# Nonlinear Continuum Mechanics with Defects Resembles Electrodynamics - A Comeback of the Aether?

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## Abstract

This article discusses the dynamics of an incompressible, isotropic elastic continuum. Starting from the Lorentz-invariant motion of defects in elastic continua (Frank 1949), MacCullagh's aether theory (1839) of an incompressible elastic solid is reconsidered. Since MacCullagh's theory, based on linear elasticity, cannot describe charges, particular attention is given to a topological defect that causes large deformations and therefore requires a nonlinear description. While such a twist disclination can take the role of a charge, the deformation field of a large number of these defects produces a microstructure of deformation related to a Cosserat continuum (1909). On this microgeometric level, a complete set of quantities can be defined that satisfies equations equivalent to Maxwell's.
**Note added in 2020.** I do not identify any longer with the entire content of this paper originally written in 2005, because the elastic continuum approach has significant difficulties in describing Dirac's large numbers. However, I think the paper contains some valuable thoughts that may stimulate further research.

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# 1 Introduction

There is a long history of elastic solid theories relating to electrodynamics, which Sir Edmund Whittaker elaborately described in his famous treatise (1951). This branch of physics is not focus of curent attention as it is widely held that aether theories and Lorentz invariance are incompatible. To disprove this prejudice, in section 1 I outline that the propagation of topological defects in elastic media even requires a description that is equivalent to the special theory of relativity (SRT), whereby the velocity of sound takes the role of $c$. Due to this little known result, which was obtained first by Frank (1949), aether theories such as the one developed by Irish physicist James MacCullagh should regain due consideration; this is done in section 2. In its classic form, MacCullagh's theory can only describe electrodynamics without charges, and it was discarded, as a result, already in the 19th century. Unfortunately, at that time, neither the knowledge of topological defects (dislocation theory started around 1950) nor on finite continuum mechanics were developed.

Therefore, I discuss the properties of a particular topological defect that can act as a charge and, at the same time, satisfies Lorentz-invariant dynamics. Section 4 raises the question regarding the deformation field of a large number of defects. While still dealing with compatible deformation, the arising microstructure appears to be a concrete example of a Cosserat continuum (Cosserat and F. 1909; Mindlin 1964; Kröner 1980; Hehl 1991). Section 5 and 6 discuss that microstructure in detail. The arising quantities are analogous to those that satisfy the equations of electrodynamics.

Though these remarks are inspired by the similarities to electrodynamics, the skeptical reader is invited to follow the discussion of a variety of effects that follow from 'pure' continuum mechanics.

# 2 Dynamics of an incompressible elastic continuum

## 2.1 Lorentz invariance

We investigate the continuum mechanics of an elastic solid with topological defects. It will be shown[1] that moving topological defects show a Lorentz contraction of their deformation fields.

The continuum is described by two quantities: the displacement[2] vector $\vec{u}$ that points from an undeformed, 'Euclidean' state to the deformed state, and its derivative, the deformation gradient $\mathbf{F}$ (Truesdell and Toupin 1960; Beatty 1987; Unzicker 2000).

The equation of motion in linear elasticity is the Navier equation (Love 1927, p. 293; Whittaker 1951, p. 139, with slight changes of notation)

$$-(\lambda+2\mu)\ grad\ \text{div}\ \vec{u} - \mu\ \text{curl}\ \text{curl}\ \vec{u} = \rho \frac{\partial^2 \vec{u}}{\partial t^2}, \quad (1)$$

where $\mu$ and $\lambda$ are the elastic constants and $\rho$ is the density of the elastic continuum. In linear approximation, incompressibility enforces div $\vec{u} = 0$, therefore, eqn. (1) reduces to

$$-\mu\ \text{curl}\ \text{curl}\ \vec{u} = \rho \frac{\partial^2 \vec{u}}{\partial t^2} \quad (2)$$

Apply now vector analysis $\Delta = \text{grad}\,\text{div} - \text{curl}\,\text{curl}$ and suppose a statically stable topological defect (that satisfies $\mu$ curl curl $\vec{u} = 0$) propagates in $x$-direction with velocity $v$.[3] Even if it causes deformations of arbitrary shape, it will be represented by a time-independent function of $x'$, $y$ and $z$, where $x' = x - vt$. With this substitution and with $\Delta = \text{curl}\,\text{curl}$, $\frac{\partial^2}{\partial t^2}$ becomes $v^2 \frac{\partial^2}{\partial x'^2}$, and the remaining terms of eqn. 1 reads:

$$\mu\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + (\mu - v^2\rho)\frac{\partial^2}{\partial x'^2} = 0 \quad (3)$$

With the further substitution

$$x'' = x'\sqrt{1 - \frac{v^2\rho}{\mu}} = (x - vt)\sqrt{1 - \frac{v^2}{c^2}}, \quad (4)$$

where $c = \sqrt{\mu/\rho}$, the propagating solution is identical to the static solution, apart from the substitution $x \to x''$, the well-known 'Lorentz contraction' by the factor $\sqrt{1 - v^2/c^2}$. The speed of light in the special theory of relativity (SR) corresponds to the velocity of transverse sound in an elastic solid.

[1] Following Frank (1949).

[2] Displacement is just a shift of material elements and must not be confused with the electric displacement $D$.

[3] The example of a screw dislocation is given in Unzicker (2000).

Similarly it can be shown that the elastic energy of a propagating solution increases with the factor $1/\sqrt{1-v^2/c^2}$ (Frank 1949, p. 132). Further details on the one-to-one correspondence to SR can be found in Eshelby (1949), Kröner (1960), Weertman and Weertman (1979), Günther (1988, 1996) and in detail in Unzicker 2000.

Can these relativistic effects, apart from being a curiosity of elasticity theory, have a deeper meaning? The famous experiments by Michelson and Morley seem to have disproved any concept of describing spacetime by continuum mechanics.

