The advance of planets' perihelion in Newtonian theory plus gravitational and rotational time dilation

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Abstract

It is shown through three different approaches that, contrary to a long-standing conviction older than 160 years, the orbit of Mercury behaves as required by Newton's equations with a very high precision if one correctly analyzes the situation in the framework of the two-body problem without neglecting the mass of Mercury. General relativity remains more precise than Newtonian physics, but the results in this paper show that Newtonian framework is more powerful than researchers and astronomers were thinking till now, at least for the case of Mercury.

The Newtonian formula of the advance of planets' perihelion breaks down for the other planets. The predicted Newtonian result is indeed too strong for Venus and Earth. Therefore, it is also shown that corrections due to gravitational and rotational time dilation, in an intermediate framework which analyzes gravity between Newton and Einstein, solve the problem. By adding such corrections, a result consistent with the one of general relativity is indeed obtained.

Thus, the most important results of this paper are two: i) It is not correct that Newtonian theory cannot predict the anomalous rate of precession of the perihelion of planets' orbit. The real problem is instead that a pure Newtonian prediction is too strong. ii) Perihelion's precession can be achieved with the same precision of general relativity by extending Newtonian gravity through the inclusion of gravitational and rotational time dilation effects. This second result is in agreement with a couple of recent and interesting papers of Hansen, Hartong and Obers. Differently from such papers, in the present work the importance of rotational time dilation is also highlighted.

1 Introduction

Based on astronomical observations, in the early 1600s Kepler established that the orbit described by a planet in the solar system is an ellipse, with the Sun occupying one of its foci. Assuming that a planet is subject only to the gravitational attraction of the Sun, Kepler's result is easily obtained mathematically in Newton's theory. But the other planets also have a gravitational attraction on the planet in question. What is the effect of their presence? If one repeats the calculation taking into account this complication, one finds that the attraction exerted by all the other planets of the solar system on the planet in question induces an advance (a precession), orbit after orbit, of the perihelion (the point of maximum approach to the Sun of the orbit of the planet). The precession of the Earth's rotation axis also gives rise to the same effect. For example, Mercury's perihelion moves slightly at the speed of 5,600 arcseconds per century, in the same direction in which the planet rotates around the Sun. However, when the contribution of the Earth's precession is removed (5,025 arcseconds), that due to the attraction of the other planets, calculated according to Newtonian physics, is not able to correctly predict what happens in reality. The balance indeed misses 43 arcseconds. It is a general conviction, supported by centennial computations, that this deviation of Mercury's orbit from the observed precession cannot be achieved by Newtonian theory. This is the famous anomalous rate of precession of the perihelion of Mercury's orbit. It was originally recognized by the French Astronomer Urbain Le Verrier in 1859 as being an important astronomical problem [1]. Starting from 1843 [2], Le Verrier indeed reanalyzed various observations of the perihelion of Mercury's orbit from 1697 to 1848, by showing that the rate of the precession of the perihelion was not consistent with the previsions of Newtonian theory. This discrepancy by $38''$ arcseconds per tropical century, which has been corrected to $43''$ by the Canadian-American astronomer Simon Newcomb in 1882 [3], seemed till now impossible to be accounted through Newton's theory. Various ad hoc and unsuccessful solutions have been proposed, but such solutions introduced more problems instead. The most famous approach by 19th century astronomers was the attempting to explain this discrepancy through the perturbing effect of a planet, Vulcan, hitherto escaped observation, smaller than Mercury and closer than this to the Sun. However, the search for this planet turned out to be unfruitful. The solution of the problem is due to Albert Einstein through his magnificent general theory of relativity in 1916 [4]. Recent analyses due to the MESSENGER data plus the Cassini mission gave a value of about $42,98''$ to the general relativistic contribution to the precession of perihelion of Mercury per tropical century [5]. If one expresses the perihelion shift in radians per revolution (in this work, polar coordinates will be used), one gets instead the general relativistic value [6]

$$\triangle\varphi \simeq \frac{24\pi^3 a^2}{T_0^2 c^2 \left(1 - e^2\right)}, \tag{1}$$

where a is the semi-major axis of the orbit, T_0 is Mercury's Newtonian orbital period, c is the speed of light, and e is the orbital eccentricity. Eq. (1) corresponds to a total angle swept per revolution by Mercury

$$\varphi \simeq \varphi_0 \left(1 + \frac{12\pi^2 a^2}{T_0^2 c^2 \left(1 - e^2\right)}\right), \tag{2}$$

where $\varphi_0 = 2\pi$ is the unperturbed (i.e. in absence of precession) total angle swept by Mercury during a complete revolution around the Sun. Inserting the numerical values in Eq. (1), see for example [7–9], one gets the well known general relativistic value $\triangle\varphi \simeq 5.02*10^{-7}$ radians per revolution which corresponds to about $0,1$ arcseconds.

In next Sections, the precession of the perihelion of Mercury's orbit will be calculated in the Newtonian framework. Three different approaches will be considered and the analysis will show that the orbit of Mercury behaves as required by Newton's equations with a very high precision if one correctly analyzes the situation in the framework of the two-body problem without neglecting the mass of Mercury. General relativity remains more precise than Newtonian physics, but the results in next Sections will show that Newtonian framework is more powerful than researchers and astronomers were thinking till now, at least for the case of Mercury. On the other hand, the Newtonian formula of the advance of planets' perihelion breaks down for the other planets. The predicted Newtonian result is indeed too strong for Venus and Earth. In fact, it will be shown that corrections due to gravitational and rotational time dilation are necessary. By adding such corrections, the same result of general relativity is retrieved.

