ON THE NUMBER OF INTERSECTIONS OF TUBES

Abstract. In this article we will always assume that the number of δ-tubes is $N=\delta^{1-n}$. Moreover, we will assume that if any two δ-tubes intersect, then they intersect in the unit ball. We will show that the number of their intersections of order μ is bounded by $\frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)2^{n-1}|B_{n-1}(0,1)|\frac{N^{n/(n-1)}}{\mu}$. After making a dyadic decomposition and summing the orders together we will find that the number of tube intersections of N tubes is bounded by $\frac{|B(0,1+N^{-1/(n-1)})|}{|B(0,1)|}2^{n-1}(1+2N^{-1/(n-1)})|B_{n-1}(0,1)|N^{n/(n-1)}$. Moreover, we will prove a generalized lemma of Córdoba and we will prove the Kakeya sets have greater dimension than $n-1$.

1. Introduction

In $\mathbf{R}^n$ a joint is formed by the intersection of n lines whose tangent vectors are linearly independent. It's a fact that the number of joints formed by N lines are bounded by $C_nN^{n/(n-1)}$. This fact has quite an elementary proof [3]. In our paper we control tube intersections in all scales. Our bound for the total more than δ-spaced line intersections is essentially sharp. An extremal example is a standard grid of N lines. A line l_i is defined as

$$l_i := \{y\in\mathbf{R}^n | \exists a,x\in\mathbf{R}^n \quad \text{and} \quad t\in\mathbf{R} \quad \text{s.t} \quad y=a+xt\}$$

We define the δ-tubes as δ neighbourhoods of lines:

$$T_i^\delta := \{x\in\mathbf{R}^n || x-y|\le\delta, \quad y\in l_i\}.$$

The order of intersection is defined as the number of tubes (lines) intersecting. We use P_μ^δ as the set of δ-tube intersections of order μ and P_μ as the set of line intersections of order P_μ. Moreover, P^δ and P mean the set of δ-tube intersections and the set of line line intersections, respectively. If $\mu>1$ then

$$P_\mu^\delta := \bigcup_{j=1}^{M_\mu^\delta}\bigcap_{i=1}^{\mu} T_{ij}^\delta$$

We define

$$\#(P_\mu^\delta)=M_\mu^\delta$$

and the total number of intersections is

$$\#(P^\delta)=\sum_\mu M_\mu^\delta.$$

In a same way we define $\#(P_\mu)$ and $\#(P)$. Our main theorem is the following:

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Theorem 1.1. *Let $N = \delta^{1-n}$. Let there be N δ-tubes. Assume that if two tubes intersect, then they intersect in the unit ball. It holds for the number of order $\mu > 1$ intersections that*

$$\#(P_\mu^\delta) \leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)2^{n-1}|B_{n-1}(0,1)|\frac{N^{n/(n-1)}}{\mu}. \tag{1.1}$$

Corollary 1.2. Let $N = \delta^{1-n}$. Let there be N δ-tubes. Assume that if two tubes intersect, then they intersect in the unit ball. It holds for the number of intersections that

$$\#(P^\delta) \leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)2^{n-1}|B_{n-1}(0,1)|N^{n/(n-1)}. \tag{1.2}$$

We consider also a special case.

Theorem 1.3. *Let $N = \delta^{1-n}$. Let there be N δ-tubes. Assume that if two tubes intersect, then their central lines intersect in the unit ball. Then it holds for the number of order $\mu > 1$ intersections that*

$$\#(P_\mu^\delta) \leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)|B_{n-1}(0,1)|\frac{N^{n/(n-1)}}{\mu}. \tag{1.3}$$

Corollary 1.4. Let $N = \delta^{1-n}$. Let there be N δ-tubes. Assume that if two tubes intersect, then their central lines intersect in the unit ball. Then it holds for the number intersections that

$$\#(P^\delta) \leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)|B_{n-1}(0,1)|N^{n/(n-1)}. \tag{1.4}$$

For lines we have the following.

Corollary 1.5. Let there be N lines. Let for any four lines l_1, l_2, l_3 and l_4 hold that $||l_1 \cap l_2 - l_3 \cap l_4|| \geq N^{-1/(n-1)}$. Then it holds for the number of intersections of order μ that

$$\#(P_\mu) \leq \frac{|B(0,1+N^{-1/(n-1)})|}{|B(0,1)|}(1+2N^{-1/(n-1)})|B_{n-1}(0,1)|\frac{N^{n/(n-1)}}{\mu}. \tag{1.5}$$

Corollary 1.6. Let there be N lines. Let for any four lines l_1, l_2, l_3 and l_4 hold that $||l_1 \cap l_2 - l_3 \cap l_4|| \geq N^{-1/(n-1)}$. Then it holds for the number of intersections

$$\#(P) \leq \frac{|B(0,1+N^{-1/(n-1)})|}{|B(0,1)|}(1+2N^{-1/(n-1)})|B_{n-1}(0,1)|N^{n/(n-1)}, \tag{1.6}$$

Our other result is the following: a generalization of a lemma of Corbóda.

