Timeless In Hilbert-Schwarzschild Metric

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Abstract

David Hilbert derived a different metric for the same manifold described by Schwarzschild's metric. The geodesic in Hilbert's manifold is a curve in Newton's manifold with radial acceleration similar to Newton's gravity. The similarity exists only if time is excluded from Hilbert's metric of Schwarzschild manifold.

I. INTRODUCTION

In 1916, Karl Schwarzschild published a paper[1,2], ”ber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie” in response to Albert Einstein's paper[3], ”Erklrung Perihelbewegung des Merkur aus”. The response presented an exact solution in comparison with Einstein's approximation solution. Schwarzschild considered the exact solution far superior to Einstein's approximation by stating:

”It is always pleasant to have strict, simple form solutions. It is more important that the calculation also gives the unambiguous certainty of the solution, about which Mr. Einstein's treatment still left doubts, and which, according to the way in which it appears below, could hardly be proved by such an approximation procedure.”.

The solution is a metric for an isotropic manifold with a static mass at the origin.

$$ds^2 = (1 - \frac{\alpha}{R})dt^2 - \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2(d\theta^2 + sin(\theta)^2 d\phi^2) \quad (1)$$

$$R = (r^3 + \alpha^3)^{1/3} \quad (2)$$

Schwarzschild's metric describes how a massless point should propagate through an empty space. With the exact solution, the path coincides with the geodesic for a manifold of isotropic symmetry. The straight geodesic in Schwarzschild manifold is a curve in Minkowski manifold. By using the same coordinate system in both manifolds, the acceleration along the curve is found to be similar to Newton's gravity under certain conditions.

David Hilbert derived a different metric[4] based on Schwarzschild's metric. The time coordinate is removed from the metric. To retain 4 dimensions for the manifold, an unknown coordinate is added to the metric. Hilbert's metric is actually a generalized version of Schwarzschild's metric without the restriction that the metric becomes Minkowski metric at the infinite radial distance.

However, the metric describes the geodesic for massless point. The presence of extra mass destroys the isotropic symmetry. Nevertheless, Hilbert's metric still resembles Newton's gravity in the radial geodesic of a positive line element under one condition: the line element is the elapsed time.

II. PROOF

A. Hilbert's Derivation

David Hilbert derived Schwarzschild's metric from the lagrangian $K\sqrt{g}$. The hypotheses on the $g_{\mu\nu}$ are the following:

1. The interval is referred to a Gaussian coordinate system - however g_{44} will still be left arbitrary; i.e. it is

$$g_{14} = 0 = g_{24} = 0 = g_{34} \quad (3)$$

2. The $g_{\mu\nu}$ are independent of the time coordinate x_4.
3. The gravitation $g_{\mu\nu}$ has central symmetry with respect to the origin of the coordinates.

Hilbert avoided Schwarzschild's fourth condition that the metric becomes Minkowski metric at infinite distance.

By replacing dt with dl, Hilbert disagreed with Schwarzschild that time is needed in the metric.

$$F(r)dr^2 + G(r)(d\theta^2 + sin^2\theta d\phi^2) + H(r)dl^2 \quad (4)$$

Hilbert stated: ”If we substitute r^* for r and then drop the symbol *, it results the expression”

$$M(r)dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 + W(r)dl^2 \quad (5)$$

and remarked: ”To this end the known expressions $K_{\mu\nu}$, K, given in my first communication, shall be calculated by comparing the differential equations of the geodesic line with variation of the integral

$$\int (M(\frac{dr}{dp})^2 + r^2(\frac{d\theta}{dp})^2 + r^2 sin^2\theta(\frac{d\phi}{dp})^2 + W(\frac{dl}{dp})^2)dp \quad (6)$$

$$\frac{d^2 w_s}{dp^2} + \sum_{\mu\nu} \left\{ {\mu\nu \atop s} \right\} \frac{d^2 w_\mu}{dp^2} \frac{d^2 w_\nu}{dp^2} = 0 \quad (7)$$