At this point I need to emphasize that the physicists of the 19th century imagined particles to be made of an external substance distinct from the 'aether', which, for some reason, can pass through the aether without (or with infinitely little) friction.[4].

In contrast to the above derivation, they never thought of 'particles' as being defects creating a displacement field. This is not astonishing, since the first examples of such defects, dislocations in solids, were discovered in 1934 by Taylor. In view of the results of Frank (1949) and others however, one realizes, *describing spacetime as an elastic continuum was not the wrong approach but a wrong or missing concept of particles moving in it*. Elastic solid theories therefore deserve reconsideration.

## 2.2 Maxwell's equations of empty space in MacCullagh's theory

Among various aether theories, MacCullagh's theory (1839) is particularly interesting. Considering an incompressible elastic solid, he identified $\mu$ curl $\vec{u}$ with the electric and $\rho\frac{\partial\vec{u}}{\partial t}$ with the magnetic field. The electric field would, thus, be related to the rotation of volume elements, and the magnetic field to their velocity.[5]

Then, equation (2) is obviously equivalent to the first Maxwell equation

$$\frac{1}{\epsilon_0}\text{curl }\vec{E} = \mu_0\frac{\partial\vec{H}}{\partial t}, \tag{5}$$

whereby we should bear in mind that linear elasticity is an approximation. With this identification, the second Maxwell equation

$$\text{div}\,\vec{H} = 0 \tag{6}$$

follows directly from the incompressibility condition div $(\rho\vec{u}) = 0$ ($\rho$ is a constant), which implies div $\frac{d\vec{u}}{dt} = 0$.

By definition

$$\text{div}\,\text{curl }\vec{u} = 0 \quad \text{and curl}\,\frac{d}{dt}\vec{u} = \frac{d}{dt}\text{curl }\vec{u} \tag{7}$$

holds, which correspond to Maxwell's second pair of equations *in vacuo*. In MacCullagh's incompressible elastic medium only transverse waves exist. Their dynamics are completely analogous to electromagnetic waves. Though only describing empty space, the r.h.s. of (5) already contains Maxwell's celebrated term $\frac{d\vec{D}}{dt}$.

# 3 The Larmor defect - a source of intrinsic rotational strain

## 3.1 Nonlinear extension of MacCullagh's theory

As Whittaker (1951, p. 287) comments on the preeding theory, 'In the analogy thus constituted, electric displacement corresponds to the twist of the elements of volume of the aether; and electric charge must evidently be represented as an intrinsic rotational strain.' On the other hand, the vector identity div curl $= 0$ seems to make charges impossible.

MacCullagh's theory is based, however, on linear elasticity, which is only an approximation. Nevertheless, one may obtain charges when dealing with large deformations that require a nonlinear treatment. Large deformations occur near topological defects that have shown the above relativistic behavior. In the following, I focus on a nontrivial topological defect I call the Larmor[6] defect.

[4] In 'The Theory of Electrons' (1915) however, Hendrik Antoon Lorentz made the following interesting statement: 'Indeed, one of the most important of our fundamental assumptions must be that the ether not only occupies all space between molecules, atoms or electrons, but that it pervades all these particles. We shall add the hypothesis that, though the particles may move, the ether always remains at rest. We can reconcile ourselves with this, at first sight, somewhat startling idea, by thinking of the particles of matter as of some local modification in the state of the ether. These modifications may of course very well travel onward while the volume-elements of the medium in which they exist remain at rest'

[5] The incompleteness of such a proposal is discussed below.

[6] See description b).

## 3.2 Defect creation

Since a proper understanding of the following is essential, I will risk some redundancy in giving different descriptions a)-d) of the same object that may help visualization. The defect can be produced as follows:

a) Cutting the elastic continuum along a (circular) surface, twisting the two faces against each other by the amount of $2\pi$ and rejoining them again by gluing.

b) On p. 227 of his article, Larmor (1900) described a nearly identical process[7]: 'if breach of continuity is produced across an element of interface in the midst of an incompressible medium endowed with *ordinary material rigidity*, for example by the creation of a lens-shaped cavity, and the material on the one side of the breach is twisted round in its plane, and continuity is then restored by cementing the two sides together..'

Of course, one creates singular deformation gradients at the circular boundary.

c) Imagine $\mathbb{R}^3$ filled with elastic material and remove a solid torus centered at the origin and with $z$ as symmetry axis (fig. 1). Then the complement is doubly connected due to the material nearby the $z$-axis. One now cuts the material along the surface bounded by the inner circle of the torus in the $x-y$-plane (hatched surface in fig. 1). Now the cut faces can be twisted, for example the face of the positive $z$-direction clockwise and the face of the negative one counterclockwise, and glued together again. If each of the twists amounts to $\pi$, the material elements meet their old neighbors again, so to speak, since the total twisting angle is $2\pi$. Returnung to the above description, now let the removed torus shrink to zero, $r \to 0$ (Unzicker 2000).

d) cut out a cylinder from the inside of the elastic material, leave the bottom of the cylinder untwisted, apply a twist of $2\pi$ to the top (see fig. 2) and put the cylinder back into the elastic continuum. Since a twist of $2\pi$ corresponds to identity, the displacement vector $\vec{u}$ is continuous at the top and bottom surface, whereas it is discontinuous on the jacket of the cylinder. Now let the jacket shrink to a circular singularity line and let the surrounding material respond elastically to the imposed deformation.

[7] However, giving another interpretation to this 'nucleus of beknottedness'. See also Whittaker 1951, p. 287.

