Infinite Twin Primes: Two Methods

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Abstract

This paper is a revision and consolidation of two different but related methods to prove that there are infinitely many twin primes. The proofs are presented in the opposite order in which they were developed, largely due to the fact that a statement used at the end of the original proof, requiring its own proof, inspired and lead to the second method. The original technique uses surfaces, parabolas, and a number of lines. The 2nd proof, presented 1st, is actually the more direct and formal method. It primarily uses 2 surfaces, and also includes the extra steps needed to prove an analogous statement to one that was treated trivially in the first. Together these proofs compliment each other, and contain my body of work on the subject in a single resource.

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1 Introduction

Over the last decade I explored finding a proof for the twin prime conjecture. During that time I developed 2 related but different approaches. One is based on dragging lines across hyperbolas and parabolas, and how that relates to certain surfaces. This approach is adequate, though in its final steps, it relies on a statement that the range of 2 surfaces is not a basis for all but a finite amount of odd numbers. While for some it may seem readily apparent from the surfaces that the statement is true, a complete proof requires something more rigorous, and so I sought to explicitly prove that same statement about the surfaces in question. This led to the other, more recent proof. It forgoes the parabolas and lines, working solely with 2 surfaces related to but different than those in the first proof, and it explicitly proves that the range of those surfaces is not a basis for the natural numbers.

When I wrote the original proof, I was still learning Latex, and instead wrote it using MS Word and the math software that I had on hand. Later, as I improved it, I wrote a small revision in an attempt to explain it more clearly. However, these were not well composed, at least to the standards of Latex formatting, and to what many people commonly expect from a proof. A few years later, when I wrote the 2nd proof, I was much more familiar with proofs in general and the mark up formatting. Eventually, I also wrote a tiny update to the 2nd proof, trying to add even more explanation.

At this point, I feel the original version and its update are inaccessible to most readers due to their format and informal style, and that since they do still offer something to the topic, that they would benefit from being more properly updated into Latex with complete explanations. I also feel that all the information from both proofs would be better served together in a single paper. As such, this paper serves as a revision and consolidation of my methods for proving infinite twin primes. If you happen to find an error or issue with either method, please consider the other separately, as though one did inspire the other, they are related but different techniques. And again, if you do find any errors, or have any questions, please contact me.

2 Proof 1 - Infinite Twin Primes from the Rows of Twin Surfaces

This part of the paper starts as it did when it was written as a standalone version, with a brief statement of the problem and its approach to the proof.

The Twin Prime Conjecture is the well known topic within the field of mathematics regarding whether or not there are an infinite number of prime numbers separated by a difference of 2. This proof uses a strategy that is most generally split into 2 halves. First, 2 surfaces are determined such that choosing any natural number which is not found on either surface generates a Twin Prime Pair. Then, the proof shows that there are infinite such natural number generators.

2.1 Infinite Twins from 2 Surfaces

Begin with the given that all primes except the number 2 are odd. This means that all primes, other than the number 2, can be expressed in the form of 2n+1 for some natural number n. Next, use the fact that all odd numbers except 1 are either prime or an odd composite. This means that all odd numbers greater than 1, which are not odd composites, are prime. Thirdly, use the fact that all odd composites are the product of 2 odd numbers greater than 1. These 3 givens are expressed as equation 1, which is interpreted as stating that primes are all odds greater than 1 that are not odd composites, for positive natural number inputs n,a, and b.

$$\text{primes} = 2n + 1 \neq (2a + 1)(2b + 1) \quad a, b, n = \mathbb{N} \tag{1}$$

Since twin primes have a difference of 2, if $2n + 1$ is the smaller prime of a pair, then $2n + 3$ is the larger. Using the same logic as equation 1 means that $2n + 3$ must also not be an odd composite. This is stated in equation 2 for positive natural inputs n, c, and d.

$$\text{upper twin} = 2n + 3 \neq (2c + 1)(2d + 1) \quad c, d, n = \mathbb{N} \tag{2}$$

Simplifying equations 1 and 2 gives equations 3 and 4 respectively.

$$n \neq 2ab + a + b \tag{3}$$

$$n \neq 2cd + c + d - 1 \tag{4}$$

Equations 3 and 4 represent basic surfaces in 3 coordinates, or rather "anti-surfaces" due to the does not equal sign, stating what the values can not be. Moreover, the second surface, eq.4, graphically represents the same surface as the first, eq.3, only slid down by a value of 1. Figures 1 and 2 show truncated tables of the positive values of the surfaces, beginning outside the origin for natural inputs ≥ 1.

