On Adiabatic Invariance and Newton's Constant

Ervin Goldfain

Abstract

We point out that Newton's constant may be interpreted as adiabatic invariant of metric oscillations in the far-field approximation of General Relativity.

Key words: Newtonian gravitation, Newton's constant, adiabatic invariance, far-field approximation, General Relativity

This note is a short sequel to [1], where it is shown that three of the four universal constants of theoretical physics ($\hbar, c, k_B$) correspond to the regime of self-similarity and adiabatic invariance of Hamiltonian dynamics. Pursuing the same line of inquiry, we highlight here that Newton's constant (G_N) is tied to the adiabatic invariance of metric oscillations in the approximation of weak gravitational fields.

We begin by recalling the issue of conserved quantities in General Relativity. The geodesic equation describing the motion of a particle of unit mass ($m=1$) in a gravitational field is given by [2-3]

$$\frac{dp_\mu}{d\tau} = \frac{1}{2} g_{\nu\rho,\mu} p^\nu p^\rho \tag{1}$$

If all components $g_{\rho\nu}$ are independent of coordinates x^μ for some fixed index μ, then the four-momentum p_μ is a constant along the trajectory of the particle. However, this condition is no longer true in a *general* gravitational field. This amounts to stating that a gravitational field cannot be stationary in any frame of reference and no conserved energy can be defined. The underlying reason is that a general metric has *ten* independent components (as for any 4 x 4 symmetric matrix), while a coordinate change involves only *four* degrees of freedom defined by functions of the form $x^{\rho}(x^{\mu})$. Conservation of four-momentum is fully restored in the local approximation of flat spacetime (tangent manifold) [2].

The key assumption of this note is that the asymptotic transition from curved to Minkowski spacetime involves inherent *metric oscillations* whose frequency and intensity scale inversely proportional with the local radius of curvature.

In the framework of linearized gravity, metric deviations are characterized by the nearly Lorentz coordinates

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \ , \quad \left| h_{\mu\nu} \right| << 1 \tag{2}$$

In particular, the far-field temporal component of (2) reads ($c = 1$) [2-3]

$$h_{00} = g_{00} - 1 = -\frac{2G_N M}{R} \tag{3}$$

where M is the mass of the gravitational source and R is the radial distance away from the source. Relation (3) fits equally well the approximate description of a spherical gravitational wave at far distances from the source, whose evolving radius of curvature in natural units is $R = t$.

A sensible "first-order" assumption is that metric oscillations are harmonic with nearly constant angular frequency $\omega_0(t) = h_{00}(t)$. Hence, one obtains the equation

$$\frac{d^2 h_{00}}{dt^2} + \omega_0^2 h_{00} = \frac{d^2 h_{00}}{dt^2} + h_{00}^3 = 0 \tag{4}$$

which represents the dynamics of a nonlinear oscillator with cubic interaction.

Appealing to the adiabatic invariance of a harmonic oscillator under a slowly varying perturbation produces a constant average action, as in [1, 4-6]

$$\frac{d\langle S\rangle}{dt} = 0 \tag{5a}$$

where, by (3),

$$\langle S\rangle = \langle h_{00} \cdot R\rangle = \langle h_{00} \cdot t\rangle \tag{5b}$$

Finally, using a dimensional representation in which

$$M = G_N = 1\ ,\ \ [G_N] = M^{-2} \tag{6}$$

and on account of (3) and (5), enables identification of Newton's constant with the *adiabatic invariant* of (4), namely,

$$\boxed{\frac{d\langle G_N\rangle}{dt} = 0 \Rightarrow |\langle G_N\rangle| = \frac{1}{2}\langle h_{00} \cdot t\rangle = \text{ const.}} \tag{7}$$

In closing, we recall that (3) generates the inverse square law of Newtonian gravitation, which, in turn, is rooted in the rotational symmetry of three-dimensional space [7]. It follows that the isotropy of classical space is ultimately responsible for the emergence of (7).

References

1. Available at the following site: https://www.researchgate.net/publication/342183452_Universal_Constants_from_Self-similarity_and_Adiabatic_Invariance

2. Schutz, B. F., “A first course in General Relativity”, Cambridge Univ. Press, 2003.

3. Landau L., Lifshitz E., “Classical Theory of Fields”, Butterworth-Heinemann, 1987.

4. https://uspas.fnal.gov/materials/02Yale/AdiabaticInv.pdf

5. Available at the following site: https://galileoandeinstein.phys.virginia.edu/7010/CM_13_Adiabatic_Invt_Actn_Angl.pdf

6. https://uspas.fnal.gov/materials/02Yale/AdiabaticInv.pdf

7. see for example, Zee, A. “Quantum Field Theory in a Nutshell”, Princeton Univ. Press, 2003.