Hence, two interesting results will be obtained: i) It is not correct that Newtonian theory cannot predict the anomalous rate of precession of the perihelion of planets orbit. The real problem is instead that Newtonian prediction is too strong. ii) Perihelion's precession can be achieved with the same precision of general relativity by extending Newtonian gravity through the inclusion of gravitational and rotational time dilation effects. This second result is in agreement with the recent interesting works [13, 14], but, differently from such works, here the importance of rotational time dilation is also highlighted.

2 Approximation of circular orbit

One starts from the case in which Mercury's mass is considered negligible with respect to the mass of the Sun, i.e. one considers the planet as being a test mass immersed in the Newtonian gravitational field of the Sun. In addition, one considers Mercury's orbit as being circular instead of elliptical. Thus, the case under consideration here is the simplest one. One takes the origin of the frame of reference in the center of the Sun. By using the traditional Newtonian equations, in order to obtain the orbital period, one merely equals the gravitational force to the centripetal one as

$$\frac{GMm}{r_0^2} = \frac{mv_0^2}{r_0}, \tag{3}$$

where G is the gravitational constant, M is the solar mass, m the mass of Mercury, r_0 the orbit's radius and v_0 the velocity of rotation of the planet. Hence, v_0 is easily obtained as

$$v_0 = \left(\frac{GM}{r_0}\right).^{\frac{1}{2}} \tag{4}$$

Then, the Newtonian orbital period is

$$T_0 = \frac{2\pi r_0}{v_0} = \frac{2\pi r_0^{\frac{3}{2}}}{(GM)^{\frac{1}{2}}}. \tag{5}$$

The corresponding angular velocity is

$$\omega_0 = \frac{2\pi}{T_0}. \tag{6}$$

Thus, in radians per revolution the angular distance that Mercury sweeps during the Newtonian orbital period T_0 is

$$\varphi_0 = \omega_0 T_0 = 2\pi. \tag{7}$$

Now one asks: what does it happen if one removes the approximation to consider Mercury's mass negligible with respect to the solar mass? One argues that a Newtonian observer set in the center of the Sun must replace Eq. (3) with

$$\frac{G\left(M+m\right)m}{r_0^2} = \frac{mv^2}{r_0}, \tag{8}$$

i.e. one must replace M with $M+m$ in Eq. (3). Let us clarify this point. The Newtonian law of universal gravitation can be written down in its general form for Mercury and the Sun as

$$\overrightarrow{F}_G = \frac{GMm}{r^2}\hat{u}_r, \tag{9}$$

where r is the distance between the Sun and Mercury and $\hat{u}_r$ is the unit vector in the radial direction. Thus, for an external inertial Newtonian observer, the Newtonian equations of motion for the Sun and Mercury are

$$Ma_s\hat{u}_r = \frac{GMm}{r^2}\hat{u}_r \Longrightarrow a_s\hat{u}_r = \frac{Gm}{r^2}\hat{u}_r \tag{10}$$

and

$$ma_m\hat{u}_r = -\frac{GMm}{r^2}\hat{u}_r \Longrightarrow a_m\hat{u}_r = -\frac{GM}{r^2}\hat{u}_r, \tag{11}$$

respectively, where a_s is the acceleration of the Sun and a_m is the acceleration of Mercury. Thus, the relative acceleration of Mercury with respect to the Sun is

$$a\hat{u}_r \equiv a_m\hat{u}_r - a_s\hat{u}_r = -\left(\frac{GM}{r^2} + \frac{Gm}{r^2}\right)\hat{u}_r = -\frac{G\left(M+m\right)}{r^2}\hat{u}_r. \tag{12}$$

Then, the total force acting on Mercury as it is seen by a Newtonian observer set in the center of the Sun is

$$F\hat{u}_r = -\frac{G\,(M+m)\,m}{r^2}\hat{u}_r, \tag{13}$$

which immediately justify Eq. (8) for a circular motion. From Eq. (8) one gets immediately the perturbed velocity of rotation of the planet as

$$v = \left[\frac{G\,(M+m)}{r_0}\right]^{\frac{1}{2}} \tag{14}$$

corresponding to a period

$$T = \frac{2\pi r_0}{v} = \frac{2\pi r_0^{\frac{3}{2}}}{\left[G\,(M+m)\right]^{\frac{1}{2}}}. \tag{15}$$

But it is also

$$(M+m)^{-\frac{1}{2}} = M^{-\frac{1}{2}}\left(1+\frac{m}{M}\right)^{-\frac{1}{2}}, \tag{16}$$

which, inserted in Eq. (15), gives

$$T = \frac{2\pi r_0^{\frac{3}{2}}\left(1+\frac{m}{M}\right)^{-\frac{1}{2}}}{\left[G\,(M)\right]^{\frac{1}{2}}} = T_0\left(1+\frac{m}{M}\right)^{-\frac{1}{2}}. \tag{17}$$

Then, the corresponding perturbed angular velocity is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{T_0}\left(1+\frac{m}{M}\right)^{\frac{1}{2}} = \omega_0\left(1+\frac{m}{M}\right)^{\frac{1}{2}}. \tag{18}$$