From equations (6,7), Hilbert determined K_{ss} as

$$K_{11} = \frac{1}{2}\frac{W''}{W} + \frac{1}{4}\frac{W'^2}{W^2} - \frac{M'}{rM} - \frac{1}{4}\frac{M'W'}{MW} \quad (8)$$

$$K_{22} = -1 - \frac{1}{2}\frac{rM'}{M^2} + \frac{1}{M} + \frac{1}{2}\frac{rW'}{MW} \quad (9)$$

$$K_{33} = sin^2\theta(-1 - \frac{1}{2}\frac{rM'}{M^2} + \frac{1}{M} + \frac{1}{2}\frac{rW'}{MW}) \quad (10)$$

$$K_{44} = \frac{1}{2}\frac{W''}{M} - \frac{1}{4}\frac{M'W'}{M^2} + \frac{1}{4}\frac{W'^2}{MW} + \frac{W'}{rM} \quad (11)$$

Calculated the Ricci scalar K and the determinant g.

$$K = \sum_{s=1}^{4} g^{ss}K_{ss} \quad (12)$$

$$g = g_{rr}g_{\theta\theta}g_{\phi\phi}g_{ll} \quad (13)$$

for lagrangian $K\sqrt{g}$

$$= ((\frac{r^2W'}{\sqrt{MW}})' - 2\frac{rM'\sqrt{W}}{M^{\frac{3}{2}}} - 2\sqrt{MW} + 2\sqrt{\frac{W}{M}})sin\theta \quad (14)$$

and remarked: "if we set"

$$M = \frac{r}{r-m} \quad (15)$$

$$W = w^2\frac{r-m}{r} \quad (16)$$

"where henceforth m and w become the unknown functions of r, we eventually obtain"

$$K\sqrt{g} = ((\frac{r^2W'}{\sqrt{MW}})' - 2wm')sin\theta \quad (17)$$

"the variation of the integral of $K\sqrt{g}$ is equivalent to"

$$\int wm'dr \quad (18)$$

"and leads to the Lagrange equations"

$$m' = 0 \quad (19)$$

$$w' = 0 \quad (20)$$

By setting $w = 1$, Hilbert remarked: "a choice that evidently does not entail any essential restriction" and derived the the interval in the form of

$$G(dr, d\theta, d\phi, dl) \quad (21)$$

$$= \frac{r}{r-m}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2 - \frac{r-m}{r}dl^2 \quad (22)$$

Hilbert was not satisfied with the singularity in this interval. He remarked: "Although in my opinion only regular solutions of the fundamental equations of physics immediately represent the reality, nevertheless just the solutions with non regular points are an important mathematical tool for approximating characteristic regular solutions."

B. Generalized Hilbert Metric

It is not necessary to change the parameter from M to m in equation (15). The lagrangian in equation (14) can be formulated as

$$K\sqrt{g} = ((\frac{r^2W'}{\sqrt{MW}})' + f(M)\sqrt{W})sin\theta \quad (23)$$

with

$$f(M) = (-\frac{rM'}{M} - M + 1)\frac{2}{\sqrt{M}} \quad (24)$$

and reduced to an equivalent lagrangian L as

$$L = f(M)\sqrt{W} \quad (25)$$

The Lagrange equation for W is

$$f(M)\frac{\partial\sqrt{W}}{\partial W} = 0 \quad (26)$$

The Lagrange equation for M is

$$\sqrt{W}\frac{\partial f}{\partial M} = \frac{d}{dr}(\sqrt{W}\frac{\partial f}{\partial M'}) \quad (27)$$

From equations (24,26), with K_a as integration constant.

$$M = \frac{r}{r+K_a} \quad (28)$$

From equations (24,27), with K_b as integration constant.

$$W = \frac{K_b}{M} \quad (29)$$

The interval from equations (5,28,29) is

$$ds^2 = \frac{r}{r+K_a}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) + K_b\frac{r+K_a}{r}dl^2 \quad (30)$$

r is not the original radial coordinate but is renamed from r^*. r is an unspecified function of radial coordinate and should be renamed as $x(r)$.

$$ds^2 = \frac{x}{x+K_a}dx^2 + x^2(d\theta^2 + sin^2\theta d\phi^2) + K_b\frac{x+K_a}{x}dl^2 \quad (31)$$

Hilbert remarked that the interval would resemble Schwarzschild's by identifying dl with dt. Hilbert had provided a general solution while Schwarzschild provided a special solution with x as a function of r.

$$x = (r^3 + \alpha^3)^{1/3} \quad (32)$$

However, Hilbert's metric is actually different from Schwarzschild's metric. x is an undefined function of r, the actual radial coordinate. dl is not the elapsed time. Time is not needed in Hilbert's metric.

