EXTENSIONS OF SOME TRIGONOMETRIC DOUBLE ANGLE AND PRODUCT FORMULAE

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ABSTRACT: In this paper, proofs of extensions of some Trigonometric double angle and Product formulae involving sine and cosine functions are presented.

Keywords: *Trigonometric double angle formulae, Trigonometric Product formulae, Binomial Triangle, Trinomial triangle, Multinomial Triangle.*

1. INTRODUCTION

The main objective of this paper is to extend the following Trigonometric double angle and Trigonometric Product formulae. However, some conjectures are stated for further research.

(1.1) $$2\text{Sin}x\text{Cos}x = \text{Sin}2x$$

(1.2) $$2\text{Cos}^2x = 1 + \text{Cos}2x$$

(1.3) $$\text{Sin}P - \text{Sin}Q = 2\text{Cos}\left(\frac{P+Q}{2}\right)\text{Sin}\left(\frac{P-Q}{2}\right)$$

(1.4) $$\text{Cos}P + \text{Cos}Q = 2\text{Cos}\left(\frac{P+Q}{2}\right)\text{Cos}\left(\frac{P-Q}{2}\right)$$

2. EXTENSIONS

(1.1) can be extended as follow:

(2.1) $$2^n Cos^n ax Sin(an+m)x = \sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

(1.2) can be extended as follow:

(2.2) $$2^n Cos^n ax Cos(an+m)x = \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x$$

(1.3) can be extended as follow:

(2.3) $$2^n Cos^n\left(\frac{P+Q}{2}\right)Sin\left(\frac{n(P-Q)}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}Sin((P+Q)k-nQ)$$

(1.4) can be extended as follow:

(2.4) $$2^n Cos^n\left(\frac{P+Q}{2}\right)Cos\left(\frac{n(P-Q)}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}Cos((P+Q)k-nQ)$$

3. PROOFS

To proof (2.1) and (2.2), note that,

(3.1) $$(r+t)^n = \sum_{k=0}^{n}\binom{n}{k}r^{n-k}t^k$$

If we let $r = e^{\left(\frac{m}{n}\right)ix}$, $t = e^{\left(2a+\frac{m}{n}\right)ix}$, we can see from (3.1) that,

$$\left(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix}\right)^n = \sum_{k=0}^{n}\binom{n}{k} e^{\left(\frac{m}{n}\right)(n-k)ix}.e^{\left(2a+\frac{m}{n}\right)ikx}$$

$$\left(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix}\right)^n = \sum_{k=0}^{n}\binom{n}{k} e^{\left(m-\left(\frac{m}{n}\right)k+2ak+\left(\frac{m}{n}\right)k\right)ix}$$

(3.2) $$\left(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix}\right)^n = \sum_{k=0}^{n}\binom{n}{k} e^{(2ak+m)ix}$$

We can see from (3.2) that,

$$\left(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix}\right)^n = \left(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax})\right)^n$$

Also. we can see from (3.2) that,

$$\sum_{k=0}^{n}\binom{n}{k} e^{(2ak+m)ix} = \sum_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x + iSin(2ak+m)x)$$

So, from (3.2), we see that,

$$\left(\mathrm{e}^{\left(\mathrm{a}+\frac{\mathrm{m}}{\mathrm{n}}\right)ix}(\mathrm{e}^{-iax}+\mathrm{e}^{iax})\right)^n = \sum_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x \;+\; iSin(2ak+m)x)$$

$$\left(2\mathrm{e}^{\left(\mathrm{a}+\frac{\mathrm{m}}{\mathrm{n}}\right)ix}\left(\frac{\mathrm{e}^{iax}+\mathrm{e}^{-iax}}{2}\right)\right)^n = \sum\nolimits_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x \;+\; iSin(2ak+m)x)$$

(3.3) $$\mathrm{e}^{(\mathrm{an}+\mathrm{m})ix}\left(\frac{\mathrm{e}^{iax}+\mathrm{e}^{-iax}}{2}\right)^{\mathrm{n}} = \sum\nolimits_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x \;+\; iSin(2ak+m)x)$$