Hence, the angle that Mercury sweeps during the period T_0 is

$$\varphi = \omega T_0 = 2\pi\left(1+\frac{m}{M}\right)^{\frac{1}{2}} \simeq 2\pi\left(1+\frac{m}{2M}\right), \tag{19}$$

in radians per revolution, where in the last step the first-order approximation in $\frac{m}{M}$ has been used, that is $\left(1+\frac{m}{M}\right)^{\frac{1}{2}} \simeq 1+\frac{m}{2M}$, because it is $m \ll M$. Therefore, in each complete revolution around the Sun, Mercury sweeps an angle larger than the unperturbed angle (7) and the difference, in radians per revolution, is

$$\triangle\varphi = \varphi - \varphi_0 \simeq \frac{\pi m}{M}. \tag{20}$$

The NASA official data give $m \simeq 3.3 * 10^{23} Kg$ [8] and $M = 1,99 * 10^{30} Kg$ [7]. Thus, one gets $\triangle\varphi \simeq 5.21 * 10^{-7}$radians per revolution which corresponds to about $0,107$ arcseconds. On the other hand, the Mercury/Earth ratio of the tropical orbit periods is 0.241[9]. Thus, one gets $44.39''$ per tropical century. This is a remarkable result that shows that, despite the above calculation has been made in the approximation of circular orbit, the correct value of the contribution of Newtonian theory to the precession of perihelion for Mercury per tropical century well approximates the value of about $42,98''$ per tropical century of general relativity [5] and the well known observational value of $43''$ per tropical century.

The physical interpretation of this nice result is that it is Mercury's back reaction, in terms of Newton's third law of motion (*to every action there is always opposed an equal reaction*), see Eqs. from (9) to (13), that generates the advance of the perihelion of Mercury in Newtonian framework.

3 Mercury's orbit as harmonic oscillator

Following [10], one recalls that each central attractive force can produce a circular orbit that should not necessarily be closed. It is closed if the radial oscillation period is a rational multiple of the orbit period. Now, let $F_c(r)$ be the total central force. Mercury's equation of motion in the radial direction is given by [10]

$$F_c(r) = m\left(\ddot{r} - \dot{\theta}^2 r\right), \tag{21}$$

where, again, r is the distance between the Sun and Mercury for an observer in the center of the Sun. The last term in Eq. (21) can be physically interpreted as a force centrifuge. Since the angular momentum J is a constant of motion, one has that

$$J = mr^2\dot{\theta}. \tag{22}$$

Solving for $\dot{\theta}$ and substituting in Eq. (21), one gets

$$F_c(r) = m\left(\ddot{r} - \frac{J^2}{m^2 r^3}\right). \tag{23}$$

In the case of a circular orbit of radius r_0, $\ddot{r} = 0$ and Eq. (23) reduces to

$$F_c(r_0) = -\frac{J^2}{m r_0^3}. \tag{24}$$

If Mercury is now slightly perturbed in the plane of its orbit and perpendicularly to its initial trajectory, it will oscillate around r_0 [10]. Then, one introduces $x = r - r_0$ and expresses the radial equation of motion in terms of x. Therefore [10]

$$\begin{aligned} F_c\left(x + r_0\right) &= m\ddot{x} - \tfrac{J^2}{m(x+r_0)^3} \\ &= m\ddot{x} - \tfrac{J^2}{m r_0^3\left(1+\frac{x}{r_0}\right)^3}. \end{aligned} \tag{25}$$

Since $\frac{x}{r_0} \ll 1$, one can use series expansion for the term in parentheses, considering only the first order terms in $\frac{x}{r_0}$. Expanding the member on the left in

Taylor series around the point $r = r_0$ one gets [10]

$$F_c(r_0) + F_c'(r_0) = m\ddot{x} - \frac{J^2}{mr_0^3}\left(1 - \frac{3x}{r_0}\right). \tag{26}$$

Inserting Eq. (24) in Eq. (26) one obtains [10]

$$\ddot{x} + m^{-1}\left[-\frac{3F_c(r_0)}{r_0} - F_c'(r_0)\right]x = 0 \tag{27}$$

One notes that this equation describes a simple harmonic oscillator if the term in parentheses is positive [10]. If this term was negative, there would be an exponential solution and the orbit would not be stable [10]. Thus, for stable orbits, the period of oscillation around $r = r_0$ is [10]

$$T_0 = 2\pi\left(\frac{m}{-\frac{3F_c(r_0)}{r_0} - F_c'(r_0)}\right)^{\frac{1}{2}}. \tag{28}$$

One defines the apse angle $\frac{\varphi_0}{2}$ as the angle swept by the radial vector between two consecutive points of the orbit where the radial vector itself takes on an extremal value [10]. The time that Mercury needs to travel this angle is $\frac{T_0}{2}$. Since the orbit can be considered approximately circular and being therefore constant r and equal to r_0, one solves Eq. (22) for $\dot{\theta}$ and finds [10]

$$\frac{\varphi_0}{2} = \frac{T_0}{2}\dot{\theta} = \pi\left(\frac{m}{-\frac{3F_c(r_0)}{r_0} - F_c'(r_0)}\right)^{\frac{1}{2}}\frac{J}{mr_0^2}. \tag{29}$$

Furthermore, observing Eq. (24), one notes that the last term of Eq. (29) can be rewritten as [10]

$$\frac{J}{mr_0^2} = \left(-\frac{F_c(r_0)}{mr_0}\right)^{\frac{1}{2}}. \tag{30}$$

Then, one gets [10]

$$\varphi_0 = 2\pi\left[3 + \frac{F_c'(r_0)}{F_c(r_0)}\right]^{-\frac{1}{2}}, \tag{31}$$

and, by setting $F_c = F_G$ in Eq. (31), where F_G is the Newtonian gravitational force given by Eq. (9), one finds $\varphi_0 = 2\pi$, which is exactly Eq. (7).

