Elegant Proof of the $3n + 1$ Problem via Modular Algebra

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Abstract: The Collatz conjecture is true, the $3n+1$ problem generates a fractal spiral from odd multiples of three to elements 5 mod 8, if they are not already, and from there onto smaller elements already known to go to one. If T is the reduced Syracuse function, then if $T(u) = x$, so too does $T(4u+1) = x$. Also 3 mod 4 elements inevitabley map to 1 mod 4. All must descend.

We consider the function $C : \mathbb{N} \to \mathbb{N}$, $C(n) \to \begin{cases} 3n+1, & \text{if } n \text{ is odd.} \\ \frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$ The question is to describe the long term behavior of a path starting from an arbitrary natural number and iterating the foregoing mappings until we enter some kind of cycle or diverge to infinitely large numbers. For example, from 3, our path takes us $3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$. The Collatz conjecture states that all paths lead to 1.

Let $\mathbb{U}$ be the set of odd numbers. Without loss of generality we collapse even cases of C to their respective odd mappings, and consider a reduced function T, mapping odd numbers u to odd numbers $T(u) \to \frac{3u+1}{2^r}$, with r maximal to produce $u \in \mathbb{N}$. This moves the uncertainty based on parity in C to the value of r. in T. Intuitively, $r(u)$ is the number of division steps that occur going from u to $T(u)$. We can conceptualize the difference between C and T as internalizing the parity dependence of the function. We prove the conjecture that iterating T from any u eventually includes 1 by demonstrating that this is a consequence of algebra.

We partition odd numbers, $u \equiv 2k = 1$ for some natural number k, into two cases modulo 4 and apply T to obtain theorem 1: $\boldsymbol{T(u) > u}$ **if and only if** $\boldsymbol{u \equiv 3+4n}$.

Peoof. Applying T to the partitions modulo 4.

I. $3(3+4n) = 9 + 12n$.

II. $9+12n+1 = 10 + 12n$.

III. $(10 + 12n)/2 = 5 + 6n$.

IV. Notice $5+6n > 3+4n$, and $r(u) = 1$ when $u \equiv 3+4n$.

V. $3(1+4n) = 3 + 12n$.

VI. $3+12n+1=4+12n$.

VII. $(4 + 12n)/4 = 1+3n$.

VIII. Notice $1+3n < 1+4n$ and $3n + 1$ regenerates T, implying $r(u) \geq 2$. ■

To examine the constraints on increasing paths, we partition the rising elements $3+4n$ into the equivalence classes modulo 8. This gives us two cases to which we apply T to obtain theorem 2. **The long term behavior of a path depends only on the constraints of descending elements** $\boldsymbol{u \equiv 1+4n}$.

Proof. Applying T to the modulo 8 subsets of elements 3 mod 4:

I. $3(7+8k) = 21 + 24k$..

II. $21+24k+1 = 22 + 24k$.

III. $(22 + 24k)/2 = 11 + 12k$.

IV. Notice $11+12k \equiv 3+4k$.

V. $3(3+8k) = 9 + 24k$.

VI. $9+24k+1=10+24k$.

VII. $(10 + 24k)/2 = 5+12k$.

VIII. Notice $5+12k \equiv 1+4k$. ■.

Remark: We have now shown that a strictly rising path must consist entirely of $u \equiv 7+8k$ because $u \equiv 3+8n$ map exclusively to $u \equiv 1+4n$. This puts the fate of infinite paths entirely on $u \equiv 1+4n$, and proves that there are no strictly increasing divergent paths to arbitrarily large $T(u)$ from a given u, all paths are forced to descend after finitely many steps.

Both modulo 8 equivalence classes of $u \equiv 1 + 4n$ map indiscriminately with respect to the modulo 8 status of $T(u)$. However, they do segregate based on $r(u)$, giving us theorem 3. **If $r(u)$ at least 3, then $u \equiv 5+8k$, [5] and if $r(u) = 2$, then $u \equiv 1+8k$ [1].**

Proof. Applying T to the modulo 8 subsets of elements 1 mod 4:

I. $3(1+8k) = 3 + 24k$.

II. $3+24k + 1 = 4 + 24k$.

III. $(4 + 24k)/4 = 1 + 6k$.

IV. Notice $r(u) = 2$ when $u \equiv 1+8k$.

V. $3(5+8\mathrm{k}) = 15 + 24\mathrm{k}$.

VI. $15+24\mathrm{k} + 1 = 16 + 24\mathrm{k}$.

VII. $(16 + 24k)/8 = 2+3k$.

VIII. Notice $r(u) \geq 3$ when $u \equiv 5+8k$. ■.

Remark: We have now partitioned the codomain of T into preimages using three equivalence classes modulo 8, those in the set, {[3], [7], [1]}. These account for what we call the *proximal preimages* of $T(u)$. These are easily demonstrated observing that mapping $5 + 6k \rightarrow 3+4k$ gives a preimage, and $1+6k \rightarrow 1+8k$, also gives a preimage, together accounting for all elements in the codomain of T with 0% error. The remaining preimages we call *distal preimages*, are all greater than the proximal preimages and follow a simple pattern of generation.

