EXTENSIONS OF SOME TRIGONOMETRIC DOUBLE ANGLE AND PRODUCT FORMULAE

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ABSTRACT: In this paper, proofs of extensions of some Trigonometric double angle and Product formulae involving sine and cosine functions are presented.

Keywords: *Trigonometric double angle formulae, Trigonometric Product formulae, binomial expansion.*

1. INTRODUCTION

The main objective of this paper is to extend the following Trigonometric double angle and Trigonometric Product formulae:

(1.1) $$2\text{SinxCosx} = \text{Sin2x}$$

(1.2) $$2\text{Cos}^2\text{x} = 1 + \text{Cos2x}$$

(1.3) $$\text{SinP} - \text{SinQ} = 2\text{Cos}\left(\frac{\text{P+Q}}{2}\right)\text{Sin}\left(\frac{\text{P−Q}}{2}\right)$$

(1.4) $$\text{CosP} + \text{CosQ} = 2\text{Cos}\left(\frac{\text{P+Q}}{2}\right)\text{Cos}\left(\frac{\text{P−Q}}{2}\right)$$

2. EXTENSIONS

(1.1) can be extended as follow:

(2.1) $$2^n Cos^n ax Sin(an+m)x = \sum_{k=0}^{n}\binom{n}{k} Sin(2ak+m)x$$

(1.2) can be extended as follow:

(2.2) $$2^n Cos^n ax Cos(an+m)x = \sum_{k=0}^{n}\binom{n}{k} Cos(2ak+m)x$$

(1.3) can be extended as follow:

(2.3) $$2^n Cos^n\left(\frac{P+Q}{2}\right) Sin\left(\frac{n(P-Q)}{2}\right) = \sum_{k=0}^{n}\binom{n}{k} Sin((P+Q)k - nQ)$$

(1.4) can be extended as follow:

(2.4) $$2^n Cos^n\left(\frac{P+Q}{2}\right) Cos\left(\frac{n(P-Q)}{2}\right) = \sum_{k=0}^{n}\binom{n}{k} Cos((P+Q)k - nQ)$$

3. PROOFS

To proof (2.1) and (2.2), we know that,

(3.1) $$(p+q)^n = \sum_{k=0}^{n}\binom{n}{k} p^{n-k} q^k$$

If we let $p = e^{\left(\frac{m}{n}\right)ix}$, $q = e^{\left(2a+\frac{m}{n}\right)ix}$, we can see from (3.1) that,

$$\left(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix}\right)^n = \sum_{k=0}^{n}\binom{n}{k} e^{\left(\frac{m}{n}\right)(n-k)ix} . e^{\left(2a+\frac{m}{n}\right)ikx}$$

$$\left(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix}\right)^n = \sum_{k=0}^{n}\binom{n}{k} e^{\left(m-\left(\frac{m}{n}\right)k+2ak+\left(\frac{m}{n}\right)k\right)ix}$$

(3.2) $$\left(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix}\right)^n = \sum_{k=0}^{n}\binom{n}{k} e^{(2ak+m)ix}$$

We can see from (3.2) that,

$$\left(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix}\right)^n = \left(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax} + e^{iax})\right)^n$$

Also. we can see from (3.2) that,

$$\sum_{k=0}^{n}\binom{n}{k} e^{(2ak+m)ix} = \sum_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x + iSin(2ak+m)x)$$

So, from (3.2), we see that,

$$\left(e^{\left(a+\frac{m}{n}\right)ix}(e^{-iax} + e^{iax})\right)^n = \sum_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x + iSin(2ak+m)x)$$

$$\left(2e^{\left(a+\frac{m}{n}\right)ix}\left(\frac{e^{iax}+e^{-iax}}{2}\right)\right)^{n} = \sum_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x\ +\ iSin(2ak+m)x)$$

(3.3) $$2^{n}e^{(an+m)ix}\left(\frac{e^{iax}+e^{-iax}}{2}\right)^{n} = \sum_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x\ +\ iSin(2ak+m)x)$$

We know that,

$$\left(\frac{e^{iax}+e^{-iax}}{2}\right)= \mathrm{Cos}(a)x$$

Also, we know that,

$$e^{(an+m)ix} = Cos(an+m)x + iSin(an+m)x$$

So, from (3.3), we can see that,

$$2^{n}(Cos(an+m)x + iSin(an+m)x)\mathrm{Cos}^{n}ax = \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x + i\sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

(3.4) $$2^{n}\mathrm{Cos}^{n}axCos(an+m)x + i(2^{n}\mathrm{Cos}^{n}axSin(an+m)x) = \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x + i\sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

Equating the real and imaginary parts of (3.4), we see that,

(3.5) $$2^{n}Cos^{n}axSin(an+m)x = \sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

This completes the proof of (2.1).