But, again, in the computation in this Section Mercury's mass has been considered negligible with respect to the mass of the Sun. A good way to take into account the presence of Mercury's mass is to work in the framework of the two-body problem. The two-body problem studies the dynamics of a system consisting of two massive objects (the Sun having mass M and Mercury having mass m in the present case) subjected to a central force. Central force is defined as a force that only depends by the modulus of the difference of the vectors position of the two objects and which is directed along the junction of the two bodies. The expression of this kind of force is well known:

$$\overrightarrow{F} = F\left(|r_m - r_M|\right)\frac{\overrightarrow{r}_m - \overrightarrow{r}_M}{|\overrightarrow{r}_m - \overrightarrow{r}_M|}, \tag{32}$$

where r_m and r_M are the positions of the two objects of mass m and M respectively, that are subject to the central force of Eq. (32) in an inertial reference system. One introduces the variables relative position, r, and position of the center of mass, R. In this way, it is always possible to approach to the general two-body problem with two independent problems through the following change of variables:

$$\begin{aligned} \overrightarrow{R} &= \tfrac{m\overrightarrow{r}_m + M\overrightarrow{r}_M}{M+m} \\ \overrightarrow{r} &= \overrightarrow{r}_m - \overrightarrow{r}_M. \end{aligned} \tag{33}$$

With this change of variables the positions of Mercury and the Sun can be written as:

$$\begin{aligned} \overrightarrow{r}_m &= \overrightarrow{R} + \tfrac{M}{M+m}\overrightarrow{r} \\ \overrightarrow{r}_M &= \overrightarrow{R} - \tfrac{m}{M+m}\overrightarrow{r}. \end{aligned} \tag{34}$$

One also defines $M_T \equiv M + m$ and $\mu \equiv \frac{Mm}{M+m}$ as the total mass and the reduced mass of the system, respectively. It is well known that the problem of the dynamics of two bodies of masses m and M interacting through one force that depends only on mutual distance is reduced to the problem of a single body of reduced mass μ that moves in space under the action of a central field. In other words, in order to have a more precise description of the Sun-Mercury system one makes the replacement $m \to \mu$ in Eqs. from (21) to (31). In particular, Eqs. (29) and (30) now read

$$\frac{\varphi}{2} = \frac{T}{2}\dot{\theta} = \pi\left(\frac{\mu}{-\frac{3F_c(r_0)}{r_0} - F_c'(r_0)}\right)^{\frac{1}{2}}\frac{J}{\mu r_0^2}. \tag{35}$$

and

$$\frac{J}{\mu r_0^2} = \left(-\frac{F_c(r_0)}{\mu r_0}\right)^{\frac{1}{2}}, \tag{36}$$

respectively. To first order in $\frac{m}{M}$ the reduced mass can be rewritten as

$$\begin{aligned} \mu &= \left(\tfrac{M+m}{Mm}\right)^{-1} = \left(\tfrac{1}{m} + \tfrac{1}{M}\right)^{-1} \\ &= m\left(1 + \tfrac{m}{M}\right)^{-1} \simeq m\left(1 - \tfrac{m}{M}\right). \end{aligned} \tag{37}$$

Thus, Eq. (35) becomes

$$\begin{aligned}\frac{\varphi}{2} = \frac{T}{2}\dot{\theta} &\simeq \pi\left(\frac{m\left(1-\frac{m}{M}\right)}{-\frac{3F_c(r_0)}{r_0}-F_c'(r_0)}\right)^{\frac{1}{2}}\frac{J}{m\left(1-\frac{m}{M}\right)r_0^2}\\ &\simeq \pi\left(1+\frac{m}{2M}\right)\left(\frac{m}{-\frac{3F_c(r_0)}{r_0}-F_c'(r_0)}\right)^{\frac{1}{2}}\frac{J}{mr_0^2}.\end{aligned} \tag{38}$$

Then, one gets

$$\varphi = 2\pi\left(1+\frac{m}{2M}\right)\left[3+\frac{F_c'(r_0)}{F_c(r_0)}\right]^{-\frac{1}{2}}, \tag{39}$$

where now it is

$$\begin{aligned}F_c &= \frac{GM_T\mu}{r^2}\\ &= \frac{G(M+m)\left(\frac{Mm}{M+m}\right)}{r^2} = F_G\end{aligned} \tag{40}$$

and one finds

$$\varphi = 2\pi\left(1+\frac{m}{2M}\right), \tag{41}$$

which is the same result of Eq. (19).