Lemma 1.7. *[A generalization of a lemma of Corbóda] For tube intersections of order $2^k > 1$ it holds that*

$$|\bigcap_{i=1}^{2^k} T_i| \lesssim \delta^{n-1}2^{-k/(n-1)}.$$

It's not hard to check that the above bound is essentially tight. Moreover, let $f \in L^1_{loc}(\mathbb{R}^n)$. For each tube in $B(0,2)$ define a as it's center of mass. Define the Kakeya maximal function
$f_\delta^* : S^{n-1} \to \mathbb{R}$ via

$$f_\delta^*(\omega) = \sup_{a\in\mathbb{R}^n} \frac{1}{T_\omega^\delta(a) \cap B(0,2)} \int_{T_\omega^\delta(a)\cap B(0,2)} |f(y)|\mathrm{d}y.$$

In this paper any constant can depend on dimension n. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

$$||f_\delta^*||_p \le C_\epsilon \delta^{-n/p+1-\epsilon}, \tag{1.7}$$

for all $\epsilon > 0$. Remarkably, a bound of the form (1.7) follows from a bound of the form

$$||\sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)}||_{p/(p-1)} \le C_\epsilon \delta^{-n/p+1-\epsilon}, \tag{1.8}$$

for all $\epsilon > 0$, and for any set of δ-separated of δ-tubes. See for example [4] or [2]. We will prove that

Theorem 1.8. *For a maximal set of δ-separated δ-tubes we have*

$$||\sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)}||_{n/(n-1)} \lesssim (\ln\frac{1}{\delta})^{(n-1)/n}\delta^{(n-1)/n},$$

It follows that, see for example [2], that

Corollary 1.9. Any Kakeya set has Hausdorff dimension at least $n-1$.

2. Previously known results

We will use the following bound for the pairwise intersections of δ-tubes:

Lemma 2.1 (Corbòda)**.** *For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n \cap B(0,2)$, we have*

$$|T^\delta_{\omega_i}(a) \cap T^\delta_{\omega_j}(b)| \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}.$$

A proof can be found for example in [2].

For any (spherical) cap $\Omega \subset S^{n-1}, |\Omega| \gtrsim \delta^{n-1}, \delta > 0$, define its δ-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an δ-separated subset of Ω.

Lemma 2.2. *In the notation just defined*

$$N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}.$$

Again, a proof can essentially be found in [2].

3. A proof of the generalization of the lemma of Corbóda

Let us define

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \le \sum_{i=1}^N 1_{T_i}(x) 1_{B(0,2)}(x) \le 2^{k+1}\}.$$

Let us suppose that $2^k = \delta^{-\beta}, 0 < \beta \le n-1$, and let‘s suppose that tube $T_{\omega'}$ intersecting $T_\omega \cap E_{2^k}$ has it‘s direction outside of a cap of size $\sim \delta^{n-1-\beta}$ on the unit sphere. Then the angle between T_ω and $T_{\omega'}$ is greater than $\sim \delta^{1-\beta/(n-1)}$. Thus by lemma 2.1 the intersection

$$|\bigcap_{i=1}^{2^k} T_i| \le |T_\omega \cap T_{\omega'} \cap E_{2^k}| \le |T_\omega \cap T_{\omega'}| \lesssim \delta^{n-1+\beta/(n-1)} = \delta^{n-1} 2^{-k/(n-1)}. \tag{3.1}$$

Thus, we can suppose that the directions in the intersection $E_{2^k} \cap T_\omega \cap T_{\omega'}$ belong to a cap of size $\sim \delta^{n-1+\beta}$. If we δ - separate the cap via lemma 2.2 we get that the cap can contain at most $\sim 2^k$ tube-directions. Thus, for any tube T_ω in the intersection there exists a tube $T_{\omega'}$, such that the angle between T_ω and $T_{\omega'}$ is $\sim \delta^{1-\beta/(n-1)}$ and the inequality (3.1) is valid. Thus we proved the lemma 1.7.