Figure 1: 2ab+a+b

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	4	7	10	13	16	19	22	25	28	31	34	37	40	43	46
2	7	12	17	22	27	32	37	42	47	52	57	62	67	72	77
3	10	17	24	31	38	45	52	59	66	73	80	87	94	101	108
4	13	22	31	40	49	58	67	76	85	94	103	112	121	130	139
5	16	27	38	49	60	71	82	93	104	115	126	137	148	159	170
6	19	32	45	58	71	84	97	110	123	136	149	162	175	188	201
7	22	37	52	67	82	97	112	127	142	157	172	187	202	217	232
8	25	42	59	76	93	110	127	144	161	178	195	212	229	246	263
9	28	47	66	85	104	123	142	161	180	199	218	237	256	275	294
10	31	52	73	94	115	136	157	178	199	220	241	262	283	304	325
11	34	57	80	103	126	149	172	195	218	241	264	287	310	333	356
12	37	62	87	112	137	162	187	212	237	262	287	312	337	362	387
13	40	67	94	121	148	175	202	229	256	283	310	337	364	391	418
14	43	72	101	130	159	188	217	246	275	304	333	362	391	420	449
15	46	77	108	139	170	201	232	263	294	325	356	387	418	449	480

Figure 2: 2cd+c+d-1

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
2	6	11	16	21	26	31	36	41	46	51	56	61	66	71	76
3	9	16	23	30	37	44	51	58	65	72	79	86	93	100	107
4	12	21	30	39	48	57	66	75	84	93	102	111	120	129	138
5	15	26	37	48	59	70	81	92	103	114	125	136	147	158	169
6	18	31	44	57	70	83	96	109	122	135	148	161	174	187	200
7	21	36	51	66	81	96	111	126	141	156	171	186	201	216	231
8	24	41	58	75	92	109	126	143	160	177	194	211	228	245	262
9	27	46	65	84	103	122	141	160	179	198	217	236	255	274	293
10	30	51	72	93	114	135	156	177	198	219	240	261	282	303	324
11	33	56	79	102	125	148	171	194	217	240	263	286	309	332	355
12	36	61	86	111	136	161	186	211	236	261	286	311	336	361	386
13	39	66	93	120	147	174	201	228	255	282	309	336	363	390	417
14	42	71	100	129	158	187	216	245	274	303	332	361	390	419	448
15	45	76	107	138	169	200	231	262	293	324	355	386	417	448	479

What these 2 surfaces dictate is the following. Surface 1, eq.3, restricted to the natural domain, is all n such that $2n+1$ will be an odd composite. Therefore, choosing any natural value for n that is not on that surface, i.e. values not from the table represented by figure 1, will make $2n+1$ a prime number. In fact, the natural numbers n that are not from the natural range of surface 1, generate all the primes except the number 2, without exception, and when ordered, generate them in order.

Similarly, surface 2 is all n such that $2n+3$ will be an odd composite. So, choosing a natural value n that is not found on either surface, ensures that both $2n+1$ and $2n+3$ are prime, and it generates a Twin Pair. In fact, like surface 1 does alone for the primes, choosing numbers not on either, generates all Twin Pairs without exception, and when ordered, generates them in order. Since choosing a natural value n that is on neither surface will always generate a unique Twin Pair, showing that there are infinite such generating values proves the Twin Prime Conjecture.

At this point, it may intuitively seem that there are an infinite number of Twin Primes, since one can always find another value not on either surface, $2ab+a+b$ or $2ab+a+b-1$, and this makes sense. As a quick initial verification, try the first few values. The first value not on either chart is the number 1. Applying $2n+1$ gives 3, corresponding to 3 and 5, the first Twin Pair. The next missing value is 2. This time $2n+1$ gives 5, corresponding to 5 and 7. Let's do 2 more. Next is $n = 5$, since 3 and 4 do appear, and $2(5) + 1$ is 11 for 11 and 13. The last example is the next missing number $n = 8$, giving $2n + 1 = 17$ for the pair 17 and 19. Remember, these charts are only portions of the full surfaces which extend infinitely, so you'll have to consult expanded versions when using them to confirm values beyond 45 in the first row.

To reiterate, in order to prove the Twin Prime conjecture, it must be shown that there is an infinite number of natural numbers not in the range of either surface when their domains are natural numbers. In concept it's straightforward, but for me, this is easier said than done. Over time, I have found a few similar strategies to do so, some better in ways, or easier to explain than others, and it is here that I describe what is currently the easiest of those for me to explain. I suspect that others know, or can devise, more direct methods to show it than compared to the technique that I offer below.

2.2 The Values Within or Outside the Range of the Surfaces

The general method that shows that there are infinite numbers outside both ranges takes the following path. Treat one variable as constant, in order to decompose the surfaces into an infinite number of lines, and thus assign each line to a row as shown in the tables. Show that there are infinite numbers outside the range of each specific row/line. Next, show that there are infinite numbers outside the range of any 2 adjacent rows. Finally, show that there are infinite numbers outside the range of any number of consecutive rows, and thus not on the surfaces.

Examine the form of the values generated in each row on both tables. These are the result of choosing a row number and setting one variable for the input of the surfaces to that value using equations 3 and 4. Notice that the surfaces are symmetric from the variable's standpoints, that is diagonalized, and so it doesn't matter which variable you use for this purpose. In this case, a and c were chosen as rows, and b and d for columns. The values can be written as 1 line per row per table, such that an infinite family of lines represent all the values on either surface.