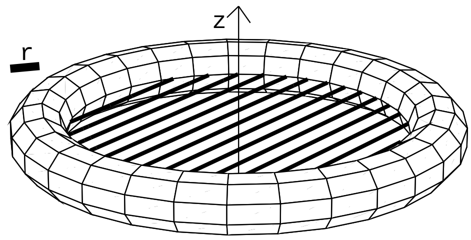


Figure 1: Schematic description of how to produce the Larmor defect in an elastic continuum. The solid torus is removed. Then the material is cut along the hatched surface. After twisting the cut faces by the amount of $2\pi$, the material is rejoined. To obtain a line defect, the solid torus can be shrunk to a singularity line. Note that after the cut, the same material elements are rejoined. Topologically speaking, the dotted line in fig. 1 represents a closed path in $SO(3)$.

The modern terminology[8] used in material science is 'twist disclination loop' with a twisting angle of $2\pi$ (Unzicker and Fabian 2003).

## 3.3 Elementary properties of Larmor defects

**Mirror-symmetric types.** Two versions of the defect exist, distinguished by the orientation of the twists. This becomes ultimately clear when you think of wringing dry a wet towel with your hands. You can do it in either by applying clockwise or counterclockwise torque, regardless of the direction in space of the twisting axis. Thus there are two physically different deformations of the continuum, mirror-symmetric to each other.

**Motion of the defect.** It is important to bear in mind that during the motion of topological defects,

[8] The Larmor defects creates deformations that are locally similar to those of a screw dislocation. The description as 'screw dislocation loop' (Unzicker 1996) is, however, no longer used.

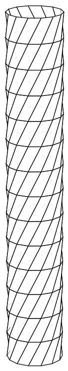

Figure 2: Deformation of a cylindrical rod to which a counterclockwise torque is applied to the top.

as in the propagation of waves, there is no travelling of material, just of structure. Therefore, the singularity line of the Larmor defect, as it is well known from the motion of dislocations, can move without any net material transport. The same holds for rotations of the defect. Thus we consider it a mobile object, like a knot in a frictionless cord.

**Energies and 'Forces'.** It is clear that the displacement field of two opposite defects compensates and yields a trivial state. Two coalescing defects, therefore, should release their stored elastic energy and transform it into elastic waves. It is further clear that two mirror-symmetric defects (of opposite sign) propagating towards each other lower their elastic energy $W$ and should experience an attracting force $-\frac{dW}{dr}$. By analogy, we conclude that defects of the same sign repel each other.[9] When using the term 'force' we should bear in mind that we are not talking about interactions among material elements but about displacement field configurations that travel unchanged in form. One may define, however, such a force using Newton's $F = m\ a$ and $m := \frac{W}{c^2}$ (cf. Frank 1949)

**The deformation field** proves intractable by linear elasticity. A solution with a finite torus (see above c)) was obtained by rather extensive numerical methods (Unzicker and Fabian 2003). The solutions that rigorously took into account the nonlinear incompressibility condition, showed an elongation along the symmetry axis of the defect. This so-called Poynting effect[10] is characteristic of the theory of finite deformations.

Instead of describing the rotation of the volume elements by the curl of the displacement vector curl $\vec{u}$, finite rotations require matrices $\mathbf{R} \in SO(3)$. How to obtain $\mathbf{R}$ from $\vec{u}$ the deformation gradient $\mathbf{F}$ (polar decomposition) and other basic concepts of nonlinear elasticity are outlined in sec. 5 of Unzicker (2000).

For our purposes, we shall restrict to an approximate solution given in fig. 3.

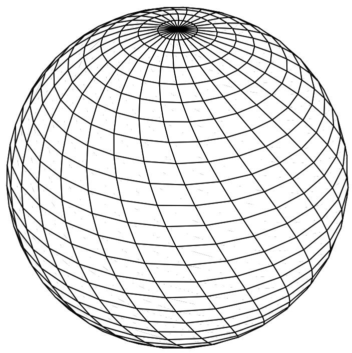

Figure 3: Qualitative description of the deformation field on the surface of a sphere surrounding a Larmor defect centered at the origin. In the first approximation, the displacement occurs along the meridians.

We shall assume the radius of the surrounding sphere to be large compared to the Larmor defect; thus the rotation of the volume elements (exaggerated in fig. 3) is satisfactorily described by curl $\vec{u}$ and we may ignore other nonlinear effects.

[9] For defects with a twisting angle of less than $2\pi$ one can offer a more rigorous argument. As obtained in Unzicker and Fabian (2003), the elastic energy increases with the square of the twisting angle. Two superimposed defects would, therefore, contain four times the elastic energy of $W$. Assuming $W$ is a continuous function of the distance $r$, equal defects must repel each other.

[10] The Poynting effect in its classic form is (Truesdell and Noll 1965, p. 193):'When an incompressible cylinder, free on its outer surface, is twisted, it experiences an *elongation* ultimately proportional to the square of the twist.'

### 3.4 Surface and volume torques.

The defect as visualized in fig. 3 is intuitively speaking a source of torque, since the regions of the poles show a clockwise twist. When trying to formulate a quantitative statement of this fact, one encounters some unexpected difficulties. Note that keeping the $z$-axis in mind, the torque is counterclockwise on the south pole, but physically the poles have equivalent ('clockwise') deformation.

The definition 'torque per area', applied to an Gaussian surface integral, can lead to contradictions. Consider the deformation in fig. 3 and try to produce it by infinitesimal torques *perpendicular to the surface*. We can start at the pole regions, apply clockwise torques, doing likewise on meridional stripes towards the equator, and obtain the desired deformation. Thus, one may claim that fig. 3 is a manifestation of clockwise torque. However, if we start by applying torques on infinitesimal areas in the equatorial region, the same deformation is produced with counterclockwise torques. The same problems arise even for flat surfaces, as one can easily verify. There is a subtle difference, however. Looking at the deformation in fig. 3, we note that the deformation in the polar regions is a *rotation*, whereas in the equatorial region we, more appropriately, speak of a *shear*. Even if curl $\vec{u}$ is, say, positive in the pole region and negative in the equatorial region, the sign does not tell us anything about the characteristic deformation of fig. 3. How do you distinguish a 'rotational' from a 'shear' curl ? We have to recover this from the topological properties of the sphere (see fig. 4).