4. On the number of intersections of given order

Define the following set

$$E_\mu := \{x \in \mathbf{R}^n | \sum_{i=1}^{N} 1_{T_i^\delta}(x) = \mu\}. \tag{4.1}$$

So that

$$\begin{aligned} \mu|E_\mu| &= \int_{[-1,1]^n \cap E_{2^k}} \sum_{i=1}^{N} 1_{T_i^\delta}(x) = \sum_{i=1}^{N} \int_{B(0,1)\cap E_{2^k}} 1_{T_i^\delta}(x) \\ &\leq \delta^{n-1}|B_{n-1}(0,1)||B(0,1)|N, \end{aligned} \tag{4.2}$$

where $B_{n-1}(0,1)$ is the $n-1$-dimensional unit ball. We define an intersection $I_{j\mu}$ of order $\mu > 1$ as

$$I_{j\mu} := \bigcap_{i=1}^{\mu} T_{ij}^\delta.$$

So that

$$E_\mu = \bigcup I_{j\mu}$$

and

$$|E_\mu| = \sum_{j=1}^{M_\mu} |I_{j\mu}|.$$

Because the central lines of tubes intersect, it holds that $I_{j\mu} \cap B(0, 1+\delta)$ contains a δ-ball. So that

$$\delta^n |B(0,1)| \leq |I_{j\mu}|. \tag{4.3}$$

It follows from above (4.3) and from (4.2) that

$$\begin{aligned} \mu\delta^n|B(0,1)|\#(P_\mu^\delta) &\leq \mu \sum_{j=1}^{M_\mu} |I_{j\mu}| \leq \mu|E_\mu| \\ &\leq (1+2\delta)|B_{n-1}(0,1)||B(0,1+\delta)|\frac{N^{n/(n-1)}}{\mu}\delta^{n-1}N, \end{aligned}$$

where we replaced the length of the tube with $1+2\delta$. Thus,

$$\mu\delta^n\#(P_\mu^\delta) \leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)|B_{n-1}(0,1)|\delta^{n-1}N,$$

which is equivalent to

$$\delta\#(P_\mu^\delta) \leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)|B_{n-1}(0,1)|\frac{N}{\mu}. \tag{4.4}$$

We assumed in our theorem 1.3 that

$$N = \delta^{1-n},$$

so that

$$N^{-1/(1-n)} = \delta \tag{4.5}$$

Thus, it follows from (4.4) and (4.5) that

$$N^{-1/(n-1)}\#(P_\mu) \leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)|B_{n-1}(0,1)|\frac{N}{\mu}. \tag{4.6}$$

which is equivalent to

$$\#(P_\mu) \leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)|B_{n-1}(0,1)|\frac{N^{n/(n-1)}}{\mu}.$$

The above implies our theorem 1.3. If we have N lines that intersect on $B(0,R)$ then we can scale $\mathbf{R}^n$ s.t if the lines intersect they intersect in $B(0,1)$. Then we choose $\delta^{1-n} = N$ and form the tubes. If each tube intersection contains only one central line intersection, then the corollary 1.5 follows. This is the case if and only if the central line intersections are spaced by strictly more than δ. On the other way the implication is trivial. To see the if part, assume that we have the lines l_1, l_2, l_3 and L_4 and intersections $l_1 \cap L_2$ and $l_3 \cap l_4$ are spaced by $||l_1 \cap l_2 - L_3 \cap L_4|| > \delta$. Then it follows that $T_1^\delta \cap T_2^\delta \cap T_3^\delta \cap T_4^\delta = \emptyset$. Otherwise, we have two tubes intersecting without their central lines intersecting, because the central line intersections are unique. In order to prove 1.4 we will take μ dyadically. So that we have

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} 1_{T_i} < 2^{k+1}\}.$$

The set of intersections are now defined as

$$P_{2^k}^\delta := \bigcup_{j=1}^{M_{2^k}^\delta} \bigcap_{i=1}^{\sim 2^k} T_{ij},$$

$\#(p_{2^k}^\delta) = M_{2^k}$, and

$$|E_{2^k}| = \sum_j |I_{j2^k}|.$$

So we have

$$2^k\delta^n|B(0,1)|\#(P_{2^k}^\delta) \leq 2^k\sum_j |I_{j2^k}| = 2^k|E_{2^k}|$$

$$\leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)|B_{n-1}(0,1)|\delta^{n-1}N.$$

Thus, like before it follows that

$$\#(P_{2^k}^\delta) \leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)|B_{n-1}(0,1)|\frac{N^{n/(n-1)}}{2^k}.$$

But if we sum above over k we have

$$\#(P^\delta) = \sum_{k\neq 0}\#(P_{2^k}^\delta) \leq \sum_{k\neq 0}\frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)|B_{n-1}(0,1)|\frac{N^{n/(n-1)}}{2^k}$$

$$\leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)|B_{n-1}(0,1)|\frac{N^{n/(n-1)}}{\mu}N^{n/(n-1)}\sum_{k=1}^{\infty}\frac{1}{2^k}$$

$$= \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)|B_{n-1}(0,1)|N^{n/(n-1)}.$$