Since the values on the 2nd surface are simply 1 less than those on the first, the y intercepts are one less for the corresponding lines. Look at the values in rows one, where a and c are held constant and set equal to 1. That is, $2(1)b+1+b$ and $2(1)d+1+d-1$. The surfaces simplify into the following lines.

All values appearing in row 1 of either surface.

$$3x + 1 \quad \text{and} \quad 3x + 0 \quad x = \mathbb{N} \tag{5}$$

This can be repeated for any row, and gives the following pattern. The next 3 rows are shown as a reference.

All values appearing in row 2 of either surface.

$$5x + 2 \quad \text{and} \quad 5x + 1 \tag{6}$$

All values appearing in row 3 of either surface.

$$7x + 3 \quad \text{and} \quad 7x + 2 \tag{7}$$

All values appearing in row 4 of either surface.

$$9x + 4 \quad \text{and} \quad 9x + 3 \tag{8}$$

Notice that the slopes are the set of odd numbers, that the y intercepts differ by 1 between surfaces per a given row, and that they also increase by 1 down the rows.

Now take a look at the form of the values NOT generated in the rows by the surfaces; that is, outside the range of the surfaces. There are an increasing odd amount of lines and values per row, determined by the row number, which represent all the values not appearing in that row on either surface.

All values not appearing in row 1 of either surface.

$$3x + 2 \tag{9}$$

All values not appearing in row 2 of either surface.

$$5x + 3, 5x + 4, 5x + 5 \tag{10}$$

All values not appearing in row 3 of either surface.

$$7x + 4, 7x + 5, 7x + 6, 7x + 7, 7x + 8 \tag{11}$$

Notice that the slopes are again the set of odd numbers, that the y intercepts span a consecutive range per row between surfaces, and that they increase by 1 for the elements of that range down the rows.

In order for an integer n to not be in the range of either surface, it must not appear in any row on either table for the surfaces. Equations 9, 10, and 11 show the general pattern for all values not in any specific row. The next step is to use that pattern to find values that do not occur in any and all rows. Also note, that due to the modular nature of the lines, y intercepts greater than or equal to the slope create congruent sets of values with another line and set of values that could be associated with that row. For example, $7x + 8$ from eq.11, generates the same range of values as $7x + 1$, except for the first value of course.

2.2.1 Values not Occurring on any Row or Rows

To find at least one set of values not in any row on either surface, and thus not on either surface, begin by noticing for rows 1, that all the values greater than the first elements of those rows, which do not appear in either row, are all the natural values of the line $3x + 2$, eq.9. That is, that the infinite set $3x + 2 =$

{5,8,11,14,17,20,...}, and so on, are not on either row 1. Put that aside for the moment and look now at the 2nd row.

Using eq.6, $5x + 1$ and $5x + 2$ are excluded, because they produce values on the 2nd rows. However, eq.10 shows that there are 3 sets of values that are never on rows 2. The first set is $5x + 3$. Repeating this for rows 3, for the first set of values never on the rows, yields $7x + 4$, for rows 4, it yields $9x + 5$, for rows 5, $11x + 6$, and so on.

This means that numbers in the intersection of the sets $3x + 2$ and $5x + 3$ are not on the first 2 rows of either surface. Numbers from that intersection that are also in the set $7x + 4$ are then not on the first 3 rows. Continuing the process means that finding an infinite number of values in the intersection of the sets of all those lines shows that there is an infinite number of values not on any row, and therefore, not on either surface.

2.3 Infinite Values not Within the Range

Going forward, it is very handy when explaining, to have a table of the values generated by these lines, in order to help visualize the relations between each row or to check some values. The general formula of the family of lines that were chosen for each row r, showing values on neither surface for that row, is equation 12.

$$(2r + 1)x + r + 1 \tag{12}$$

In Figure 3, values that would continue to the right have been broken into 2 more groups and moved below so that more values could be shown. The lines for the rows are labeled on the left column.

Figure 3: (2r+1)x+r+1

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	3x+2	5	8	11	14	17	20	23	26	29	32	35	38	41	44	47
2	5x+3	8	13	18	23	28	33	38	43	48	53	58	63	68	73	78
3	7x+4	11	18	25	32	39	46	53	60	67	74	81	88	95	102	109
4	9x+5	14	23	32	41	50	59	68	77	86	95	104	113	122	131	140
5	11x+6	17	28	39	50	61	72	83	94	105	116	127	138	149	160	171
6	13x+7	20	33	46	59	72	85	98	111	124	137	150	163	176	189	202
7	15x+8	23	38	53	68	83	98	113	128	143	158	173	188	203	218	233
8																
9																
10	3x+2	50	53	56	59	62	65	68	71	74	77	80	83	86	89	92
11	5x+3	83	88	93	98	103	108	113	118	123	128	133	138	143	148	153
12	7x+4	116	123	130	137	144	151	158	165	172	179	186	193	200	207	214
13	9x+5	149	158	167	176	185	194	203	212	221	230	239	248	257	266	275
14	11x+6	182	193	204	215	226	237	248	259	270	281	292	303	314	325	336
15	13x+7	215	228	241	254	267	280	293	306	319	332	345	358	371	384	397
16	15x+8	248	263	278	293	308	323	338	353	368	383	398	413	428	443	458
17																
18																
19	3x+2	95	98	101	104	107	110	113	116	119	122	125	128	131	134	137
20	5x+3	158	163	168	173	178	183	188	193	198	203	208	213	218	223	228
21	7x+4	221	228	235	242	249	256	263	270	277	284	291	298	305	312	319
22	9x+5	284	293	302	311	320	329	338	347	356	365	374	383	392	401	410
23	11x+6	347	358	369	380	391	402	413	424	435	446	457	468	479	490	501
24	13x+7	410	423	436	449	462	475	488	501	514	527	540	553	566	579	592
25	15x+8	473	488	503	518	533	548	563	578	593	608	623	638	653	668	683

