\mathbf{W}$ is the tensor representation of the vorticity $2\boldsymbol{\omega}$ in the three dimensional physical vector space, where $\mathbf{W}_{ij} = -\varepsilon_{ijk}\omega_k$ (ε_{ijk} is the permutation tensor). Coope et al. (1965) and Coope and Snider (1970) noted that a general second-rank tensor, like $\nabla\mathbf{u}$, can be decomposed to three unique, *irreducible* second-rank tensors of various weights

$$\nabla\mathbf{u} = \mathbf{D_0} + \frac{1}{3}\mathrm{Tr}\left(\nabla\mathbf{u}\right)\mathbf{U} + \mathbf{W} \tag{2}$$

where $\mathrm{Tr}(\nabla\mathbf{u})$ is the trace of the velocity gradient tensor and $\mathbf{D_0}$ is a symmetric, traceless (called *natural*) second-rank tensor of weight two. $\mathbf{W}$ is an second-rank tensor of weight one, whereas $\mathbf{U}$ is the zeroth-weight, second-rank isotropic tensor. $\mathbf{D_0}$, $\mathbf{W}$ and $\mathbf{U}$ are all irreducible second-rank tensors under the three-dimensional rotational group. The weight of an *irreducible* tensor is k if its dimension is $2k+1$. Dimensions of $\mathbf{D_0}$, $\mathbf{W}$ and $\mathbf{U}$ are 5, 3 and 1 respectively when represented via, say, an orthogonal basis in a three dimensional vector space (roughly speaking, the number of independent components is the dimension of an irreducible, k-weight, second-rank tensor in a three-dimensional vector space). Any other possible decompositions are necessarily reducible. Undoubtedly, eq.(2) was known before the works of Coope et al. (1965) and Coope and Snider (1970), but they are significant in the sense that it provides a general algorithm to find such irreducible decompositions of any arbitrary ranked cartesian tensor in three dimensional vector space, and under the three-dimensional group of rotations.

One such reason of interest in decompositions of $\nabla\mathbf{u}$ is the need to identify vortices in fluid flows. In the quest to find characteristics that define a vortex, the vorticity field has been found lacking due to a variety of reasons (Epps, 2017). An interesting, alternative proposition of a novel decomposition of the velocity gradient tensor is presented by Sun (2019) based on the Lie algebra of the special orthonormal Lie group $SO(3)$. This decomposes the velocity gradient tensor into a component which is a rotation tensor instead of the usual spin tensor. As a sidenote, Kundu and Cohen (2002) could be a potential source of confusion as $\mathbf{W}$ is called the rotation tensor in the book, whereas here it is called the spin tensor (technically it is the rate-of-rotation tensor). Sun (2019) had noted that a deeper significance of this decomposition is not yet clear and further investigations are necessary in that direction. The comments here are intended to interpret and rectify some of the aspects of Sun (2019).

Sun (2019) decomposes the velocity gradient tensor as,

$$\nabla \mathbf{u} = \mathbf{K} + \mathbf{Q} \tag{3}$$

where $\mathbf{Q} \in SO(3)$ is a rotation tensor and $\mathbf{K}$ is the residual. It has to be noted that this decomposition is not irreducible under $SO(3)$. Anti-symmetric tensors like $\mathbf{W}$ belong to the Lie algebra $so(3)$ of the Lie group $SO(3)$. There exists an exponential map from $so(3) \to SO(3)$. Exploiting this, Sun (2019) expresses a rotation tensor $\mathbf{Q} \in SO(3)$ as,

$$\mathbf{Q} = e^{\mathbf{W}} \tag{4}$$

First, Sun (2019) does not address the issue of dimensional inconsistency in eq.(4). Physical dimension of $\mathbf{W}$ is sec^{-1} - there are obvious problems and one cannot exponentiate a dimensional quantity. It is unclear if all the physical quantities in Sun (2019) are non-dimensional. Second, presuming that $\nabla \mathbf{u}, \mathbf{D}$ and $\mathbf{W}$ are non-dimensional right from the outset, there is a more basic problem with eq.(4). $\mathbf{Q}$ in eq.(4) does not represent a *one-parameter subgroup* of $SO(3)$ in the neighborhood of $\mathbf{I}$ (Fegan, 1991; Hall, 2015), where $\mathbf{I}$ is identity element of $SO(3)$. This is an essential requirement for an isomorphism from $so(3)$ to $SO(3)$ in the neighborhood of identity. The second objection is fundamental because fixing it will seamlessly fix the first objection and not vice-versa. In other words, even if all the quantities were dimensionless, eq.(4) would still not represent a one-parameter subgroup of $SO(3)$. Equation(4) needs rectification. Due to eq.(4), there are discrepancies in the inferences that can be drawn from Sun's exposition. For example, a question raised by Sun (2019): under what condition(s) $\mathbf{K}$ is symmetric? To further his arguments, $\mathbf{K}$ can be symmetric, at best, for vortical flows with vanishing vorticity ($\omega \to 0$). It will, however, be shown here that it is impossible for $\mathbf{K}$ to be symmetric in a flow with vorticity, if $\mathbf{Q}$ is a one-parameter subgroup of $SO(3)$. Such inconsistencies occur due to the disregard of this fundamental property of the mapping from Lie algebras to the corresponding Lie groups. A natural justification for the consideration of a one-parameter subgroups is provided in what follows along with the implications and limitations of this decomposition.