(3.6) $$2^{n}Cos^{n}axCos(an+m)x = \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x$$

This completes the proof of (2.2).

If we set $m\ =\ np-an\ and\ x\ = 1$ in (3.5)and (3.6), $we\ see\ that,$

(3.7) $$2^{n}Cos^{n}axSin(np) = \sum_{k=0}^{n}\binom{n}{k}Sin((2k-n)a+np)$$

(3.8) $$2^{n}Cos^{n}axCos(np) = \sum_{k=0}^{n}\binom{n}{k}Cos((2k-n)a+np)$$

If we set $a\ =\ \left(\frac{P+Q}{2}\right),\ p\ =\ \left(\frac{P-Q}{2}\right)\ in$ (3.7), $we\ see\ that,$

$$\begin{aligned}2^{n}Cos^{n}\left(\frac{P+Q}{2}\right)Sin\left(\frac{n(P-Q)}{2}\right) &= \sum_{k=0}^{n}\binom{n}{k}Sin((2k-n)\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right))\\ &= \sum_{k=0}^{n}\binom{n}{k}Sin((2k)\left(\frac{P+Q}{2}\right)-n\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right))\\ &= \sum_{k=0}^{n}\binom{n}{k}Sin((2k)\left(\frac{P+Q}{2}\right)+n\left(\frac{-p-Q+P-Q}{2}\right))\\ 2^{n}Cos^{n}\left(\frac{P+Q}{2}\right)Sin\left(\frac{n(P-Q)}{2}\right) &= \sum_{k=0}^{n}\binom{n}{k}Sin((P+Q)k-nQ)\end{aligned}$$

This completes the proof of (2.3).

Also, if we set $a\ =\ \left(\frac{P+Q}{2}\right),\ p\ =\ \left(\frac{P-Q}{2}\right)\ in$ (3.8), $we\ see\ that,$

$$\begin{aligned}2^{n}Cos^{n}\left(\frac{P+Q}{2}\right)Cos\left(\frac{n(P-Q)}{2}\right) &= \sum_{k=0}^{n}\binom{n}{k}Cos((2k-n)\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right))\\ &= \sum_{k=0}^{n}\binom{n}{k}Cos((2k)\left(\frac{P+Q}{2}\right)-n\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right))\\ &= \sum_{k=0}^{n}\binom{n}{k}Cos((2k)\left(\frac{P+Q}{2}\right)+n\left(\frac{-p-Q+P-Q}{2}\right))\\ 2^{n}Cos^{n}\left(\frac{P+Q}{2}\right)Cos\left(\frac{n(P-Q)}{2}\right) &= \sum_{k=0}^{n}\binom{n}{k}Cos((P+Q)k-nQ)\end{aligned}$$

This completes the proof of (2.4).

4. SOME OTHER NEW IDENTITIES

$$2^n Cosh^n(a)xSinh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} Sinh(2ak+m)x$$

$$2^n Cosh^n(a)xCosh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} Cosh(2ak+m)x$$

$$2^n(-1)^{\frac{n}{2}} Sin^n axSin(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Sin(2ak+m)x \qquad (n\ is\ even)$$

$$2^n(-1)^{\frac{n-1}{2}} Sin^n axSin(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Cos(2ak+m)x \qquad (n\ is\ odd)$$

$$2^n(-1)^{\frac{n+1}{2}} Sin^n axCos(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Sin(2ak+m)x \qquad (n\ is\ odd)$$

$$2^n(-1)^{\frac{n}{2}} Sin^n axCos(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Cos(2ak+m)x \qquad (n\ is\ even)$$

$$2^n Sinh^n axSinh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Sinh(2ak+m)x \qquad (n\ is\ even)$$

$$-2^n Sinh^n axSinh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Cosh(2ak+m)x \qquad (n\ is\ odd)$$

$$-2^n Sinh^n axCosh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Sinh(2ak+m)x \qquad (n\ is\ odd)$$

$$2^n Sinh^n axCosh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Cosh(2ak+m)x \qquad (n\ is\ even)$$

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