Quantum Entropy Conservation and Photon Quantum Equation

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Abstract: I try to prove that the entropy is a invariant in a isolated system. I write a possible photon quantum equation to obtain the associated flow of entropy in the gauge bosons.

The energy and entropy (for reversible processes, or time reversal symmetry) of an isolated classical system is conserved.

I think that this is true for an isolated-reversible quantum system.

The Von Neumann quantum entropy is:

$$
\begin{aligned}
&S(t) = -\operatorname{Tr}\left[\widehat{\rho}(t)\ln(\widehat{\rho}(t))\right] = -\operatorname{Tr}\left[\widehat{U}\,\widehat{\rho}\,\widehat{U}^\dagger \ln\left(\widehat{U}\,\widehat{\rho}\,\widehat{U}^\dagger\right)\right] = \\
&= -\operatorname{Tr}\left[\widehat{U}\,\widehat{\rho}\,\widehat{U}^\dagger \sum_n \frac{(-1)^{n-1}}{n}\left(\widehat{U}\,\widehat{\rho}\,\widehat{U}^\dagger - \widehat{U}\,\widehat{U}^\dagger\right)^n\right] = -\operatorname{Tr}\left[\sum_n \frac{(-1)^{n-1}}{n}\widehat{U}\,\widehat{\rho}\,(\widehat{\rho}-1)^n\,\widehat{U}^\dagger\right] = \\
&= -\operatorname{Tr}\left[\sum_n \frac{(-1)^{n-1}}{n}\widehat{\rho}\,(\widehat{\rho}-1)^n\right] = -\operatorname{Tr}\left[\widehat{\rho}(t_0)\ln(\widehat{\rho}(t_0))\right] = S(t_0)
\end{aligned}
$$

so that the entropy of an isolated-reversible system, with the unitary transformation $\widehat{U}(\widehat{H}) = e^{-i\frac{\widehat{H}}{\hbar}t}$, is conserved[1]; this is true for each analytic function $f(t) = \operatorname{Tr}\left[\widehat{\rho}(t) f(\widehat{\rho}(t))\right]$, so that there is an infinity of invariants of motion.

If there is a variation of the measured quantum entropy, then there must be unobserved particles that hold the total value: the gauge bosons.

For example the black hole emits gauge bosons (Hawking radiations, gravitons) that contain information on the evolution of the black hole.

This is true for the Universe, that contain itself, so that the entropy of the Universe, with the gauge bosons, could be an invariant.

The simplest gauge boson is the photon, but it is necessary a quantum equation for the photon to evaluate the entropy from a wavefunction.

I write the photon quantum equation deriving it from the Lagrangian[2] of the electromagnetic field

$$
\begin{aligned}
&F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \\
&F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_z \\ E_z & -B_y & B_z & 0 \end{pmatrix} \\
&\mathcal{L} = -\frac{1}{16\pi}F^{\alpha\beta}F_{\alpha\beta} = -\frac{1}{16\pi}g_{\gamma\alpha}g_{\delta\beta}\left(\partial^\gamma A^\delta - \partial^\delta A^\gamma\right)\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) \\
&p_\mu = \frac{\partial\mathcal{L}}{\partial\partial^0 A^\mu} = -\frac{F_{0\mu}}{4\pi} \\
&\mathbf{p} = -\frac{\mathbf{E}}{4\pi} \\
&\mathcal{H} = \partial_0 A_l \frac{\partial\mathcal{L}}{\partial\partial_0 A_l} - \mathcal{L} = -\frac{1}{4\pi}F^{0\mu}\partial_0 A_\mu + \frac{1}{16\pi}F^{\alpha\beta}F_{\alpha\beta} = \frac{\mathbf{E}^2}{8\pi} + \frac{\mathbf{B}^2}{8\pi}
\end{aligned}
$$

[1] This is a simple generalization of the work of Ansari, Steensel and Nazarov

[2] I use the book Classical Electodynamics of Jackson

this result[3] permit to write the photon quantum equation:

$$\boxed{i\hbar\partial_t\Psi(A^\mu) = -2\pi\hbar^2\Delta_{\mathbf{A}}\Psi(A^\mu) + \frac{\nabla\times\mathbf{A}}{8\pi} = -2\pi\hbar^2\Delta_{\mathbf{A}}\Psi(A^\mu) + \frac{\mathbf{B}^2}{8\pi}}$$

I write three solution of this equation, to verify the exactness of the equation. The first wavefunction is for a constant electric field:

$$\begin{aligned}
&A^\mu = (-Ex, 0, 0, 0)\\
&\mathbf{E} = \hat{\mathbf{x}}E\\
&\mathbf{B} = 0\\
&i\hbar\partial_t\Psi(A^\mu) = -2\pi\hbar^2\Delta_{\mathbf{A}}\Psi(A^\mu)\\
&\Psi = e^{-i\frac{W}{\hbar}t}\psi(A^\mu)\\
&W\psi(A^\mu) = -2\pi\hbar^2\Delta_{\mathbf{A}}\psi(A^\mu)\\
&\Psi(t, A^\mu) = \tfrac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}\left(\mathbf{p}\cdot\mathbf{A}-2\pi\mathbf{p}^2t\right)}\\
&\Delta_{\mathbf{A}}\Psi - \tfrac{B^2}{16\pi^2\hbar^2}\Psi + \tfrac{i}{2\pi\hbar}\partial_t\Psi = 0\\
&\Delta_{\mathbf{A}}\Psi^* - \tfrac{B^2}{16\pi^2\hbar^2}\Psi^* - \tfrac{i}{h}\partial_t\Psi^* = 0\\
&\Psi^*\Delta_A\Psi - \Psi\Delta_A\Psi^* + \tfrac{i}{h}\Psi^*\partial_t\Psi + \tfrac{i}{h}\Psi\partial_t\Psi^* = 0\\
&\nabla_{\mathbf{A}}\cdot\mathbf{J} + \partial_t(\Psi^*\Psi) = 0\\
&\mathbf{J}_{\mathbf{A}} = -ih\left(\Psi^*\nabla_{\mathbf{A}}\Psi - \Psi\nabla_{\mathbf{A}}\Psi^*\right)\\
&\mathbf{E} = \Psi^*(-i\hbar\nabla_{\mathbf{A}})\Phi = \tfrac{\mathbf{p}}{h}\\
&\Psi(t, A^\mu) = \tfrac{1}{\sqrt{2\pi\hbar}}e^{2\pi i\left(\mathbf{E}\cdot\mathbf{A}-2\pi h\mathbf{E}^2t\right)}\\
&\mathbf{J}_{\mathbf{A}} = 4\pi\mathbf{E}
\end{aligned}$$