Due to Brouwer's fixed point theorem, for any deformation there are at least two points on the sphere where $\vec{u}$ vanishes. In our special case fig. 3, it vanishes indeed at the poles, and additionally, at the equator. We shall call the deformation around the poles rotational curl and what happens at the equator we shall call shear curl. Now we can easily define the rotational curl linked to the 0-dimensional objects as the relevant one for our purposes, that is to determine the sign of the Larmor defect inside the sphere fig. 3. Correspondingly, we shall talk about rotational and shear torques, referring to that topological definition.

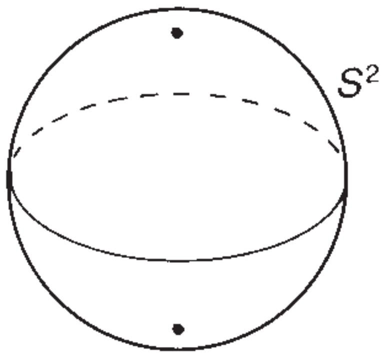


Figure 4: Positions on the sphere surrounding a Larmor defect where the displacement vector $\vec{u}$ vanishes. If $\vec{u}$ vanishes at a point, curl $\vec{u}$ in the vicinity is called rotational, in the vicinity of the line, it is called a shear type.

### 3.5 Relations to the Cosserat continuum

In classical elasticity theory, at a boundary of a body only forces are assumed to act per unit area. Then, in the static case, the stress tensor $\sigma_{ik}$[11] is shown to be symmetric (Cauchy 1827; Love 1927, p. 78; Truesdell and Toupin 1960; Beatty 2000).

In 1909, the brothers E. and F. Cosserat considered an elastic medium in which moment stresses $\tau_{ik}$, that is torque $i$ per unit area $k$, are allowed. The equilibrium law in modern notation (e.g. Kröner 1980, eqn. 42) is

$$\frac{\partial}{\partial k}\tau_{ik} = \operatorname{div}\tau_{ik} = \bar{\sigma}_{ik} = \sigma_{ik} - \sigma_{ki}. \qquad (8)$$

$\bar{\sigma}_{ik}$ is a vector perpendicular to both $i$ and $k$. Since it will be relevant for the following, I shall visualize this equation in its integral form[12] by considering a cylinder-shaped volume element

$$\int_{jacket} \vec{\sigma} \times \vec{r}\, d\vec{f} = \int_{circles} \vec{\tau} d\vec{f} \qquad (9)$$

inside an elastic material (see fig. 5). Generally, both surface integrals have to be taken over the entire surface. Here, if we apply a torque to both ends of the rod towards a given orientation, this has to be compensated by surface tractions on the jacket of the

[11] Force $i$ per unit area $k$.

[12] Using a generalized Gauss' theorem for tensors to div $\tau_{ik}$ and Stokes' theorem for the curl-like quantity $\bar{\sigma}_{ik}$ to all circular slices.

cylinder. It is important to note that the contributions of $\tau_{yy}$ at the front and the back do not cancel out since the opposite amount of torque is applied on differently oriented surfaces. Thus the quantity $\tau_{yy}$ changes along the rod, as it is clear from div $\tau_{ik} \neq 0$ in eqn. (9), too. It is also evident that given a constant $\sigma_{xz}$ and $r$, the total torque applied to the rod increases with its length $l$, thus one can reasonably define a volume density $t_i$ of torque - in the above case $2\tau_{yy}/l$.

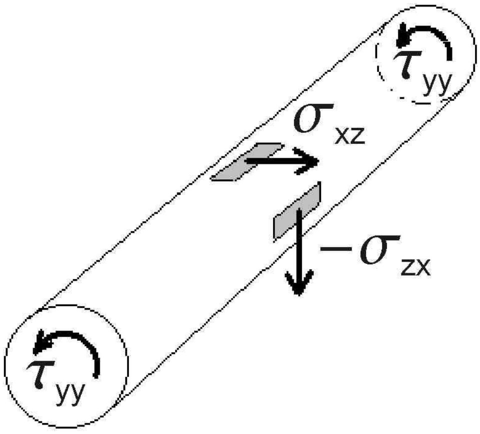


Figure 5: Visualization of eqn. (9), a volume element in an elastic material consisting of a cylinder on whose covers torques act. These torques are compensated by surface tractions on the jacket.

It is clear that these theories of more general elastic continua require a microstructure of the material and, therefore, frequently are applied to incompatible deformations. In the above example, their application can be justified if the size of the regions of interest is larger than the volume element (fig. 5). On the microscopic level, the above deformation is compatible and still obeys classical continuum mechanics.

# 4 Distribution of a large number of Larmor defects

## 4.1 Paradoxes with isotropy

Inspired by the hypothesis that Larmor defects, acting as 'sources of intrinsic rotational strain', could serve as charges, one may wonder about the nature of a macroscopic electric field. However, even without that motivation, as a riddle of pure continuum mechanics, we may consider a large number, such as $10^{15}$, of (positive) Larmor defects distributed isotropically inside a sphere. Then the question arises: what deformation field do we observe at a distance? Due to incompressibility, the displacement field $\vec{u}$ has no radial component, and if *we assume* a radial symmetry of $\vec{u}$, due to isotropy curl $\vec{u}$ must also vanish; with a trivial derivative, the deformation itself however would be trivial. Larmor's defect of a negative sign should still experience an attracting force, however. How is this force transmitted? Where has all the stored elastic energy gone, if the deformation is trivial?

## 4.2 Continuum mechanics with microstructure -'texture'

To avoid this counterintuitive consequence, the only possible solution is that radial symmetry is broken and the anisotropy of a single Larmor defect transforms into an anisotropy of the macroscopic deformation even if one tries to arrange the defects in the highest possible symmetric order.