A Lie group, such as the $SO(3)$, has the structure of a differentiable manifold in the vector space of real matrices. On any integral curve induced by the tangent tensor field like $\mathbf{W}$ on $SO(3)$, the following hold (Hall, 2015) in the neighborhood of $\mathbf{I}$,

$$\frac{d\sigma(\tau)}{d\tau} = \mathbf{W} \tag{5}$$

where τ is the parameter in the map $\sigma : I_R \to SO(3)$, with $\tau \in I_R = [a,b] \in \mathbb{R}$ and I_R contains 0 ($a, b \in \mathbb{R}$). τ might be interpreted as the time increment/decrement, $t - t_0$, where $\sigma(0) = \mathbf{I}$ for some reference time t_0. Along the integral curve $\sigma(\tau) \in SO(3)$, eq.(5) demands,

$$\sigma(\tau) = \mathbf{Q}(\tau) = e^{\mathbf{W}\tau} \tag{6}$$

Equation(6) would describe a family of rotations parameterised by τ: a one-parameter subgroup of $SO(3)$. Equation(6) can also be derived by a much simpler consideration of the orthonormal, rotation tensor $\mathbf{Q}(\tau)$. Time derivative of $\mathbf{Q}\mathbf{Q}^T$ $(= \mathbf{I})$ is

$$\frac{d(\mathbf{Q}\mathbf{Q}^T)}{d\tau} = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T = \dot{\mathbf{I}} = \mathbf{0} \tag{7}$$

where $\dot{\mathbf{Q}}, \dot{\mathbf{I}}$ denote the time derivatives of $\mathbf{Q}$ and $\mathbf{I}$ respectively, and $\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$ (with $\mathbf{Q}^T = \mathbf{Q}^{-1}$). From eq.(7), it is obvious that $\dot{\mathbf{Q}}\mathbf{Q}^T$ is anti-symmetric. Thus for any $\mathbf{Q}$, there always exists a $\dot{\mathbf{Q}}$ such that,

$$\dot{\mathbf{Q}} = \mathbf{W}\mathbf{Q} \tag{8}$$

This tensorial differential equation is equivalent to eq.(5), and the following satisfies eq.(8),

$$\mathbf{Q}(\tau) = e^{\mathbf{W}\tau}\mathbf{Q}_0 \tag{9}$$

Consider the integral curve in $SO(3)$ through the identity with $\mathbf{Q}_0 = \mathbf{I}$, thereby reducing eq.(9) to eq.(6), reiterating the fact that $\mathbf{Q}(\tau)$ is a one-parameter sub-group of $SO(3)$ near $\mathbf{I}$. This is mathematically and dimensionally a more consistent and correct exponential map from $so(3) \to SO(3)$ than eq.(4). If $\mathbf{W}$ is independent of time, there are no restrictions on τ in eq.(6), and eq.(4) is recovered only for a special case of $\tau = 1$. But, in a generic fluid flow field, $\mathbf{W}$ must be a function of time for a material fluid parcel. Therefore, this limits the validity of eq.(6) to $|\tau| \to 0$.

The Rodrigues' formula used by Sun (2019) (eq.16 of the paper) is valid only for $\tau = 1$. Based on this modified exponential map in eq.(6), the complete Rodrigues' formula for $\mathbf{Q}$ is,

$$\mathbf{Q} = \mathbf{I} + \frac{\sin\omega\tau}{\omega}\mathbf{W} + \frac{1-\cos\omega\tau}{\omega^2}\mathbf{W}^2$$

And, if decomposition (3) for a non-dimensional $\nabla\mathbf{u}$ is demanded such that $\mathbf{K}$ is symmetric, then the following must hold,

$$\left(1 - \frac{\sin\omega\tau}{\omega}\right)\mathbf{W} = \mathbf{0} \tag{10}$$

Equation(10) can be satisfied for any allowable τ ($|\tau| \to 0$), if and only if $\mathbf{W} = \mathbf{0}$ identically. Thus, $\mathbf{K}$ can never be symmetric in a vortical flow. This is in distinction to the possibility of a symmetric $\mathbf{K}$ from Sun's exposition, where symmetric $\mathbf{K}$ is allowable for vortical flows with $\omega \to 0$. It is clear that incorrect use of the transformation from the Lie algebra to Lie group, $so(3) \to SO(3)$ (eq.(4)) is the source of such discrepancies.

For a turbulent flow, as mentioned earlier, $\mathbf{W}$ would have erratic dependence on time, and the exponential map would be valid just for infinitesimal time durations, i.e., $|\tau| \to 0$. In that limit, $\mathbf{Q} = \mathbf{I}$ and $\mathbf{K} = \mathbf{D} + \mathbf{W} - \mathbf{I}$, severely restricting the applicability of this new and not irreducible decomposition of the velocity gradient tensor.

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