EXTENSIONS OF SOME TRIGONOMETRIC DOUBLE ANGLE AND PRODUCT FORMULAE

Suaib Lateef

A student of Ekiti State University in affiliation with Emmanuel Alayande College of Education, Oyo, Oyo State, Nigeria.

ABSTRACT: In this paper, proofs of extensions of some Trigonometric double angle and Product formulae involving sine and cosine functions are presented.

Keywords: *Trigonometric double angle formulae, Trigonometric Product formulae, binomial expansion.*

1. INTRODUCTION

The main objective of this paper is to extend the following Trigonometric double angle and Trigonometric Product formulae:

(1.1) $$2\mathrm{Sin}x\mathrm{Cos}x = \mathrm{Sin}2x$$

(1.2) $$2\mathrm{Cos}^2x = 1 + \mathrm{Cos}2x$$

(1.3) $$\mathrm{SinP} - \mathrm{SinQ} = 2\mathrm{Cos}\left(\frac{P+Q}{2}\right)\mathrm{Sin}\left(\frac{P-Q}{2}\right)$$

(1.4) $$\mathrm{CosP} + \mathrm{CosQ} = 2\mathrm{Cos}\left(\frac{P+Q}{2}\right)\mathrm{Cos}\left(\frac{P-Q}{2}\right)$$

2. EXTENSIONS

(1.1) can be extended as follow:

(2.1) $$2^n Cos^n ax Sin(an+m)x = \sum_{k=0}^{n}\binom{n}{k} Sin(2ak+m)x$$

(1.2) can be extended as follow:

(2.2) $$2^n Cos^n ax Cos(an+m)x = \sum_{k=0}^{n}\binom{n}{k} Cos(2ak+m)x$$

(1.3) can be extended as follow:

(2.3) $$2^n Cos^n\left(\frac{P+Q}{2}\right) Sin\left(\frac{n(P-Q)}{2}\right) = \sum_{k=0}^{n}\binom{n}{k} Sin((P+Q)k-nQ)$$

(1.4) can be extended as follow:

(2.4) $$2^n Cos^n\left(\frac{P+Q}{2}\right) Cos\left(\frac{n(P-Q)}{2}\right) = \sum_{k=0}^{n}\binom{n}{k} Cos((P+Q)k-nQ)$$

3. PROOFS

To proof (2.1) and (2.2), we start by noting that,

(3.1) $$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax}))^n = (e^{\left(ai-ai+i\frac{m}{n}\right)x}+e^{\left(ai+ai+i\frac{m}{n}\right)x})^n$$

(3.2) $$= (e^{\left(\frac{m}{n}\right)ix}+e^{\left(2a+\frac{m}{n}\right)ix})^n$$

Expanding (3.2) using binomial expansion, we see that,

$$(e^{\left(\frac{m}{n}\right)ix}+e^{\left(2a+\frac{m}{n}\right)ix})^n = \binom{n}{0}.(e^{n\left(\frac{m}{n}\right)ix}) + \binom{n}{1}.(e^{(n-1)\left(\frac{m}{n}\right)ix}).e^{\left(2a+\frac{m}{n}\right)ix} + \binom{n}{2}.(e^{(n-2)\left(\frac{m}{n}\right)ix}).e^{2\left(2a+\frac{m}{n}\right)ix} + \cdots + \binom{n}{n}.e^{n\left(2a+\frac{m}{n}\right)ix}$$

$$= \binom{n}{0}.(e^{imx}) + \binom{n}{1}.(e^{\left(2ai+\left(\frac{im}{n}\right)+\left(\frac{inm}{n}\right)-\left(\frac{im}{n}\right)\right)x}) + \binom{n}{2}.(e^{\left(4ai+\left(\frac{2im}{n}\right)+\left(\frac{inm}{n}\right)-\left(\frac{2im}{n}\right)\right)x}) + \cdots + \binom{n}{n}.e^{(2an+m)ix}$$

(3.3) $$= \binom{n}{0}.(e^{imx}) + \binom{n}{1}.(e^{(2a+m)ix}) + \binom{n}{2}.(e^{(4a+m)ix}) + \cdots + \binom{n}{n}.e^{(2an+m)ix}$$

We can see from (3.3) that,

(3.4) $$\binom{n}{0}.(e^{imx}) + \binom{n}{1}.(e^{(2a+m)ix}) + \binom{n}{2}.(e^{(4a+m)ix}) + \cdots + \binom{n}{n}.e^{(2an+m)ix} = \sum_{k=0}^{n}\binom{n}{k}e^{(2ak+m)ix}$$