Note that,

$$(\frac{\mathrm{e}^{iax}+\mathrm{e}^{-iax}}{2})= \mathrm{Cos(a)x}$$

Also note that,

$$\mathrm{e}^{(\mathrm{an}+\mathrm{m})ix} = Cos(an+m)x + iSin(an+m)x$$

So, from (3.3), we can see that,

$$2^n(Cos(an+m)x + iSin(an+m)x)\mathrm{Cos}^n ax = \sum\nolimits_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x + i\sum\nolimits_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

(3.4) $$2^n\mathrm{Cos}^n axCos(an+m)x + i(2^n\mathrm{Cos}^n axSin(an+m)x) = \sum\nolimits_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x + i\sum\nolimits_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

Equating the real and imaginary parts of (3.4), we see that,

(3.5) $$2^nCos^n axSin(an+m)x = \sum\nolimits_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

This completes the proof of (2.1).

(3.6) $$2^nCos^n axCos(an+m)x = \sum\nolimits_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x$$

This completes the proof of (2.2).

If we set $m \;=\; ny-an \; and \; x \;=1$ in (3.5)and (3.6), $we \; see \; that,$

(3.7) $$2^nCos^n axSin(ny) = \sum\nolimits_{k=0}^{n}\binom{n}{k}Sin((2k-n)a+ny)$$

(3.8) $$2^nCos^n axCos(ny) = \sum\nolimits_{k=0}^{n}\binom{n}{k}Cos((2k-n)a+ny)$$

If we set $a \;=\; \left(\frac{P+Q}{2}\right),\; y \;=\; \left(\frac{P-Q}{2}\right) \; in$ (3.7), $we \; see \; that,$

$$\begin{aligned}
2^nCos^n\left(\frac{P+Q}{2}\right)Sin\left(\frac{n(P-Q)}{2}\right) &= \sum\nolimits_{k=0}^{n}\binom{n}{k}Sin((2k-n)\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right))\\
&= \sum\nolimits_{k=0}^{n}\binom{n}{k}Sin((2k)\left(\frac{P+Q}{2}\right)-n\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right))\\
&= \sum\nolimits_{k=0}^{n}\binom{n}{k}Sin((2k)\left(\frac{P+Q}{2}\right)+n\left(\frac{-p-Q+P-Q}{2}\right))\\
2^nCos^n\left(\frac{P+Q}{2}\right)Sin\left(\frac{n(P-Q)}{2}\right) &= \sum\nolimits_{k=0}^{n}\binom{n}{k}Sin((P+Q)k-nQ)
\end{aligned}$$

This completes the proof of (2.3).

Also, if we set $a \;=\; \left(\frac{P+Q}{2}\right),\; y \;=\; \left(\frac{P-Q}{2}\right) \; in$ (3.8), $we \; see \; that,$

$$\begin{aligned}
2^nCos^n\left(\frac{P+Q}{2}\right)Cos\left(\frac{n(P-Q)}{2}\right) &= \sum\nolimits_{k=0}^{n}\binom{n}{k}Cos((2k-n)\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right))\\
&= \sum\nolimits_{k=0}^{n}\binom{n}{k}Cos((2k)\left(\frac{P+Q}{2}\right)-n\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right))\\
&= \sum\nolimits_{k=0}^{n}\binom{n}{k}Cos((2k)\left(\frac{P+Q}{2}\right)+n\left(\frac{-p-Q+P-Q}{2}\right))\\
2^nCos^n\left(\frac{P+Q}{2}\right)Cos\left(\frac{n(P-Q)}{2}\right) &= \sum\nolimits_{k=0}^{n}\binom{n}{k}Cos((P+Q)k-nQ)
\end{aligned}$$

This completes the proof of (2.4).