C. Equatorial Geodesics

Schwarzschild applied the rotational symmetry to his metric by setting

$$\phi = \frac{\pi}{2} \tag{33}$$

and remarked: "If one also restricts himself to the motion in the equatorial plane ($\theta = 90^o$,$d\theta = 0$)".

With the geodesic confined in equatorial plane, equation (31) is simplified as

$$ds^2 = \frac{x}{x+K_a}dx^2 + x^2 d\phi^2 + K_b\frac{x+K_a}{x}dl^2 \tag{34}$$

Define L as

$$L = \frac{x}{x+K_a}(\frac{dx}{ds})^2 + x^2(\frac{d\phi}{ds})^2 + K_b\frac{x+K_a}{x}(\frac{dl}{ds})^2 = 1 \tag{35}$$

L is a valid lagrangian according to "Formulation And Validation Of First Order Lagrangian"[5].

The Lagrange equation for x is

$$2x(\frac{d\phi}{ds})^2 - K_b\frac{K_a}{x^2}(\frac{dl}{ds})^2 \tag{36}$$

$$= \frac{K_a}{(x+K_a)^2}(\frac{dx}{ds})^2 + \frac{2x}{x+K_a}\frac{d^2x}{ds^2} \tag{37}$$

The Lagrange equation for ϕ is

$$x^2\frac{d\phi}{ds} = const. = K_\phi \tag{38}$$

The Lagrange equation for l is

$$K_b\frac{x+K_a}{x}\frac{dl}{ds} = const. = K_l \tag{39}$$

From equations (35,38,39), the lagrangian is

$$L = \frac{x}{x+K_a}(\frac{dx}{ds})^2 + K_\phi\frac{d\phi}{ds} + K_l\frac{dl}{ds} = 1 \tag{40}$$

$$\frac{x+K_a}{x}(1-\frac{K_\phi^2}{x^2}) = \frac{K_l^2}{K_b} + (\frac{dx}{ds})^2 \tag{41}$$

From equations (36,37,38,39), the Lagrange equation for x is

$$(x+K_a)\frac{2K_\phi^2}{x^3} - \frac{K_a}{x+K_a}(\frac{K_l^2}{K_b} + (\frac{dx}{ds})^2) = 2x\frac{d^2x}{ds^2} \tag{42}$$

From equations (41,42), the radial acceleration is

$$\frac{d^2x}{ds^2} = -\frac{K_a}{2x^2} + \frac{K_\phi^2}{x^4}(x+\frac{3}{2}K_a) \tag{43}$$

D. Newton's Gravity

For the metric to describe gravitation, the radial geodesic must represent Newton's gravity if such geodesic exists under the condition:

$$\frac{d^2\phi}{ds^2} = 0 = \frac{d\phi}{ds} = 0 = K_\phi \tag{44}$$

From equations (43,44), the radial acceleration is

$$\frac{d^2x}{ds^2} = -\frac{K_a}{2x^2} \tag{45}$$

ds becomes the elapsed time. dx can be set to dr since $x(r)$ is an undefined function of r. The constant K_a can be determined from the mass at the origin according to Newton's gravity.

$$a = -\frac{GM}{r^2} \tag{46}$$

From equations (39,41), with $K_\phi = 0 = d\phi$,

$$(\frac{dx}{ds})^2 = \frac{K_a}{x} + 1 - \frac{K_l^2}{K_b} \tag{47}$$

The radial speed from equation (47) can be identified with the energy equation in Newtonian mechanics.

$$v^2 = \frac{2GM}{r} + \frac{2E}{m} \tag{48}$$

In Schwarzschild metric, dl is the elapsed time. From equation (35), with $d\phi = 0$,

$$\frac{x}{x+K_a}(\frac{dx}{ds})^2 + K_b\frac{x+K_a}{x}(\frac{dl}{ds})^2 = 1 \tag{49}$$

From equations (39,49), with dl as elapsed time,

$$(\frac{dx}{dl})^2 = (\frac{x+K_a}{x})^3(\frac{K_b}{K_l})^2 - K_b(\frac{x+K_a}{x})^2 \tag{50}$$

Differentiate with respect to l for the acceleration.

$$\frac{d^2x}{dl^2} = -\frac{K_a}{2x^2}(3(\frac{x+K_a}{x})^2(\frac{K_b}{K_l})^2 - 2K_b\frac{x+K_a}{x}) \tag{51}$$

The acceleration is similar to Newton gravity only if $\frac{K_a}{x}$ approaches 0.

$$\frac{d^2x}{dl^2} = -\frac{K_a}{2x^2}(3(\frac{K_b}{K_l})^2 - 2K_b) \tag{52}$$

Therefore, dl is not a good choice for the elapsed time. ds is a better choice.