**Approximate numerical implementation.** As a first approach, the deformations of six Larmor defects (see fig. 3), with axes oriented in dodecahedral symmetry were superimposed, yielding the result as shown in fig. 6. The black and white regions correspond to the rotational and shear curl of the displacement field, respectively. As in fig. 4, this definition is possible since we can draw a pattern of dots and lines indicating $\vec{u} = 0$.

We imagine the black regions to be tubes of twisted material, two of them coming out of each Larmor defect. Since a complete numerical treatment of this case would go beyond any computing power[13], one can just suspect that a texture like the one shown in fig. 6 is a configuration of minimal elastic energy. To test this hypothesis, random distributions of the defect axes that showed consistently higher energies were analyzed. This also turned out to be true for little random modifications of the configuration shown in fig. 6. The existence of a repulsive force between 'tubes' of even sign seems reasonable, therefore. For the following

[13]The solution for a single defect in Unzicker and Fabian (2003) could be obtained by a reduction to two dimensions.

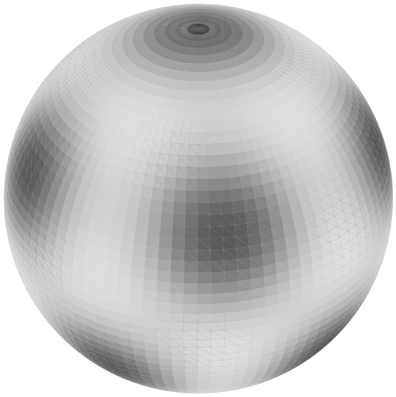

Figure 6: Qualitative description of the deformation field on the surface of a sphere surrounding several Larmor defects. In the first approximation, we can identify regions of the rotational (black) and the shear (white) curl of the displacement vector $\vec{u}$ on the surface and a medium gray level indicating 0. The black spots correspond to clockwise twisted areas in fig. 3.

we shall assume that ensembles of Larmor defects create a microstructure of such tubes. Tubes piercing through a surface element appear as a 'texture' sketched in fig. 7, the little arrows now indicating the orientation.

Extending the analogy, one is tempted to consider the twisted tubes as electrical field lines. However, even without that motivation, it is quite evident from the above considerations on energy that more defects add further tubes at the microstructural level rather than causing the deformation field to vanish. It is surprising that, in a material with compatible deformation, these textures of deformation appear at large distances from the topological defects.

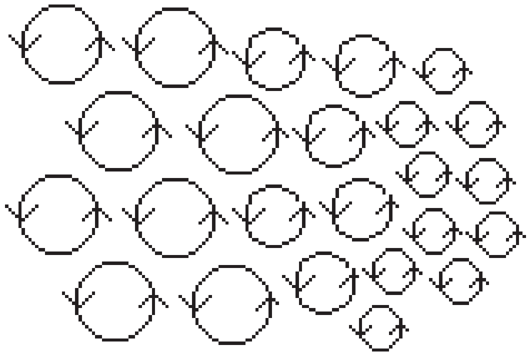

Figure 7: Schematic picture of the twisted regions in fig. 6. From left to right, the 'density' of tubes increases.

# 5 Analogs to electromagnetic quantities

## 5.1 Gaussian surface integral

Given that Larmor defects cause a microscopic deformation, as shown in fig. 7, it is easy to deduce the net number[14] $n$ of Larmor defects included within a closed surface by counting the $2n$ tubes piercing through it. Equivalently, one may speak of integration of the density of piercing tubes over a surface, this density having the unit $\frac{1}{m^2}$. Using the analogy to the Gaussian surface integral, this density could be a measure of the electric displacement.[15] $\vec{D}$ would then be proportional to $1/r^2$, $r$ being the radius of the tubes.

## 5.2 Electric field and electric displacement

I shall attempt, however, a related, but not identical definition of $\vec{E}$ and $\vec{D}$. Though if one must be careful while 'integrating' torques over surfaces, it makes sense to address rotational and shear torque, as outlined in section 3.4. In addition to the surface torque, we can define a volume torque ($N/m^2$) as outlined in the example fig. 5. For example, the black spots in fig. 6, continued towards the inside of

[14] We do not count pairs of opposite signs, since they may cancel out.

[15] This must not be confused with the displacement vector $\vec{u}$, which is the shift of material elements.

the sphere, are regions of a nonvanishing rotational torque density.

We may also consider the rotational torque on the surface fig. 7. The divergence of this surface torque is a torque density with unit $N/m^2$. We will see below that this rotational torque density $T/V$ obeys the same equations as the electric field $\vec{E}$. In the simple example of cylindrical rods, for small deformations the torque is proportional to the twisting angle and the shear modulus $\mu$. We may, therefore, divide the torque density by $\mu$ and obtain a dimensionless measure of strain, $\frac{T}{\mu V}$ which is intuitively related to a twisting angle. This definition proves to be analogous to the corresponding quantities in MacCullagh's theory. In an incompressible isotropic material, the deformation $\epsilon_{ik}$ and the stress tensor $\sigma_{ik}$ are related by

$$\sigma_{ik} = 2\mu\epsilon_{ik}, \tag{10}$$

$\mu$ being the shear modulus. Since according to eqn. 8, we may express the rotational torque density as $\bar{\sigma}_{ik}$, (10) generalizes to

$$\bar{\sigma}_{ik} = 2\mu\bar{\epsilon}_{ik}. \tag{11}$$

The anti-symmetric part of the deformation tensor, $2\bar{\epsilon}_{ik}$, is however nothing other than $\text{curl}\,\vec{u}$, the quantity that MacCullagh assigned to the electric displacement. We shall, therefore, take the quantity $\frac{T}{\mu V} = \frac{E}{\mu}$, the rotational part of $\text{curl}\,\vec{u}$, as analogous to the electrical displacement $\vec{D}$.

## 5.3 Coulomb's law

Consider a flat surface pierced by tubes of the same size $r$. In the case of a 'homogeneous electric field', these tubes continue in a direction perpendicular to the pierced surface.