4. SOME OTHER NEW IDENTITIES

$$2^nCosh^n(a)xSinh(an+m)x = \sum_{k=0}^{n}\binom{n}{k}Sinh(2ak+m)x$$

$$2^n Cosh^n(a)xCosh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} Cosh(2ak+m)x$$

$$2^n(-1)^{\frac{n}{2}} Sin^n axSin(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Sin(2ak+m)x \qquad (n\ is\ even)$$

$$2^n(-1)^{\frac{n-1}{2}} Sin^n axSin(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Cos(2ak+m)x \qquad (n\ is\ odd)$$

$$2^n(-1)^{\frac{n+1}{2}} Sin^n axCos(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Sin(2ak+m)x \qquad (n\ is\ odd)$$

$$2^n(-1)^{\frac{n}{2}} Sin^n axCos(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Cos(2ak+m)x \qquad (n\ is\ even)$$

$$2^n Sinh^n axSinh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Sinh(2ak+m)x \qquad (n\ is\ even)$$

$$-2^n Sinh^n axSinh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Cosh(2ak+m)x \qquad (n\ is\ odd)$$

$$-2^n Sinh^n axCosh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Sinh(2ak+m)x \qquad (n\ is\ odd)$$

$$2^n Sinh^n axCosh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Cosh(2ak+m)x \qquad (n\ is\ even)$$

5. UNDERSTANDING MULTINOMIAL TRIANGLES

In order to understand Multinomial Triangle, we need to take a quick look at the Binomial Triangle; Binomial Triangle is a triangle formed by arranging the coefficients of the expansion of two variables which is denoted by,

(1.0) $$(a+x)^p = \sum_{k=0}^{p} \binom{n}{k} a^{p-k} x^k$$

We need to know that a and x in (1.0) are the two variables and $\binom{n}{k}$ stands for all the coefficients of $(a+x)^p$ when expanded. For example, the expansion of $(a+x)^3$ gives,

(1.1) $$(a+x)^3 = a^3 + 3a^2x + 3a^2x + x^3$$

We can see that the coefficients of the expansion of $(a+x)^3$ are (1, 3, 3, 1) For a better clarification, we explain the meaning of Binomial Triangle using Fig 1.0 below.

BINOMIAL TRIANGLE

P = 0					0		1		0					Z_1
P = 1				0		1		1		0				Z_1---Z_2
P = 2			0		1		2		1		0			Z_1---Z_3
P = 3		0		1		3		3		1		0		Z_1---Z_4
P = 4	0		1		4		6		4		1		0	Z_1---Z_5

Fig 1.0

From fig 1.0, we can easily generate the coefficients of the expansion of $(a+x)^p$ by adding together two consecutive numbers of the expansion of $(a+x)^{p-1}$. For example, to generate the coefficients of the expansion of $(a+x)^5$, we look at where P = 4 and add the first two numbers which are 0 and 1, which gives us 1. To get the second coefficient, we look at where p = 4 again and add the second and the third numbers which are 1 and 4, which gives us 5. Doing this on and on, we see that the coefficients of the expansion of $(a+x)^5$ are (1, 5, 10, 10, 5, 1). So, we have the next table to be,

P = 0						0		1		0						Z_1
P = 1					0		1		1		0					Z_1---Z_2
P = 2				0		1		2		1		0				Z_1---Z_3
P = 3			0		1		3		3		1		0			Z_1---Z_4
P = 4		0		1		4		6		4		1		0		Z_1---Z_5
P = 5	0		1		5		10		10		5		1		0	Z_1---Z_6

Fig 1.1

Fig 1.1 explains how Binomial Triangle is formed.

Note that all the zeros in all the tables are meant for explaining how the next coefficients are generated, so are ignored when we want to use the coefficients. It means the first coefficient of the expansion of $(x_1+x_2+x_3+...+x_r)^n$ will always be 1 when the coefficients of $x_1, x_2, x_3,...,x_r$ are 1.

TRINOMIAL TRIANGLE

Trinomial Triangle is formed just like that of Binomial Triangle except that three consecutive numbers are added. For quadrinomial, four consecutive numbers are added. Fig 1.3 below is called Trinomial Triangle.

P = 0					0	0	1	0	0					Z_1
P = 1				0	0	1	1	1	0	0				Z_1---Z_3
P = 2			0	0	1	2	3	2	1	0	0			Z_1---Z_5
P = 3		0	0	1	3	6	7	6	3	1	0	0		Z_1---Z_7
P = 4	0	0	1	4	10	16	19	16	10	4	1	0	0	Z_1---Z_9

Fig 1.2

Looking at Fig 1.2, we see that three consecutive numbers are added to give the number immediately below the middle number. For example, the arrows in Fig 1.2 means (0+0+1 = 1), (2+1+0 = 3) and (1+3+6 = 102). In the case of Binomial Triangle, the next coefficient to be generated is placed below the middle of the two consecutive numbers; see the arrows in Fig 1.0 for more clarification.