Let D be a function $D : \mathbb{U} \rightarrow \mathbb{U}$, $D(u) = 4u + 1$. Now $T(D(u)) = T(u)$, and in fact $T(u) = T(u')$ if and only if $u' = D(u)$, for some j where j denostes the iterations of $D(u)$ applied. This has been known in the litterature for some time. We will show that [5] mod 8 equivalence class contains all distal preimages, that is, the predictability of D, theorem 4. **All u become congruent to 5 mod 8 under D, i.e. $D(u)\ {}^{8}{\equiv}_{8}\ 5$, and map to $T(u)$, i.e $T(D(u) = T(u)$.**

Proof. Applying D to arbitrary u, and T to arbitary $D(u)$:

I. $4(2x - 1) = 8x - 4$.

II. $8x - 4 + 1 = 8x - 3$.

III. $8x - 3 \equiv 8x + 5$.

IV. Thus, all distal preimages are congruent to 5 mod 8.

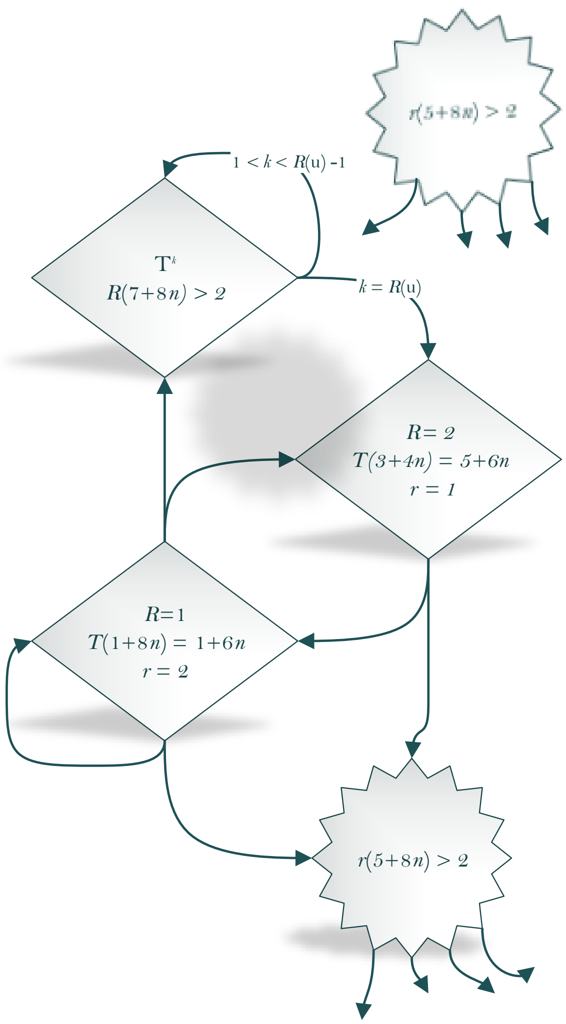
V. $3(4u+1) = 12u+3$

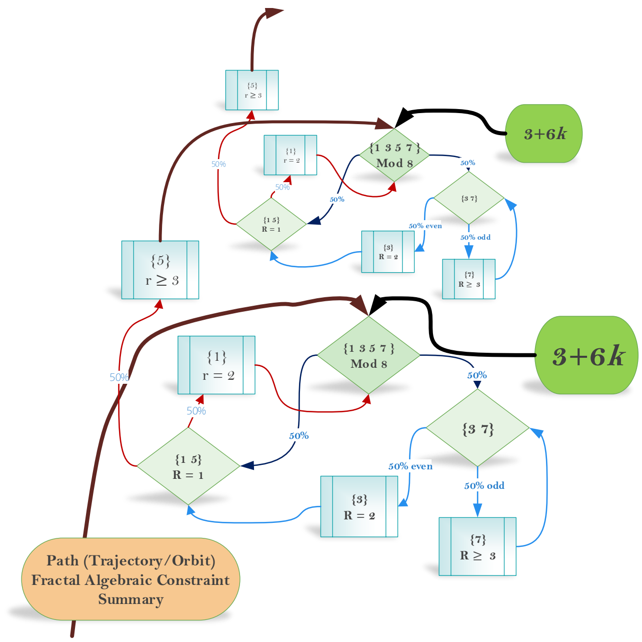
VI. $12u + 3 + 1 = 12u + 4$

VII. $(12u + 4)/4 = 3n + 1$

VIII. And all distal preimages map onto the image of u. ■.

This completely solves the $3n+1$ problem. Every element $6k \pm 1$ has exactly one proximal preimage determined by a linear mapping, and an infinitude of distal preimages that descend upon its' image from above. Since these are always from above, the paths are always descending from a macroscopic perspective. The entire problem reduces to a unidirectional flow based on the template given by the path from 3. This is a fractal expansion of $3{+}6k \to 5{+}8k \to 1$ to $3{+}6k \to 5{+}8k \to 6k{\pm}1 \to 1$. Putting together the foregoing gives us the constraints on path generation that give us our summary diagram. Following the arrows in the theoreitcal summary figures below from any starting position eventually results in the bottom position, which practice corresponds to a descent on 1.





In these figures, the function R: $\mathbb{U} \to \mathbb{N}$, maps $R(u)$ to the number of consecutive non-descending iterates starting from u in its path iterating $T(u)$. Paths from u with immediate descent have $R(u) = 1$.

Supplamental material.

To remove any doubt that we have complete mastery of the $3n+1$ problem, we advise the reader to observe the underlying structure of the $an+1$ problems in general. There we find the $3n+1$ problem sandwiched between the uniform divergence of $2n+1$ and uniform convergence of $2n+0$. Since $3 < \pi$, it lacks the escape velocity of $5n+1$, and being unidirectional, cannot enter nontrivial loops the way $3n-1$ can.