It is clear that since the same torque is applied to a tube of any size, the torque per volume $\tau_i$ decreases inversely proportional to the square of the radius of the tube. In the weak-field-limit, where we expect Hooke's law to be valid, for the twisting angle $\varphi \sim D \sim 1/r^2$ holds. This is also in agreement with the energy density of the electric field $w \sim E^2$.

## 5.4 Dynamical deformations analogous to the magnetic field

As is well-known from classical electrodynamics, the magnetic field creating a Lorentz force can be understood as an electric field arising from a Lorentz contraction of distances between charges in a transformed inertial system (e.g. Landau and Lifshitz 1972, par. 24). I shall use this approach to define a quantity analogous to the magnetic field.

Assume the $x$-axis to be a wire with net charge 0 in the rest system $S$ in which Larmor defects of opposite sign move in opposite directions, respectively. Then, at a distance $r$ the deformation field will be a superposition of 'tubes', moving and oriented oppositely (see fig. 8). When discussing motion, we bear in mind that there are just structures of microstrain propagating, while the elastic material remains at rest, like a water wave.

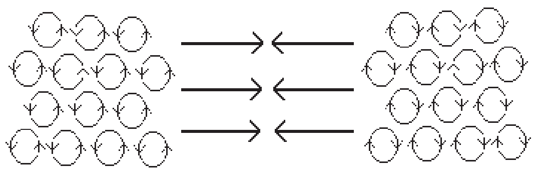

Figure 8: Model of a magnetic field created by charges of different sign moving in opposite directions. The respective 'electric' fields that are Lorentz-contracted, cancel out because the number of twisted circles per area is equal.

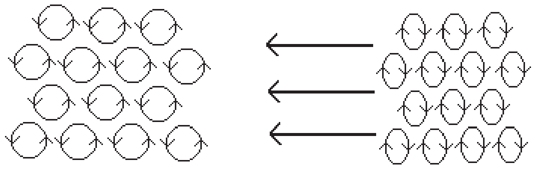

Figure 9: The same situation as in fig. 8, from a system that moves with velocity $v$ to the right with respect to the wire. The charges moving to the right are also no longer Lorentz-contracted, but the incoming charges are still more contracted. A greater net density of clockwise twisted tubes is perceived, which can be interpreted as an 'electric' field.

Note that because of the velocity $v$ in $x$-direction, the tubes must show a slight Lorentz contraction, due to the result in section 2.1. Since the net number of tubes cancels out[16], there is no 'electric' field and no force. The situation changes how-

[16] This does not mean that the time-dependent displacement field $u(t)$ is trivial, see below.

ever, if a Larmor defect moves with the same velocity $v$ parallel to the defects of the same sign (System $S'$). These will acquire their original distances in *their* rest system $S'$, whereas the defects of opposite sign, moving with even greater velocity, relatively, will shorten in $x$-direction (see fig. 9). Consequently, the moving defect will, at a distance, be perceived as a net number of (contracted) tubes carrying an opposite 'charge' and experience an attractive force towards the $x$-axis due to the extra torque arising.

The completely analogous situation is commonly described with a (static) magnetic field parallel to concentric circles around the current and a Lorentz force $e\ (v \times \mathbf{B})$.

With this indirect argument, we can define a field analogous to the magnetic flux $\vec{B}$ as the amount of net (extra) torque per volume created by such a motion of defects,

$$\vec{B} = \frac{1}{c^2} \vec{E} \times \vec{v}, \tag{12}$$

as it is well known from electrodynamics. The direction of the field is perpendicular to both the moving tube axes and their direction of motion. Since $c^2$ equals $\frac{\mu}{\rho}$ in the continuum mechanical case, eq. 12 transforms to

$$\vec{B} = \rho \vec{D} \times \vec{v} \tag{13}$$

Following MacCullagh, we once more define the magnetic field strength $\vec{H}$ as $\frac{1}{\rho}\vec{B}$, thus yielding the simple relation $\vec{H} = \vec{D} \times \vec{v}$. It is important to bear in mind that $v$, in this case, is not a material velocity as in MacCullagh's case but a velocity of structures - the twisted tubes may well travel onward while the volume elements of the medium remain at the same position. $H$ acquires an intuitive meaning as advected rotational curl, directed perpendicular to both the curl and the advection velocity. To facilitate visualization, one more remark is given.

**The magnetic flux,** in terms of the analogy, equals the electric field gained by the velocity $v$, due to the relativistic effects with a factor $v/c^2$, yielding the unit $\frac{Ns}{m^3}$. Since this is a momentum density (as in MacCullagh's theory), there is another possible interpretation. A propagating texture of twisted tubes as shown in fig. 8 produces a time-dependent displacement field $\vec{u}(t)$. For the situation given above, we have to imagine the advection of tubes. While the inside of the tube passes (comparable long period), the velocity vector $\frac{d}{dt}\vec{u}$ is pointing downwards, followed by a short period in which upward velocity is advected. This behavior is sketched in fig. 10

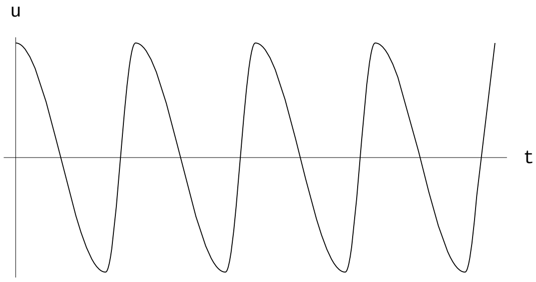


Figure 10: Time-dependent displacement vector $\frac{d}{dt}\vec{u}$ in the situation of fig. 8.