4 Weak deviation from third Kepler's law

In order to work again in the framework of the two-body problem, one starts by replacing $m \to \mu$ in Eq. (22), obtaining

$$J = \mu r^2\dot{\theta} = 2\mu\dot{A}_0, \tag{42}$$

where A is the area swept by $\vec{r}$ during the orbital motion. Thus, one obtains

$$J = 2\mu\frac{dA}{dt} \quad and \quad dt = 2\mu\frac{dA}{J}. \tag{43}$$

Then, by integration over a period, one obtains

$$T = 2\mu\frac{A}{J}. \tag{44}$$

Recalling that the generic expression for the area of a conic is given by

$$A = \pi a^2\left(1-e\right)^{\frac{1}{2}}, \tag{45}$$

where a and e are the semi-major axis and the eccentricity of the ellipse, respectively, one substitutes for (44) and gets

$$T = 2\pi\mu\frac{a^2\left(1-e\right)^{\frac{1}{2}}}{J}. \tag{46}$$

Also remembering that it is

$$\frac{J^2}{\mu k} = a\,(1-e)\,, \tag{47}$$

one obtains

$$J = [\mu a k\,(1-e)]^{\frac{1}{2}}\,. \tag{48}$$

Then, by inserting Eq. (48) in Eq. (46) and by using a bit of algebra, one gets

$$T = 2\pi\left(\frac{a^3\mu}{k}\right)^{\frac{1}{2}}. \tag{49}$$

As it is $k = GMm$ for the gravitational system of Mercury and the Sun, Eq. (49) becomes

$$T = 2\pi\left(\frac{a^3}{GM_T}\right)^{\frac{1}{2}}, \tag{50}$$

where $M_T = M + m$ is the total mass of the system. Hence, from Eq. (50) one easily obtains

$$\begin{aligned}\tfrac{a^3}{T^2} &= \tfrac{GM_T}{4\pi^2}\\ &= \tfrac{G(M+m)}{4\pi^2} = \tfrac{GM}{4\pi^2}\left(1+\tfrac{m}{M}\right),\end{aligned} \tag{51}$$

and one immediately sees that Kepler's third law, that is "*the ratio between* T^2 *and* a^3 *is constant for each planet in the solar system, depending only on the mass of the Sun and not from that of the planet*", i.e.

$$\frac{a_0^3}{T_0^2} = \frac{GM}{4\pi^2}, \tag{52}$$

is strictly correct only in the approximation $m \ll M$, when the mass of the planet is considered negligible with respect to the solar mass. a_0 and T_0 in Eq. (52) are the unperturbed semi-major axis and the unperturbed period of revolution of the ellipse, respectively. Therefore, if one considers the mass of the planet as being not negligible with respect to the solar mass, Eq. (51) shows that there is a weak deviation from Kepler's third law in Newtonian gravitation. Combining Eqs. (52) and (51) one obtains

$$\begin{aligned}\tfrac{a^3}{T^2} &= \tfrac{a_0^3}{T_0^2}\left(1+\tfrac{m}{M}\right)\\ \Longrightarrow \tfrac{a^3}{a_0^3} &= \tfrac{T^2}{T_0^2}\left(1+\tfrac{m}{M}\right).\end{aligned} \tag{53}$$

On the other hand, the variation of the angle merely makes the ellipse precess [11]. This means that the shape and area of the ellipse must remain unchanged. Hence, a necessary condition is $a = a_0$. By inserting this in Eq. (53), one immediately gets

$$T = \frac{T_0}{\left(1+\frac{m}{M}\right)^{\frac{1}{2}}}, \tag{54}$$

which is exactly the result of Eq. (17) that was obtained in Section 2 in the approximation of circular orbity. One also easily checks that Eq. (54) is consistent with Eq. (38) in Section 3 too. Thus, the corresponding perturbed angular velocity is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{T_0}\left(1+\frac{m}{M}\right)^{\frac{1}{2}} = \omega_0\left(1+\frac{m}{M}\right)^{\frac{1}{2}}. \tag{55}$$

Hence, the angle that Mercury sweeps during the period T_0 is

$$\varphi = \omega T_0 = 2\pi\left(1+\frac{m}{M}\right)^{\frac{1}{2}} \simeq 2\pi\left(1+\frac{m}{2M}\right), \tag{56}$$

in radians per revolution, where in the last step the first-order approximation in $\frac{m}{M}$ has been used exactly like in previous Sections. The result of Eq. (56) is the same as that of Eqs. (19) and (41), but the analysis in this Section is more precise because it has been performed in the framework of the two-body problem and considering the exact elliptical orbit of Mercury.

5 Breakdown of Newtonian formula

One applies Eq. (20) to Venus trajectory. The mass of Venus is $m_V \simeq 4.87 * 10^{24} Kg$ [15]. Thus, one gets a value of $\triangle\varphi \simeq 7.68 * 10^{-6}$ radians per revolution that corresponds to about 1.6 arcseconds. The Venus /Earth ratio of the tropical orbit periods is $0,615$ [15]. Hence, one gets $258,16''$ per tropical century that is about 30 times larger than the observational value of $8.62''$ [12]. Similar results are obtained if one considers Heart's data. In that case, the mass of Heart is $m_T \simeq 5.97 * 10^{24} Kg$, that implies a precession value of $\triangle\varphi \simeq 9.42 * 10^{-6}$ radians per revolution that corresponds to about 1.94 arcseconds. Then, one gets $194''$ per tropical century that is about 50 times larger than the observational value of $3.83''$ [12].