2.3.1 Infinite Values in the Intersection of Adjacent Rows

When comparing 2 rows, the full intersection of the sets is of interest, requiring each line to get its own variable. For the first 2 rows, the relationship is equation 13.

$$3x + 2 = 5y + 3 \tag{13}$$

This has the integer solutions in eq.14 for an integer s.

$$x = 5s + 2 \quad y = 3s + 1 \tag{14}$$

The integer solutions and adjacent row relation are used to calculate the positions and values of the intersection between those rows. Starting with $s = 0$ in eq.14, and using the resulting values in eq.13, shows that the first element of row 2, 8 in this case, will map to the 2nd element of row one, again 8, and that every 3rd element thereafter on row 2, will map to every 5th element thereafter on row 1. Checking the $s = 1$ case gives $y = 4$ and $x = 7$, and we indeed see the 4th element on row 2, value 23, mapping to the 7th element of row 1. The next shared value in this instance would be $s = 2$ with row 2 element 7 value 38, and row 1 element 12 value 38.

This establishes an infinite number of values in the intersection of these 2 rows, 1 for each s, and therefore shows an infinite number of values not within the first rows of the 2 surfaces. Using the general formula for rows from eq.12 allows for the solution between any 2 adjacent rows. Using rows r and $r+1$, in the same way as was done in eq.13 for rows 1 and 2, gives eq.15.

$$(2r + 1)x + r + 1 = (2(r + 1) + 1)y + (r + 1) + 1 \tag{15}$$

This has the integer solutions in eq.16 for a row r and an integer s.

$$x = (2r + 3)s + r + 1 \quad y = (2r + 1)s + r \tag{16}$$

This verifies that there is an infinite number of solutions in the intersection of any two adjacent rows, and therefore an infinite number of values not on those rows of the 2 surfaces.

2.3.2 Specific Shared Values For the First 3 Rows

The goal is to show an intersection common to all rows. To do this, it helps to show the common values through the first 3 rows. When the term position is used going forward, it generally refers to the ordinal value or location of a member within a set, not the actual value of that element. This can also be thought of as the column value in Figure 3, though not to confuse things, rather what would be those column's proper values were they not all offset by 1 to the right due to the labels in column A as pictured.

It has already been shown that all values from the first row, $3x + 2$, are not on the first rows of the surfaces. It was also shown that an infinite number of elements will map between any row r and row $r + 1$. The question is now which specific elements map between rows? Start by examining the second row, $5x+3$. Eq.14 showed that the elements that map from $5x+3$ in the second row, into the first row, are in the $3s+1$ positions of the second row. That is, the positions {1,4,7,10,...} of row 2 that correspond to the values {8,23,38,53,...}.

Now repeat the question, this time asking not only which values will map from row 3 to row 2, but which values will map from row 3 to the specific values in row 2 that were mentioned, and thus allow them to intersect with row 1 also? Using eq.16 with $r = 2$ gives the location of the overall elements that map from row 3 to row 2 as the elements in the $y = 5s + 2$ positions of the 3rd row. It also shows that they map to $x = 7s + 3$ positions in the 2nd row. Since it was shown that all $3s + 1$ locations in the 2nd row map to the first, this means that whenever $3x + 1 = 7y + 3$, eq.17, an element in a $5s + 2$ position of row 3 maps to a position in row 2 that will then go on to map to row 1.

$$3x + 1 = 7y + 3 \tag{17}$$

This has the integer solutions in eq.18 for an integer s.

$$x = 7s + 3 \quad y = 3s + 1 \tag{18}$$

Since there are integer solutions of $x = 7s + 3$ and $y = 3s + 1$, it also means that there is an infinite number of such elements. However, because of the $3x + 1 = 7y + 3$ requirement, it must now be noted that though an infinite number do map through, not all of the elements in $5s + 2$ positions on the 3rd row will map to a location on the 2nd row that continues on to row 1.