E. Null Geodesics

For null geodesic, $ds = 0$. From equation (34), with $d\phi = 0$ for the radial geodesic.

$$\frac{x}{x+K_a}(dx)^2 + K_b\frac{x+K_a}{x}(dl)^2 = 0 \tag{53}$$

$$(\frac{dx}{dl})^2 = -K_b(1+\frac{K_a}{x})^2 \tag{54}$$

Differentiate with respect to l to get radial acceleration.

$$\frac{d^2x}{dl^2} = K_b(1+\frac{K_a}{x})\frac{K_a}{x^2} \tag{55}$$

The acceleration is similar to Newton gravity only if $\frac{K_a}{x}$ approaches zero.

The acceleration with affine parameter, dn, as the elapsed time can be obtained from equations (39,53),

$$\frac{x}{x+K_a}(\frac{dx}{dn})^2 + \frac{x}{x+K_a}\frac{K_l^2}{K_b} = 0 \tag{56}$$

$$(\frac{dx}{dn})^2 = -\frac{K_l^2}{K_b} \tag{57}$$

$$\frac{d^2x}{dn^2} = 0 \tag{58}$$

If null geodesic does represent the path of light, the speed of light from equation (58) would remain constant under the gravity from the mass at the origin. Einstein had apparently chosen the wrong elapsed time.

F. Kepler's Law

Another important test of Newton's gravity is Kepler's third law which measures the gravity at a great distance.

From equation (38), with ds as the elapsed time,

$$\frac{d\phi}{ds} = \frac{K_\phi}{x^2} \tag{59}$$

From equations (38,39), with dl as the elapsed time,

$$\frac{d\phi}{dl} = \frac{x+K_a}{x^3}K_b\frac{K_\phi}{K_l} \tag{60}$$

The metric fails to describe Kepler's third law with any possible choice of the elapsed time.

III. CONCLUSION

Neither Hilbert's metric nor Schwarzschild's metric represents Newton's gravity. The manifold is not compatible with Kepler's law for any possible circular orbit. The acceleration is similar to Newton's gravity only in the radial geodesic with time coordinate excluded from the metric.

The line element is proportional to the affine parameter by a constant and is a good choice for the elapsed time which is conserved in all reference frames[6,7,8].

Hilbert's metric proves that the concept of space time manifold is not feasible for the solar system. The time coordinate is not needed. As Hilbert had stated, the metric does not describe physical system but remains as an interesting mathematical tool.

Furthermore, the metric requires all mass to be at the origin. The presence of any mass outside the origin will destroy the isotropic symmetry which the metric is based on. The metric is not suitable for the solar system.

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[1] K. Schwarzschild: ber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. https://de.wikisource.org/w/index.php?title=das Gravitationsfeld eines

[2] S.Antoci and A.Loinger: On the gravitational field of a mass point according to Einstein's theory. https://arxiv.org/abs/physics/9905030

[3] A. Einstein: Erklrung Perihelbewegung des Merkur aus. http://www.relativitycalculator.com/pdfs/einstein_mercury_perihelion_german.pdf

[4] Antoci, Salvatore: David Hilbert and the origin of the "Schwarzschild solution". http://fisica.unipv.it/antoci/0310104.pdf

[5] Su, Eric: Formulation And Validation Of First Order Lagrangian. viXra: Relativity and Cosmology/2007.0003 (2020). http://vixra.org/abs/2007.0003

[6] Su, Eric: Conservation Property of Standing Wave. viXra: Relativity and Cosmology/2006.0009 (2020). http://vixra.org/abs/2006.0009

[7] Su, Eric: Error in Modern Astronomy. viXra: Relativity and Cosmology/1908.0005 (2019). http://vixra.org/abs/1908.0005

[8] Eric Su: List of Publications, http://vixra.org/author/eric_su