Thus, it has been shown that, contrary to a longstanding conviction older than 160 years, the real problem of Newtonian theory concerning the anomalous rate of precession of the perihelion of planets orbit is not the absence of a prediction. Instead, the real problem with Newtonian physics is that such a prediction is too strong. A key point is the following. In Newtonian physics, time is absolute, so time passes in the same way in one reference frame as in the other reference frame. It has been recently shown [13, 14] that, in the framework of the anomalous rate of precession of the perihelion of planets orbit as well as in the other two classical tests of general relativity, namely deflection of light and gravitational redshift, gravitational time dilation effects must be taken into account. Here, one realizes a different analysis, in an intermediate framework which analyzes gravity between Newton and Einstein. It will be indeed shown in the following Sections that an improved version of Eq. (20) that takes into account both gravitational and rotational time dilation will achieve with a high precision the advance of perihelion completely consistent with the general relativistic result. For the sake of simplicity, this approach will be performed in the approximation of circular orbit.

6 Gravitational time dilation

One starts by considering Eq. (17). The period T in such an equation is measured by a Newtonian observer that sees time as absolute. But, based on gravitational time dilation, the period that is measured by a relativistic observer is *different*. In a general relativistic framework, gravitational time dilation is well approximated by [11]

$$t_g = \sqrt{g_{00}(r_0)}t_l, \tag{57}$$

where g_{00} is the coefficient of the coordinate time in the metric describing the gravitational field, t_g is the proper time between two events for an observer deep within the gravitational field and t_l is the coordinate time between the events for an observer at an arbitrarily large distance r_0 from the source of the gravitational field. Thus, t_l has a role similar to the role of the absolute Newtonian time because it is not affected by the gravitational field. Following [16], in a weak field approximation one considers a locally inertial, non-rotating, freely falling coordinate system with origin at the Sun's center, and writes an approximate solution of Einstein's field equations in isotropic coordinates

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)(cdt)^2 - \left(1 + \frac{2GM}{rc^2}\right)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{58}$$

where r, θ, ϕ are spherical polar coordinates. By using the coordinates of Eq. (58), Eq. (57) for the the Sun's gravitational field reads

$$t_g = \sqrt{1 - \frac{r_g}{r_0}}t_P \simeq \left(1 - \frac{1}{2}\frac{r_g}{r_0}\right)t_l \tag{59}$$

where $r_g = \frac{2M}{c^2}$ is the Sun's gravitational radius and r_0 the radial distance between the Sun and the planet. One considers the variations due to time dilation as being corrections with respect to the Newtonian observer. In other words, one defines a new observer having again the origin of the frame of reference in the center of the Sun, which sees again the spatial directions as being a Newtonian observer, but measures the time between two events by using t_g in Eq. (59) instead of the absolute Newtonian time. One defines this new observer as "Corrected Newtonian Observer" (CNO). Then, one also notes that, together with this time dilation, there is also a variation of the proper radial distance between the Sun and the planet with respect to a Newtonian observer. If this latter observer measures a distance, say $r_0 = ct_0$, the CNO will measure a distance $ct_0\sqrt{1 - \frac{r_g}{r_0}} \simeq r_0\left(1 - \frac{1}{2}\frac{r_g}{r_0}\right)$. The correction to the proper radial distance between the Sun and the planet due to spatial curvature, which is given by the opposite of the coefficient $g_{11} = -\left(1 + \frac{2GM}{rc^2}\right)$ in the line element of Eq. (58), see [22], will not be considered because one is assuming that the CNO sees the spatial directions as being a pure Newtonian observer, which means that the spatial directions are considered Euclidean. Then, one performs the following replacements in Eq. (17)

$$r_0 \to r_0\left(1 - \frac{1}{2}\frac{r_g}{r_0}\right) \qquad T_0\left(1 + \frac{m}{M}\right)^{-\frac{1}{2}} \to T_0\left(1 + \frac{m}{M}\right)^{-\frac{1}{2}}\left(1 - \frac{1}{2}\frac{r_g}{r_0}\right), \tag{60}$$

that means that the CNO must replace in Eq. (17) the original time and distances with time and distances corrected by Eq. (60). Therefore, for the CNO Eq. (17) becomes

$$
\begin{aligned}
T_F &= \frac{2\pi r_0^{\frac{3}{2}}\left(1-\frac{1}{2}\frac{r_g}{r_0}\right)^{\frac{3}{2}}\left(1+\frac{m}{M}\right)^{-\frac{1}{2}}}{[G(M)]^{\frac{1}{2}}} \\
&\simeq \left(1-\frac{3}{4}\frac{r_g}{r_0}\right)\left(1-\frac{1}{2}\frac{r_g}{r_0}\right)T_0\left(1-\frac{m}{2M}\right),
\end{aligned}
\tag{61}
$$

where T_F is the *final* perturbed orbital period measured by the CNO and $T_0\left(1-\frac{m}{2M}\right)$ is the unperturbed (with respect to the relativistic corrections of Eq. (60)) orbital period. Then, the corresponding perturbed angular velocity, which is seen by the CNO, is

$$
\omega_F = \frac{2\pi}{T_F} \simeq \omega\left(1+\frac{3}{4}\frac{r_g}{r_0}\right)\left(1+\frac{1}{2}\frac{r_g}{r_0}\right), \tag{62}
$$