there is a constant probability current in the direction of the electric field[4].

[3]The step-by-step calculus in The classical theory of field of Landau and Lifshitz

[4]this could be measured to obtain the wavefunction's phase to verify the theory

The second wavefunction is for a constant magnetic field:

$$A^\mu = (0, 0, By, 0)$$
$$\mathbf{B} = (B, 0, 0)$$
$$\mathbf{E} = 0$$
$$i\hbar\partial_t\Psi = -2\pi\hbar^2\Delta_\mathbf{A}\Psi + \frac{B^2}{8\pi}\Psi$$
$$\Psi(A^\mu) = e^{-i\frac{W}{\hbar}t}\psi(A^\mu)$$
$$-2\pi\hbar^2\Delta_A\psi + (\frac{B^2}{8\pi} - W)\psi = 0$$
$$\Psi(A^\mu) = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}\left(\mathbf{p}\cdot\mathbf{A} - 2\pi\mathbf{p}^2 t + \frac{\mathbf{B}^2}{8\pi}t\right)}$$
$$\mathbf{E} = -i\hbar\Psi(A^\mu)^*\nabla_\mathbf{A}\Psi(A^\mu) = \frac{\mathbf{p}}{h}$$
$$\Psi(A^\mu) = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}\left(h\mathbf{E}\cdot\mathbf{A} - 2\pi h^2\mathbf{E}^2 t + \frac{\mathbf{B}^2}{8\pi}t\right)} = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}\frac{\mathbf{B}^2}{8\pi}t}$$
$$\mathbf{J_A} = 0$$

there is not a flow of probability in a constant magnetic field.

The third wavefunction is for a circular polarized photon:

$$\phi = 0$$
$$\mathbf{A} = A(0, -c\sin(kx - \omega t), c\cos(kx - \omega t))$$
$$\mathbf{A}^2 = A^2c^2$$
$$\mathbf{B} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & A\sin(x - ct) & A\cos(x - ct) \end{pmatrix} = kcA\ (0, \sin(kx - \omega t), \cos(kx - \omega t)) =$$
$$= \omega A\ (0, \sin(kx - \omega t), \cos(kx - \omega t))$$
$$\mathbf{B}^2 = \omega^2 A^2$$
$$i\hbar\partial_t\Psi = -\pi\hbar^2\Delta_A\Psi + \frac{\omega^2 A^2}{8\pi}$$
$$W = -2\pi\hbar^2\Delta_A\Psi + \frac{\omega^2 A^2}{8\pi}$$
$$\Delta_A\Psi + \frac{W}{2\pi\hbar^2} - \frac{\omega^2}{16\pi^2\hbar^2}A^2 = 0$$
$$\xi = \alpha\mathbf{A}$$
$$\alpha^2\Delta_\xi\Psi + \frac{W}{2\pi\hbar^2} - \frac{\omega^2}{16\pi^2\hbar^2\alpha^2}\xi^2 = 0$$
$$\alpha = \sqrt[4]{\frac{\omega^2}{16\pi^2\hbar^2}} = \sqrt{\frac{\nu}{2\hbar}}$$
$$\Delta_\xi\Psi + \frac{2W}{h\nu} - \xi^2 = 0$$
$$\Psi\left(h\nu\left(n_y + n_z + \frac{1}{2}\right)\right) = e^{-\frac{\alpha^2A^2c^2}{2}}\prod_j\left(\frac{\alpha}{\pi^{1/2}2^{n_j}n_j!}\right)^{1/2}H_{n_j}(\alpha cA^\mu) =$$
$$= e^{-\frac{\alpha^2A^2c^2}{2}}\frac{\alpha^{3/2}}{\pi^{3/2}2^{n_x+n_y+n_z}n_x!n_y!n_z!}H_{n_x}(\alpha cA_x)H_{n_y}(\alpha cA_y)H_{n_z}(\alpha cA_z) =$$
$$= (-1)^{n_y}e^{-\frac{\alpha^2A^2c^2}{2}}\frac{\alpha^{3/2}}{\pi^{3/2}2^{n_x+n_y+n_z}n_x!n_y!n_z!}H_{n_y}(\alpha cA\sin(kx - \omega t))H_{n_z}(\alpha cA\cos(kx - \omega t))$$

this is a three-dimensional quantum harmonic oscillator for a photon[5].

The photon wavefunction is a A^μ function, then:

$$\Psi(A^\mu) = \sum_{n_x,n_y,n_z} a_{n_x,n_y,n_z} A^{n_x} A^{n_y} A^{n_z} = a_{000} + a_{100}A^1 + a_{010}A^2 + a_{001}A^3 + \cdots$$

so that the interaction in the quantum equations must contain[6] the product of vector potential and particle wave function.

[5] $n_x = n_y$ for the quantum number

[6] to the first order of approximation