Since the average value of the displacement velocity has to be zero, for the definition of $\vec{B}$ one may consider only the longer periods with slower motion. This is not as arbitrary as it seems. In the electric case, we were integrating torques over the tube areas only because of the problems mentioned in section 3.4. In a situation as shown in fig. 10, the medium velocity during the slow phase should be closely related to the magnetic field. Again, we have to distinguish carefully the velocity of structures and material.

## 5.5 Purpose of the above definitions.

Regarding the 'classical' fields appearing in MacCullagh's theory, we have noted differences as well as similarities. The main difference is that the motion of the propagating microstrain assumes the role of the material velocity. However, we still can assign the same physical units to the microgeometric quantities. If both classical and microgeometric fields satisfy Maxwell's equations, we would face a situation in which *two* different physical quantities are hidden in each of the conventional electric and magnetic fields. That means, our perception of an electric (magnetic) field could be a mixture of fields describing wave propagation (MacCullagh) and charges/currents (topology). I shall address this possibility now.

# 6 Maxwell's equations for the microgeometric fields

In the following, the notations $\vec{E}$ and $\vec{B}$ will be reserved for the fields in MacCullagh's theory ($\frac{1}{\epsilon_0}$curl $\vec{u}$ and $\rho\frac{d\vec{u}}{dt}$), while the microgeometric quantities are denoted as $\vec{E}^*$ and $\vec{B}^*$ ($\vec{B}^* = \mu_0\vec{H}^*$, $\vec{D}^* = \epsilon_0\vec{E}^*$).

We first consider the second Maxwell equation and check whether it is also satisfied for the microgeometric magnetic field $\vec{B}^*$. Since $\vec{B}^*$ arises from a Lorentz transformation of $\vec{E}^{*'}$ from the system $S'$ to $S$, vector analysis rules yield

$$\text{div } \vec{B} = \frac{1}{c^2}\text{div}\,(\vec{v}\times\vec{E}^{*'}) = \frac{1}{c^2}(\vec{v}\cdot\text{curl } \vec{E}^{*'} + \vec{E}^{*'}\cdot\text{curl } \vec{v}). \quad (14)$$

since both fields $\vec{E}^{*'}$ and $v$ are irrotational, the r.h.s. of (14) vanishes and div $\vec{B} = 0$ holds for the microgeometric case, too.

## 6.1 Ampere's equation for the microgeometric magnetic field

If $\vec{B}$ is identified as above with the velocity $\rho\frac{d\vec{u}}{dt}$, the fourth Maxwell equation *in vacuo* (7) is satisfied by definition. A static magnetic field for instance of a coil however, cannot be represented by a velocity, because the steady flux through the coil would create increasingly larger deformations of any elastic material.

To visualize Maxwell's fourth equation we consider the situation sketched in fig. 11. A circular region is pierced by twisted tubes (electrical field lines).

In addition, further tubes are crossing into the circle from the right.[17] Tubes moving perpendicular to their axes create a microgeometric magnetic field in the perpendicular direction, as outlined in fig. 8 above. It is clear that any tube crossing into the circle increases the line integral $\int \vec{H}^* d\vec{s}$ along the circle, which according to Stokes' theorem is a measure of curl $\vec{H}^*$. Increasing the number of tubes in the circle is, however, nothing other than increasing the microgeometric electric displacement, $\frac{dD^*}{dt}$.

[17] Bear in mind that the structures are crossing, not the material.

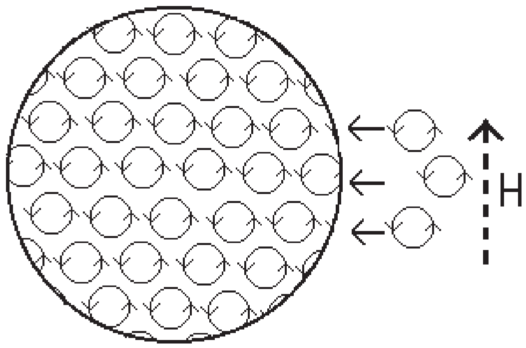


Figure 11: Electric field lines (tubes) crossing into a circular region increase the density of tubes ($\frac{dD^*}{dt}$) and can be interpreted as a line integral of $\vec{B}$.

When trying to visualize Maxwell's fourth equation, it is useful to imagine the magnetic field $H$ as follows: $D$ is the number of curls per area ($1/m^2$, cf. fig. 7 and 8), or, if one prefers a dimensionless unit, the average twisting angle) The moving curls generate a field $H = Dv$, thus a product of displacement field and velocity, whereby the direction is perpendicular to $\vec{v}$ and the field lines (tubes).

Of course, adding electrical field lines could be accomplished by a Larmor defect piercing the circular surface, too. This demonstrates that the current density $j$ and $\epsilon_0\frac{dD}{dt}$ are of the same nature. Thus we obtain

$$\text{curl}\,\vec{H}^* = \vec{j} + \frac{d\vec{D}^*}{dt} \quad (15)$$

or, after multiplication with the shear modulus $\mu$ ($\frac{1}{\epsilon_0}$),

$$c^2\text{curl}\,\vec{B}^* = \frac{\vec{j}}{\epsilon_0} + \frac{d\vec{E}^*}{dt} \quad (16)$$

## 6.2 Faraday's equation for the microgeometric magnetic field

In MacCullagh's theory, the equation of motion (2) proved to be equivalent to the first Maxwell equation. This seems to work quite well as far as electromagnetic waves are concerned, but runs into serious problems when dealing with slowly changing magnetic fields of nonoscillatory character (Whittaker 1951, p. 280). While explaining the validity of curl $\vec{E}^* = -\frac{d}{dt}\vec{B}^*$, I shall again invoke the visual imagination. In fig. 12 a visualization of $\mathrm{curl}\,\vec{E}^*$ is given, that means a situation of closed circular tubes (field lines), sketched as tori. The arrangement is symmetric with respect to the $z$-axis. Assume all tubes to be twisted symmetrically, e.g. clockwise on the r.h.s. and counterclockwise on the left (see fig. 12). Since these 'field lines' are closed, they do not end as usual in Larmor defects.