where ω is given by Eq. (18). Hence, the final angle that Mercury sweeps during the period $T_0\left(1-\frac{m}{2M}\right)$ is

$$
\begin{aligned}
\varphi_F = \omega_F T_0\left(1-\frac{m}{2M}\right) &\simeq 2\pi\left(1+\frac{3}{4}\frac{r_g}{r_0}\right)\left(1+\frac{1}{2}\frac{r_g}{r_0}\right) \\
&\simeq 2\pi\left(1+\frac{5}{4}\frac{r_g}{r_0}\right),
\end{aligned}
\tag{63}
$$

in radians per revolution, where in the above computations the first-order approximation in $\frac{r_g}{r}$ and $\frac{m}{M}$ have been used. Thus, one finally obtains

$$
\triangle\varphi_F \simeq \frac{5\pi}{2}\frac{r_g}{r_0}. \tag{64}
$$

7 Rotational time dilation

Also a rotational effect has to be considered. Rotation generates indeed another dilation effect, see for example [17–20]. In order to understand the necessity of an additional effect of rotational dilation, one stresses that time differences of the previous Section have been calculated by the CNO that sees the planet as being at rest. Then, a key point is that the planet is moving instead. Thus, the CNO must consider an additional effect due to rotational time dilation, which has a very longstanding history. Such a history started from a famous paper of Einstein [21]. In the context of general relativity, from the historical point of view it was during his analysis of the rotating frame that Einstein had the intuition to represent the gravitational field in terms of space-time curvature [21]. Einstein indeed wrote, verbatim [21]:

"*The following important argument also speaks in favor of a more relativistic interpretation. The centrifugal force which acts under given conditions of a body is determined precisely by the same natural constant that also gives its action in* *a gravitational field. In fact we have no means to distinguish a centrifugal field from a gravitational field. We thus always measure as the weight of the body on the surface of the earth the superposed action of both fields, named above, and we cannot separate their actions. In this manner the point of view to interpret the rotating system K' as at rest, and the centrifugal field as a gravitational field, gains justification by all means. This interpretation is reminiscent of the original (more special) relativity where the pondermotively acting force, upon an electrically charged mass which moves in a magnetic field, is the action of the electric field which is found at the location of the mass as seen by the reference system at rest with the moving mass.*"

This interpretation by Einstein of the rotating system in terms of a gravitational field permitted various general relativistic analysis of Mössbauer rotor experiments [17–19] and Sagnac experiments [20]. The key point of the above highlighted interpretation by Einstein is the Einstein's Equivalence Principle (EEP) which enables the equivalence between gravitation and inertial forces [17–20]. Following [17–20], in a full general relativistic analysis one considers a transformation from an inertial coordinate system, with the $z-axis$ perpendicular to the plane of the rotational motion, to a second coordinate system, which rotates around the $z-axis$ in cylindrical coordinates. For a flat Lorentzian coordinate system the metric is [17–20]

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2. \tag{65}$$

One performs a transformation to a reference frame $\{t', r', \phi' z'\}$, which has constant angular velocity ω around the $z-axis$, obtaining [17–20]

$$t = t' \quad r = r' \quad \phi = \phi' + \omega t' \qquad z = z' \ . \tag{66}$$

Thus, one gets the well known Langevin line-element for the rotating reference frame [17–20]

$$ds^2 = \left(1 - \frac{r'^2\omega^2}{c^2}\right) c^2 dt'^2 - 2\omega r'^2 d\phi' dt' - dr'^2 - r'^2 d\phi'^2 - dz'^2. \tag{67}$$

Through the above discussed EEP, the metric (67) can be interpreted as a static "gravitational field" [17–20], following Einstein's original idea [21]. Then, the EEP permits to consider the inertial force that a rotating observer experiences as if the same observer is subjected to a gravitational "force" [17–21].

Then, one can apply Eq. (57) to the Langevin line-element of Eq. (67) obtaining,

$$d\tau = \sqrt{\left(1 - \frac{r^2\omega^2}{c^2}\right)} dt \simeq \left(1 - \frac{1}{2}\frac{r^2\omega^2}{c^2}\right) dt, \tag{68}$$

where, in the rotating frame, τ is the proper time between two events for the rotating observer at a distance r from the origin and having angular velocity ω, and t is the coordinate time between the events for an observer at the origing of the coordinate system. The primes on t' and r' has been dropped in Eq.

(68) in order to just use the symbols t and r'. In fact, from Eq. (66) one has $t = t'$ $r = r'$. In the current case one fixes $r = r_0$, $\omega = \omega_0$ obtaining

$$\tau = \sqrt{1 - \frac{{r_0}^2\omega_0^2}{c^2}} t \simeq \left(1 - \frac{1}{2}\frac{{r_0}^2\omega_0^2}{c^2}\right) t. \tag{69}$$

Thus, the CNO defined in the previous Section must also consider the correction of Eq. (69).