In essence, the rules that apply to Binomial Triangle also apply to all Multinomial Triangles whose numbers form the coefficients of the expansion of even number of variables, i.e. $(x_1+x_2+x_3+...+x_r)^n$, where r is any positive even number including zero(Mononomial). Also, the rules that apply to Trinomial Triangle also apply to all Multinomial Triangles whose numbers form the coefficients of the expansion of odd number of variables, i.e. $(x_1+x_2+x_3+...+x_r)^n$, where r is any positive odd number. The number of zeros to be added to both sides of the coefficients is (r-1). This means no zero is needed for any Monomial expansion because the expansion will always result in a single term, i.e. $(x)^n=x^n$ and therefore, its coefficient will always be 1 except the coefficient of x is not 1.

WHAT IS Z_n?

The number of coefficients of a particular Multinomial expansion is needed to know the number of terms that the expansion has. Let n be the number of terms of that particular multinomial expansion, then Z_n is the nth coefficient of the expansion. For example, in Fig 1.2 where we have P = 3. We could see that we have

Z_1---Z_7 at the other end. Since we have to ignore all the zeros, Z_1---Z_7 means there are 7 coefficients in a Trinomial Triangle when $(a+b+c)^3$ is expanded and that, $Z_1=1$, $Z_2=3$, $Z_3=6$, $Z_4=7$, $Z_5=6$, $Z_6=3$, and $Z_7=1$.

CONJECTURES

Let n = the number of the coefficients of a Multinomial expansion,

r = number of variables of a Multinomial expansion,

p = power of a Multinomial expansion,

Z_k = kth coefficient of a Multinomial expansion with respect to p and r,

Then

$$\sum_{k=1}^{n} Z_k . cos\,(a + 2(k-1)d) = \left(\frac{sinrd}{sind}\right)^p . cos(a + (r-1)pd) \qquad (r > 2)$$

$$\sum_{k=1}^{n} Z_k . sin(a + 2(k-1)d) = \left(\frac{sinrd}{sind}\right)^p . sin(a + (r-1)pd) \qquad (r > 2)$$

$$\sum_{k=1}^{n} Z_k . cosh(a + 2(k-1)d) = \left(\frac{sinhrd}{sinhd}\right)^p . cosh(a + (r-1)pd) \qquad (r > 2)$$

$$\sum_{k=1}^{n} Z_k . sinh(a + 2(k-1)d) = \left(\frac{sinhrd}{sinhd}\right)^p . sinh(a + (r-1)pd) \qquad (r > 2)$$

PRACTICAL EXAMPLES

Suppose r = 3, p = 3, from Fig 1.2, we see that the coefficients are (1, 3, 6, 7, 6, 3, 1). So, we have,

$$\cos\,(a) + 3cos(a+2d) + 6cos(a+4d) + 7cos(a+6d) + 6\cos(a+8d) + 3cos(a+10d) + \cos(a+12d) = \left(\frac{sin3d}{sind}\right)^3 . cos(a+6d)$$

$$\sin\,(a) + 3sin(a+2d) + 6sin(a+4d) + 7\,sin(a+6d) + 6sin(a+8d) + 3sin(a+10d) + sin(a+12d) = \left(\frac{sin3d}{sind}\right)^3 . sin(a+6d)$$

$$\cosh\,(a) + 3cos\,h(a+2d) + 6cos\,h(a+4d) + 7cos\,h(a+6d) + 6cosh(a+8d) + 3cosh(a+10d) + cosh(a+12d) = \left(\frac{sinh3d}{sinhd}\right)^3 . cosh(a+6d)$$

$$\sinh\,(a) + 3sin\,h(a+2d) + 6sin\,h(a+4d) + 7sinh(a+6d) + 6sinh(a+8d) + 3sin\,h(a+10d) + \sin h(a+12d) = \left(\frac{sinh3d}{sinhd}\right)^3 . sinh(a+6d)$$

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