As an example, the first integer solution for y is $y = 3s + 1$ with $s = 0$, giving $y = 1$. Since the $5s + 2$ positions on the 3rd row map to the $7s + 3$ spots on the 2nd, it means that the $5(1) + 2 = 7$th element on row 3 maps to the $7(1) + 3 = 10$th element on row 2. When checked, the value 53 maps between those locations, and also continues on and is found in the first row. The next solution, with $s = 1$, gives $y = 4$. This translates to the $5(4) + 2 = 22$nd element on row 3 mapping to the $7(4) + 3 = 31$st element on the 2nd row. Again, when checked, the value 158 maps between those locations, and also continues on and is found in the first row.

2.3.3 Intersections Across Subsequent Rows

In order to proceed to the 4th and subsequent rows, the question must first be answered of which specific subset of elements from the $5s + 2$ positions in the 3rd row map to the proper positions in the 2nd row. Also note that the $s = 0$ position actually represents the first member of a set, $s = 1$ the second, and so on, and as such, that the ordinal location value of an element within a set is 1 greater than that integer s when spoken of in terms of being the first, second, or "xth" element of that set. As shown in the above example, the first value on row 3 that meets all the requirements is 53. Inserting the $5s + 2$ positions mapping from row 3 to row 2 into its row value of $7x + 4$ for x, gives $35s + 18$. Set it equal to the first intersection of all 3 rows, 53, eq.19.

$$7(5s + 2) + 4 = 35s + 18 = 53 \tag{19}$$

Solving for s gives $s = 1$, which corresponds to the 2nd member of the $5s+2$ subset, which remember, are the positions of values that map from row 3 to row 2. Repeating the process for the next value of 158, gives $s = 4$, corresponding to the 5th member of that subset. Solving for all values that map back to row 1 gives $s = 3t + 1$ for a generic integer t, which again could also be spoken of as being the $3x+2$ member of the $5s + 2$ subset for an integer x. That is, in terms of the members, that the $3x + 2$ positions of the $5s + 2$ locations in row 3, are those that continue to row 2 in positions that will then continue on to row 1. To help avoid confusion, and to make it more explicit, the $5s + 2$ locations are columns {2,7,12,17,22,27,32,37,42,...}, and the $3x + 2$ elements {2,5,8,11,...}, of those locations, are then the corresponding columns {7,22,37,...}.

This can now be put in terms of the row 3, $7x + 4$ set directly, and can answer the question at the beginning of this section. Inserting the $3s+1$ integer relation that selects continuing $5s + 2$ positions into $5s + 2$, which remember represents all positions that map from row 3 to 2 in general, now puts directly in terms of the $7x + 4$ set, only those locations which also map to row 1. This is equation 20.

$$5(3s + 1) + 2 = 15s + 7 \tag{20}$$

This states that the $15s + 7$ elements of the $7x + 4$ set are those that map through row 2 to row 1; that is, row 3, columns {7,22,37,52,...}. Now that it is known which values on row 3 intersect with both rows 1 and 2, the process can be repeated asking which values on row 4 will map to those specific locations on row 3.

Row 4 Using $r = 3$ in eq.16 gives the location of the overall elements that map from row 4 to row 3 as the elements in the $y = 7s + 3$ positions of the 4th row. It also shows that they map to the $x = 9s + 4$ positions in the 3rd row. This means that wherever $9x + 4 = 15y + 7$, an element in a $7s + 3$ position of row 4 maps to a position in row 3 that will then go on to map through to row 1 and intersect all 4 rows.

$$9x + 4 = 15y + 7 \tag{21}$$

This has the integer solutions in eq.22 for an integer s.

$$x = 5s + 2 \quad y = 3s + 1 \tag{22}$$

Like before, the question is which specific subset of elements from the $7s+3$ positions of the 4th row map to the proper positions in the 3rd row. For $s = 0$ in eq.22 gives $y = 1$, and then $15y + 7 = 22$. The 22nd value of row 3, and the first value to intersect all 4 rows, is 158. Insert the $7s + 3$ position value that maps from row 4 to row 3 into its row 4 element value of $9x + 5$ for x, and set it equal to 158, eq.23.

$$9(7s+3)+5 = 63s+32 = 158 \tag{23}$$

Solving for s in eq.23 gives $s = 2$, which corresponds to the 3rd member of the $7s+3$ subset. Repeating the process for $s = 1$ in eq.22 gives the next value of 473, and gives $s = 7$ when it's used in eq.23, corresponding to the 8th member of that subset. Solving for all values gives the relation $s = 5t+2$, for a generic integer t, which again could also be spoken of as being the $5x+3$ member of the set for an integer x.

This can now be put in terms of the row 4, $9x+5$ set directly. Inserting the $5s+2$ integer relation that selects continuing $7s+3$ positions into $7s+3$, now puts directly in terms of the $9x+5$ set, only those locations which also map to row 1. This is equation 24.

$$7(5s+2)+3 = 35s+17 \tag{24}$$

This states that the $35s+17$ elements of the $9x+5$ set are those that intersect the first 4 rows. By now, you may begin to see the pattern, and/or, it begins to emerge. Continue the technique for the transition from row 5 to row 4.