8 Total correction due to time dilation

Using Eqs. (5) and (6), Eq. (69) can be rewritten as

$$\tau = \sqrt{1 - \frac{1}{2}\frac{r_g}{r_0}} t \simeq \left(1 - \frac{1}{4}\frac{r_g}{r_0}\right) t. \tag{70}$$

This means that the CNO must make the replacement $T_F \rightarrow T_T = T_F \left(1 - \frac{1}{4}\frac{r_g}{r_0}\right)$ in Eq. (61), which now becomes

$$\begin{aligned} T_F &= \frac{2\pi r_0^{\frac{3}{2}} \left(1 - \frac{1}{2}\frac{r_g}{r_0}\right)^{\frac{3}{2}} \left(1 + \frac{m}{M}\right)^{-\frac{1}{2}} \left(1 - \frac{1}{4}\frac{r_g}{r_0}\right)}{[G(M)]^{\frac{1}{2}}} \\ &\simeq \left(1 - \frac{3}{4}\frac{r_g}{r_0}\right)\left(1 - \frac{1}{2}\frac{r_g}{r_0}\right)\left(1 - \frac{1}{4}\frac{r_g}{r_0}\right) T_0 \left(1 - \frac{m}{2M}\right) \end{aligned} \tag{71}$$

Then, Eq. (62)

$$\begin{aligned} \omega_F = \frac{2\pi}{T_F} &\simeq \omega \left(1 + \frac{3}{4}\frac{r_g}{r_0}\right)\left(1 + \frac{1}{2}\frac{r_g}{r_0}\right)\left(1 + \frac{1}{4}\frac{r_g}{r_0}\right) \\ &= \omega \left(1 + \frac{3}{2}\frac{r_g}{r_0}\right), \end{aligned} \tag{72}$$

where, again, in the above computations the first-order approximation in $\frac{r_g}{r}$ and $\frac{m}{M}$ have been used. Therefore, Eqs. (63) and become

$$\varphi_F = \omega_F T_0 \left(1 - \frac{m}{2M}\right) \simeq 2\pi \left(1 + \frac{3}{2}\frac{r_g}{r_0}\right), \tag{73}$$

and

$$\triangle\varphi_F \simeq 3\pi \frac{r_g}{r_0}. \tag{74}$$

One also stresses that there is no variation of the proper radial distance between the Sun and the planet due to rotational time dilation. To understand this, one notes that, despite in the above analysis the EEP has been used, Eq. (66) represents a coordinate transform between two different observers. If the observer in the rotating frame uses the EEP in order to realize a general relativistic analysis, the Lorentzian observer that uses the coordinates of Eq. (65) sees no "gravitational field". Instead, such a Lorentzian observer sees the opposite time dilation predicted by special relativity and one well knows from the Lorentz transformations that, in special relativity, one has proper space dilation *only* in the direction of the motion. In the current case, the planet moves in a transverse direction with respect to the radial coordinate. Thus, the variation of the proper distance, which is measured by the CNO, is not in the radial direction. This is consistent with the issue that for the general relativistic rotating observer that uses the coordinates of Eq. (67) the propagation of light is not radial, see [17]. This means that such a general relativistic rotating observer sees variation of the proper distance in the transverse direction, i.e. in the direction which is perpendicular to the radial direction. In fact, also the opposite of the coefficient g_{11} in the line element of Eq. (67) is equal to one.

One notes that the final result of Eq. (74) is completely consistent with the result of general relativity. In fact, Eq. (1) can be rewritten as [11]

$$\triangle\varphi \simeq \frac{3\pi r_g}{a\left(1-e^2\right)}, \tag{75}$$

and, for a circular motion, it is $a\left(1-e^2\right)=r_0$.

9 Conclusion remarks

It has been shown through three different approaches that, contrary to a long-standing conviction, older than 150 years, the orbit of Mercury behaves as required by Newton's equations with a very high precision if one correctly analyzes the situation in the framework of the two-body problem without neglecting the mass of Mercury. The results obtained are remarkable. The real value predicted by Newtonian theory concerning the advance of the perihelion of Mercury is of $44.39''$ per tropical century that well approximates the value of about $42,98''$ per tropical century of general relativity and the well known observational value of $43''$ per tropical century. Thus, the real difference between Einstein's and Newton's prevision concerning the advance of the perihelion of Mercury is not of about $43''$ as astronomers and researchers were thinking for more than 100 years. Instead, such a difference is only $1.41''$ per tropical century. The physical interpretation of this result is that it is Mercury's back reaction, in terms of Newton's third law of motion, that generates the advance of the perihelion of Mercury in Newtonian framework. General relativity remains more precise than Newtonian theory regarding the precession of Mercury's perihelion, but the difference is very little.

The Newtonian formula of the advance of planets' perihelion breaks down for the other planets. This means that the predicted Newtonian result is too strong for Venus and Earth. Thus, it has also been shown that corrections due to gravitational and rotational time dilation, in an intermediate framework which analyzes gravity between Newton and Einstein, solve the problem. In fact, by adding such corrections, a result consistent with the one of general relativity has been obtained.

Summarizing, the most important results of this paper are two: i) It is not correct that Newtonian theory cannot predict the anomalous rate of precession of the perihelion of planets' orbit. The real problem is instead that a pure Newtonian prediction is too strong. ii) Perihelion's precession can be achieved with the same precision of general relativity by extending Newtonian gravity through the inclusion of gravitational and rotational time dilation effects. This second result is in agreement with a couple of recent and interesting papers of Hansen, Hartong and Obers [13, 14]. The difference with such papers is that in the present work the importance of rotational time dilation has been additionally highlighted.

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