This proves 1.5. And again if we have N lines that intersect on $B(0,R)$ then we can scale $\mathbf{R}^n$ s.t the lines intersect in $B(0,1)$. Then we choose $\delta^{1-n}=N$ and put the lines as central lines of the tubes, and if each tube intersection contains only one central line intersection, then the corollary 1.6 follows from 1.5. This is the case when the line intersections are spaced by more than δ.
Now, let us scale δ to 2δ. Define the scaled versions $I_{j\mu}$ and E_μ as $I'_{j\mu}$ and E'_μ, respectively. It holds that $I'_{j\mu}\cap B(0,1+\delta)$ contains a δ-ball, even if the central lines of the tubes never meet. So that

$$\delta^n|B(0,1)|\leq |I'_{j\mu}|.$$

We define M'_u as the number of intersection of order μ of 2δ-tubes. Clearly

$$\#(P^\delta_\mu)\leq \#(P^{2\delta}),$$

because each intersection in E_μ is an intersection in $\bigcup_{\mu>1}E'_\mu$. Just like before it follows that

$$\mu\delta^n|B(0,1)|\#(P^\delta_\mu)\leq \int_{E_\mu\cap B(0,1+\delta)}\sum_{i=1}^N 1_{T_i^\delta}\leq \int_{\bigcup_{\mu>1}E'_\mu\cap B(0,1+\delta)}\sum_{i=1}^N 1_{T_i^{2\delta}}$$
$$\leq (1+2N^{-1/(n-1)})2^{n-1}|B_{n-1}(0,1)||B(0,1+\delta)|\delta^{n-1}N,$$

where we replaced the length of the tube with $1+2\delta$. Because $N=\delta^{1-n}$ it follows that

$$N^{-1/(n-1)}\#(P_\mu)\leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)2^{n-1}|B_{n-1}(0,1)|\frac{N}{\mu}.$$

which is equivalent to

$$\#(P_\mu)\leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)2^{n-1}|B_{n-1}(0,1)|\frac{N^{n/(n-1)}}{\mu}.$$

The above implies our main theorem 1.1.
In order to prove 1.2 we will again take μ dyadically. So that we have

$$E_{2^k}:=\{x\in\mathbf{R}^n|2^k\leq \sum_{i=1}^N 1_{T_i}\leq 2^{k+1}\}.$$

The set of intersections are now defined as

$$P^\delta_{2^k}:=\bigcup_{j=1}^{M^\delta_{2^k}}\bigcap_{i=1}^{\sim 2^k}T_{ij},$$

$\#(p^\delta_{2^k})=M_{2^k}$, and

$$|E'_{2^k}|=\sum_j |I'_{j2^k}|.$$

So we have

$$2^k\delta^n|B(0,1)|\#(P^\delta_{2^k})\leq \int_{E_{2^k}\cap B(0,1+\delta)}\sum_{i=1}^N 1_{T_i^\delta}\leq \int_{\bigcup_{k>0}E'_{2^k}\cap B(0,1+\delta)}\sum_{i=1}^N 1_{T_i^{2\delta}}$$
$$\leq (1+2N^{-1/(n-1)})2^{n-1}|B_{n-1}(0,1)||B(0,1+\delta)|\delta^{n-1}N$$

where I'_{j2^k} and E'_{2^k} are scaled versions of I_{j2^k} and E_{2^k}, respectively. Thus, like before it follows that

$$\#(P^{\delta}_{2^k}) \leq \frac{|B(0,1+\delta)|}{|B(0,1)|}2^{n-1}(1+2\delta)2^{n-1}|B_{n-1}(0,1)|\frac{N^{n/(n-1)}}{2^k}.$$

But if we sum above over k we have

$$\begin{aligned}\#(P^{\delta}) = \sum_{k\neq 0}\#(P^{\delta}_{2^k}) &\leq \sum_{k\neq 0}\frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)2^{n-1}|B_{n-1}(0,1)|\frac{N^{n/(n-1)}}{2^k}\\ &\leq \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)2^{n-1}|B_{n-1}(0,1)|\frac{N^{n/(n-1)}}{\mu}N^{n/(n-1)}\sum_{k=1}^{\infty}\frac{1}{2^k}\\ &= \frac{|B(0,1+\delta)|}{|B(0,1)|}(1+2\delta)2^{n-1}|B_{n-1}(0,1)|N^{n/(n-1)}.\end{aligned}$$

This proves 1.2.

5. The $\delta^{(n-1)/n}$ bound for the Kakeya maximal function

We notice that the dyadic decomposition contains about logarithmic many terms 2^k. We have from (1.8), from generalize lemma of Córdoba and from the theorem 1.1 that

$$||\sum_{\omega\in\Omega}1_{T_\omega(a_\omega)}||^{n/(n-1)}_{n/(n-1)} \sim \sum_k 2^{kn/(n-1)}|E_{2^k}| \lesssim \sum_k 2^{kn/(n-1)}N\delta^{n-1}2^{-k/(n-1)} \lesssim \ln\frac{1}{n}\delta^{-1}.$$

The above is equivalent to our theorem 1.8.

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