Row 5 Using $r = 4$ in eq.16 gives the location of the overall elements that map from row 5 to row 4 as the elements in the $y = 9s+4$ positions of the 5th row. It also shows that they map to the $x = 11s+5$ positions in the 4th row. This means that whenever $11x+5 = 35y+17$, an element in a $9s+4$ position of row 5 maps to a position in row 4 that will then go on to map through to row 1 and intersect all 5 rows.

$$11x+5 = 35y+17 \tag{25}$$

This has the integer solutions in eq.26 for an integer s.

$$x = 35s+17 \quad y = 11s+5 \tag{26}$$

As previously, the question is which specific subset of elements from the $9s+4$ positions of the 5th row map to the proper locations in the 4th row. For $s = 0$ in eq.26 gives $y = 5$, and then $35y+7 = 192$. The 192nd value of row 4, and the first value to intersect 5 rows, is 1733. Insert the $9s+4$ position value that maps from row 5 to row 4 into its row 5 element value of $11x+6$ for x, and set it equal to 1733, eq.27.

$$11(9s+4)+6 = 99s+50 = 1733 \tag{27}$$

Solving for s in eq.27 gives $s = 17$, which corresponds to the 18th member of the $9s+4$ subset. Repeating the process for $s = 1$ in eq.26 gives the next value of 5198, and gives $s = 52$ when it's used in eq.27, corresponding to the 53rd member of that subset. Solving for all values gives the relation $35t+17$, for a generic integer t, which again could also be spoken of as being the $35x+18$ member of the set for an integer x.

This can now be put in terms of the row 5, $11x+6$ set directly. Inserting the $35s+17$ integer relation that selects continuing $9s+4$ positions into $9s+4$, now puts directly in terms of the $11x+6$ set, only those locations which also map to row 1. This is equation 28.

$$9(35s+17)+4=315s+157 \tag{28}$$

This states that the $315s+157$ elements of the $11x+6$ set are those that intersect the first 5 rows.

Row 6 At this point, the technique for finding the next set of values is established, and a general formula can be described. When doing so, it is also helpful to have the information from row 6, and rather than walk through the procedure again, the associated equations for row 6 are simply provided as follows.

$$13x+6=315y+157 \tag{29}$$

$$x=315s+157 \quad y=13s+6 \tag{30}$$

$$13(11s+5)+7=22523 \tag{31}$$

$$11(315s+157)+5=3465s+1732 \tag{32}$$

2.3.4 The General Formula for All Rows

To generate the formula for all rows, examine equations 17, 21, 25, and 29. For ease, these are relisted as eq.33, which also includes the next corresponding relation from row 7.

$$\begin{array}{c} \text{Row 3} \quad 3x+1=7y+3 \\ \text{Row 4} \quad 9x+4=15y+7 \\ \text{Row 5} \quad 11x+5=35y+17 \\ \text{Row 6} \quad 13x+6=315y+157 \\ \text{Row 7} \quad 15x+7=3465y+1732 \end{array} \tag{33}$$

Now ask, from where do the values in these equations emerge, and what is being compared between the left and right side of the equations? The process began with all values of $3x+2$ on the first row not being on the first rows of the surfaces. From there, it was determined that the $3x+1$ values on row 2 were the ones that intersected the first row. Using $r=2$ with x in eq.16 showed the members mapping in from row 3 to row 2 into the $7x+3$ positions. In eq.17, the set was arbitrarily assigned by me into the right side of the relation as $7y+3$ as to set the precedent going forward of the lesser valued parameters on the left when comparing sets. These are the sets represented and compared in Row 3 of eq.33, and they went on to generate the $15y + 7$ set, using equations 17-20, as seen in the right side of Row 4 of eq.33.

From that point forward, the left equations of the comparisons are from eq.16 for x with $r = R - 1$. That is, the left side of the Row 4 comparison in eq.33 uses the $r = 3$ value with x in eq.16, the Row 5 uses $r = 4$, and so on.

As for the right sides of the comparisons, the Row 4 sets went on to generate the $35y + 17$ set, using equations 21-24, as seen in Row 5 of eq.33. From that point forward, the right equations are generated using the right set from the previous row as the input for the sets from eq.16, but this time for y and with $r = R - 2$. That is, the right side of the Row 6 comparison in eq.33 is generated by inserting the right side of Row 5 into the y value in eq.16 with $r = 4$, the right side of the Row 7 comparison in eq.33 is generated by inserting the right side of Row 6 into the y value in eq.16 with $r = 5$, and so on. Because of the r to $r + 1$ relation in eq.16, this turns out to be the same as inserting a given Right side set into the previous Left side for x. For example, $9(35y + 17) + 4 = 315y + 157$, and $11(315y + 157) + 5 = 3465y + 1732$.

The Left side sets for comparison in row R, from Row 4 onward, are simply $(2R + 1)x + R$. For the Right side sets and Row 5 onward, the slopes m of the sets are the products of the first $R - 3$ consecutive odd numbers beginning with the odd number 5, and the y intercepts are $(m - 1)/2$. That is, the slope for Row 5 is $5x7$, for Row 6 it's $5x7x9$, for Row 7 it's $5x7x9x11$, and so on, and the intercepts are half of 1 less than those slopes.