Now, equating (3.1) and (3.4), we get,

(3.5) $$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax}))^n = \sum_{k=0}^{n}\binom{n}{k}e^{(2ak+m)ix}$$

From (3.5), we see that,

$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax}))^n = (2e^{\left(a+\frac{m}{n}\right)ix}(\frac{e^{iax}+e^{-iax}}{2}))^n$$

We know that,

$$\mathrm{Cos(a)x} = \left(\frac{e^{iax}+e^{-iax}}{2}\right)$$

So,

$$\left(e^{\left(a+\frac{m}{n}\right)ix}.\left(e^{-iax}+e^{iax}\right)\right)^n = 2^n e^{n\left(a+\frac{m}{n}\right)ix}\cos^n ax$$
$$= 2^n e^{(an+m)ix}\cos^n ax$$

We know that,

$$e^{(an+m)ix} = Cos(an+m)x + iSin(an+m)x$$

Therefore,

$$\left(e^{\left(a+\frac{m}{n}\right)ix}.\left(e^{-iax}+e^{iax}\right)\right)^n = 2^n(Cos(an+m)x + iSin(an+m)x)\mathrm{Cos}^n ax$$

(3.6) $$= 2^n\mathrm{Cos}^n axCos(an+m)x + i(2^n\mathrm{Cos}^n axSin(an+m)x)$$

Also from (3.5), we see that,

$$\sum_{k=0}^{n}\binom{n}{k}e^{(2ak+m)ix} = \sum_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x + iSin(2ak+m)x)$$

(3.7) $$= \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x + i\sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

Since (3.6) is equal to (3.7), we have,

(3.8) $$2^n\mathrm{Cos}^n axCos(an+m)x + i(2^n\mathrm{Cos}^n axSin(an+m)x) = \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x + i\sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

Equating the real and imaginary parts of (3.8), we see that,

(3.9) $$2^nCos^n axSin(an+m)x = \sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

This completes the proof of (2.1).

(3.10) $$2^nCos^n axCos(an+m)x = \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x$$

This completes the proof of (2.2).

If we set $m = np - an$ and $x = 1$ in (3.9)and (3.10), $we\ see\ that$,

(3.11) $$2^nCos^n axSin(np) = \sum_{k=0}^{n}\binom{n}{k}Sin((2k-n)a+np)$$

(3.12) $$2^nCos^n axCos(np) = \sum_{k=0}^{n}\binom{n}{k}Cos((2k-n)a+np)$$

If we set $a = \left(\frac{P+Q}{2}\right)$, $p = \left(\frac{P-Q}{2}\right)$ in (3.11), $we\ see\ that$,

$$2^nCos^n\left(\frac{P+Q}{2}\right)Sin\left(\frac{n(P-Q)}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}Sin((2k-n)\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right))$$
$$= \sum_{k=0}^{n}\binom{n}{k}Sin((2k)\left(\frac{P+Q}{2}\right)-n\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right))$$
$$= \sum_{k=0}^{n}\binom{n}{k}Sin((2k)\left(\frac{P+Q}{2}\right)+n\left(\frac{-p-Q+P-Q}{2}\right))$$
$$2^nCos^n\left(\frac{P+Q}{2}\right)Sin\left(\frac{n(P-Q)}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}Sin((P+Q)k-nQ)$$

This completes the proof of (2.3).

Also, if we set $a = \left(\frac{P+Q}{2}\right)$, $p = \left(\frac{P-Q}{2}\right)$ in (3.12), $we\ see\ that$,

$$2^nCos^n\left(\frac{P+Q}{2}\right)Cos\left(\frac{n(P-Q)}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}Cos((2k-n)\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right))$$
$$= \sum_{k=0}^{n}\binom{n}{k}Cos((2k)\left(\frac{P+Q}{2}\right)-n\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right))$$
$$= \sum_{k=0}^{n}\binom{n}{k}Cos((2k)\left(\frac{P+Q}{2}\right)+n\left(\frac{-p-Q+P-Q}{2}\right))$$
$$2^nCos^n\left(\frac{P+Q}{2}\right)Cos\left(\frac{n(P-Q)}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}Cos((P+Q)k-nQ)$$

This completes the proof of (2.4).

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AUTHOR'S BIOGRAPHY

Suaib Lateef is a student of Ekiti State University in affiliation with Emmanuel Alayande College of Education, Oyo, Oyo State, Nigeria. His discipline is computer science but he has an immense passion for mathematics.
Phone Number: +2349032779723.

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