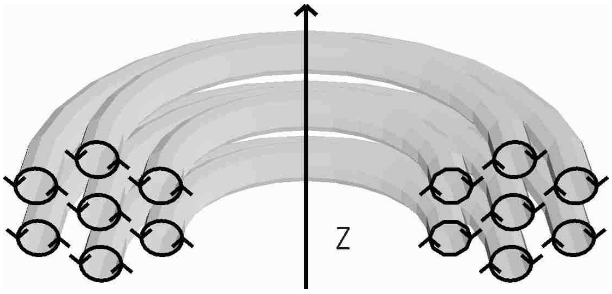


Figure 12: Each of the tori stands for a region of twisted material, which is analogous to an electrical field line.

Evidently, such a situation can never be stable. Rather the twisted tori should relax and continue twisting back until their internal orientation is reversed. Naturally, this corresponds to reversing the electric field. Imagine such an opposite situation at $t = T/2$, whereas $t = 0$ is shown in fig. 12. How should we visualize the deformations at $t = T/4$ and $t = 3T/4$? We *can* imagine the tori at rest while their inside rotates back. Alternatively, the tubes may maintain their orientation but propagate towards the $z$-axis, that is the tori would shrink to an interval on the $z$-axis. Of course, after having crossed the axis, the tori would expand again to the size shown in fig. 12, with reversed orientation. Both scenarios of the transition $t = 0$ to $t = T$ are indistinguishable, however. Remember that there is no net material transport but just a propagation of structures, and it does not make sense to see a difference between an extinction/creation process and a propagation. In the latter scenario, we observe twisted tubes crossing the $z$-axis from all directions in the $xy$-plane. Bearing in mind the above definition of $\vec{H}^*$, this is nothing other than a microgeometric magnetic field strength.

Similar to a pendulum, the situation in fig. 12 changes from a maximum value of $\mathrm{curl}\,\vec{E}^*$ ($t = 0, T/2$, $\vec{B}^* = 0$) to a maximal $\vec{B}^*$ ($t = T/4, 3T/4$, $\mathrm{curl}\,\vec{E}^*$), thus the rotation of the tubes generate a microgeometric magnetic field and

$$\mathrm{curl}\,\vec{E}^* = -\frac{d}{dt}\vec{B}^* \tag{17}$$

holds.

## 6.3 Overview

Table I gives an overview of the quantities that fulfill Maxwell's equations and the respective units.

| Symb. | MacCullagh | Microg. | Units |
|---|---|---|---|
| E | $\frac{1}{\mu}\mathrm{curl}\,\vec{u}$ | $\frac{T}{V} = \mu\bar{\varphi}$ | $Nm^{-2}$ |
| B | $\rho\frac{du}{dt}$ | $\frac{v}{c^2}\mu\bar{\varphi}$ | $Nsm^{-3}$ |
| D | $\mathrm{curl}\,\vec{u}$ | $\bar{\varphi}$ | 1 |
| H | $\frac{du}{dt}$ | $\bar{\varphi}\frac{du}{dt}$ | $ms^{-1}$ |

Table I.

As far as constants are concerned, we shall identify dielectricity $\epsilon_0$ with the inverse of the shear modulus $1/\mu$ and the magnetic permeability $\mu_0$ with the density $\rho$ of the elastic continuum. One should not worry about measuring these quantities, since only the speed of light ($c^2 = \frac{1}{\epsilon_0\mu_0}$) is accessible to observation and corresponds to the velocity of transverse sound $\sqrt{\mu/\rho}$. Further considerations on units are given in section 4 of Unzicker (1996).

While in MacCullagh's theory this applies to the vacuum case only, the topological and microgeometric quantities also describe charges and currents. Thus the combinations $\vec{E} + \vec{E}^*$, $\vec{B} + \vec{B}^*$ etc. do satisfy the basic equations of electrodynamics and correspond to the classical fields $\vec{E}$ and $\vec{B}$ that we observe. Remember that the classical fields are merely theoretical constructs and we may be yet unaware that they consist of two different quantities. Experimentally, the wave dynamics (MacCullagh) usually appears separated from the static or slow-motion regime. There are, however, situations in which new effects should occur if the micromechanic analogy is indeed appropriate. If finite rotations are related to the electric field, the superposition principle is violated for strong fields. A proposal to test this has been worked out in Unzicker (2019).

## 7 Outlook

The above analysis of an isotropic, incompressible elastic continuum with topological defects was inspired by the far-reaching analogy to a spacetime with elementary particles. The first ideas in this direction emerged in the 19th century: 'On this view, electrons, and hence all material bodies built up from them, are of the nature of structures in the aether....' (Whittaker 1951, p. 287). These hypotheses are now supported by further coincidences. First, equations equivalent to those of special relativity follow directly from elasticity theory. As shown here for the first time, one can define quantities derived from the elastic deformation that satisfy equations that are equivalent to Maxwell's. Regarding energies and forces, a quantitative analysis still must be done. Important progress with respect to the aether theories, however, is that the contradictions arising from material velocities were resolved by substituting them with velocities of deformation structures.

The Larmor defect, a twist disclination, was shown to describe charges and therefor, should be considered a natural extension of MacCullagh's theory. Larmor defects of different sign cancel out and release elastic energy. On the other hand, given a sufficient amount of elastic energy, the creation of a defect pair should be possible. Thus, it does not make sense to assign an 'identity' to topological defects,[18] a well-known behavior from quantum statistics.

Furthermore, the analysis of topological defects led to the appearance of a microstructure of the deformation field. This interesting phenomenon of microstrain is a result of pure continuum mechanics that does not necessarily need to be interpreted in 'electrodynamic' terms. If one accepts such an interpretation however, it may answer the question why quantized structures necessarily appear in electrodynamics.

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[18] See a detailed discussion in Unzicker (2002).

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