The left set for row R$\geq$ 4.

$$(2R + 1)x + R \tag{34}$$

The right set for row R$\geq$ 5.

$$\left(\prod_{k=2}^{R-2}(2k + 1)\right) y + \frac{\left(\prod_{k=2}^{R-2}(2k + 1)\right) - 1}{2} \tag{35}$$

The product can also be expressed as:

$$\frac{(2R - 3)!}{3 * 2^{R-2} * (R - 2)!} \tag{36}$$

Setting eq.34 equal to eq.35 has the following integer solutions in eq.37 for Row R and an integer s.

$$x = \left(\prod_{k=2}^{R-2}(2k + 1)\right) s + \frac{\left(\prod_{k=2}^{R-2}(2k + 1)\right) - 1}{2} \qquad y = (2R + 1)s + R \tag{37}$$

Equation 38 is eq.37 in terms of the factorial expression.

$$x = \left(\frac{(2R-3)!}{3*2^{R-2}*(R-2)!}\right)s + \frac{\left(\frac{(2R-3)!}{3*2^{R-2}*(R-2)!}\right)-1}{2} \quad y = (2R+1)s + R \quad (38)$$

This shows that there is an infinite number of natural number solutions in the intersection of any and all rows R. Note that it does not give the specific elements in that set, but simply proves the existence of the set. It shows that there is no row that exists such that an infinite subset can not be mapped from the first row, through the intersection of all subsequent rows, to that row. Because these are numbers that are not in the range of either surface, there is an infinite amount of natural numbers not in the range of those surfaces. Therefore, since all numbers not in the range of those surfaces generate unique Twin Pairs, there is an infinite number of Twin Prime Pairs.

Some Notes About the Integer Solutions Some notes should be included about the nature of the integer solutions. The first note involves the behaviors of eq.35 and eq.37/38. Eq.37/38 gives an infinite set of integer solutions based on the integer s, which is enough for generating infinite sets for a given Row, however, in cases where the slopes in comparison between x and y have a common divisor, it does not give all integer solutions. This happens every 3rd row. The full integer solutions for those Rows simplify to have smaller slopes, and thus generate even more members to the sets, however, adjusting to include all of those complicates eq.35, and it is not necessary, since it's still using every 3rd solution from those infinite sets for those rows as stated; which is of course still an infinite subset. It actually means that there are even more elements in the infinite intersection than the ones shown by eq.37/38. This is similar to the second note.

Remember that this entire process was done using only the first set of values that were not in each row. Recall from equations 10 and 11 how each row has an increasing number of sets not in that row. While the first row must use $3x + 2$, there are an infinite number of other combinations with different sets from other rows that generate their own infinite intersections.

Lastly, using a given row R for eq.37/38, and then generating integer solutions with integers s for all rows $\leq R$, does significantly increase the chance that the corresponding n will be a Twin Pair generator, but does not guarantee it. However, because the set is diagonalized, for any n that is generated for a given R, it is never needed to check for values past row x, where x is the ceiling to the solution for x of $2x^2 + 2x + 1 = n$, the equation for the diagonal.

2.4 Conclusion

In summary, the proof used the fact that all Prime numbers greater than 2 are odd numbers that are not odd composites. It generated 2 surfaces from that requirement, one for each member of a Twin Pair, and showed that numbers not in the range of either surface always generate unique Twin Pairs. It then showed that there is an infinite number of elements in the set outside of that range, and therefore that there are infinite Twin Prime pairs.

I hope you enjoyed the proof. If you know or find a more concise method to show an infinite number of natural elements not on the surfaces, or would like to discuss the proof in some other manner, such as improvements, corrections, or errors, I am interested to know.

Q.E.D.

3 Proof 2 - Infinite Twin Primes from Surfaces, Lines, and Parabolas

This proof begins with 2 basic surfaces of z(x,y) above the first quadrant. Natural values of z are looked at in terms of y, using the x-y plane, by "scanning" those values with lines using a given range of y-intercept. The intersection of the lines with the value functions leads to a map through various equations to 2 additional surfaces, which are used along with some parabolas to show the existence of infinite twins.

3.1 Defining and Analyzing the Initial Surface

Begin with the first quadrant of the surface $z = xy$, and require x, y, and z to be natural numbers. Along the slices $x = 1$ and $y = 1$, the range of the surface gives the counting numbers, and over the rest of the field, where $x \geq 2$ and $y \geq 2$, it returns all the composites.

$$z = xy \quad x, y, z = \mathbb{N} \tag{1}$$

Therefore, if a specific value for z has a natural solution within that field, apart from the first row and column, it is composite, and likewise, if it has no natural solution, it is prime. Going forward, when the field of z is mentioned, it is assumed to refer to the composite portion outside of the first row and column as needed, unless otherwise noted. Analyze this surface by treating each height z as a value to be checked, and then solving for y. This yields equation 2.

$$y = \frac{z}{x} \tag{2}$$

Next, a line $y = -x + b$ is "scanned" and moved across equation 2 for any given value of z by incrementing the natural y-intercept b with $4 \leq b \leq \frac{z}{2} + 2$.

$$y = -x + b \quad \text{for} \quad 4 \leq b \leq \frac{z}{2} + 2 \quad b = \mathbb{N} \tag{3}$$

The lower limit on b ensures that the (1,1) answer is ignored, and more specifically comes from the intersection of the line with the first composite in the field, namely the number z = 4. The lower limit can and will be refined further later. The upper limit comes from there being no factors greater than

1/2 of a number, other than the number, and making b greater than that would lead to natural solutions on the first row and column which are supposed to be ignored. This is shown in the picture for z = 4.

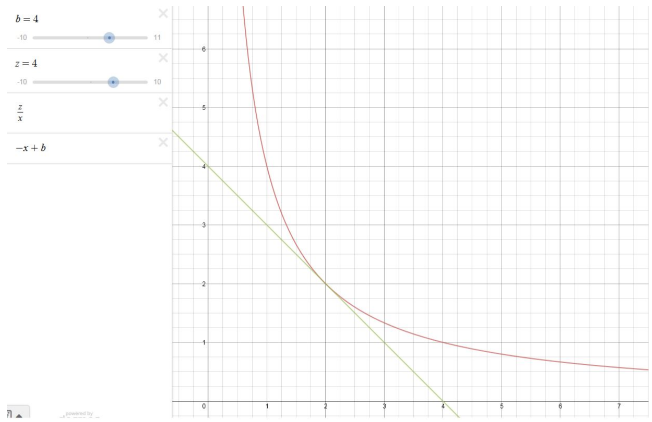

Figure 4: Scan reaching a curve with a natural solution

Set equations 2 and 3 equal and solve for x. If there are natural solutions for x within b's domain, then z is composite, and if there are no natural solutions, then it is prime. Equations 4 to 6 show the solutions using the quadratic formula.

$$\frac{z}{x} = -x + b \tag{4}$$

$$x^2 - bx + z = 0 \tag{5}$$

$$x = \frac{b \pm \sqrt{b^2 - 4z}}{2} \tag{6}$$

As another example, z = 7 is shown along with the lines within the domain of b. Notice that the lines do not intersect the curve at any of the natural number grid intersections, and that the curve only crosses the major grid at (1,7) and (7,1).

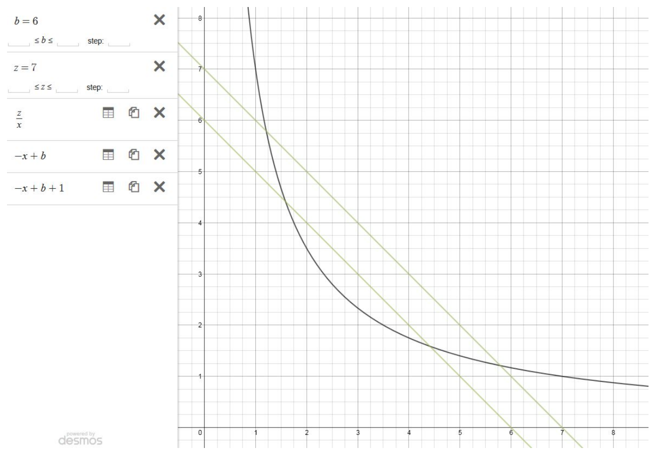

Figure 5: A prime showing no natural solutions

Lastly, z = 12 is shown, displaying multiple solutions on multiple lines, namely at x equals 2, 3, 4, and 6.

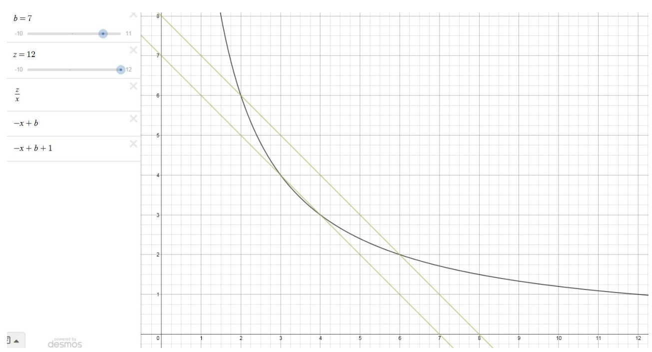

Figure 6: A curve with multiple natural solutions

3.2 The Twin Surface

Next allow $z - 2 = xy$ to represent a 2nd surface that will check values 2 less than a given number z, in this case looking for the smaller prime of a twin pair. Graphically, this is the first surface slid up the z-axis by 2. Solving for x by repeating the process in the previous section is shown in equations 7 to 11.

$$z - 2 = xy \tag{7}$$

$$y = \frac{z-2}{x} \tag{8}$$

$$\frac{z-2}{x} = -x + b \tag{9}$$

$$x^2 - bx + z - 2 = 0 \tag{10}$$

$$x = \frac{b \pm \sqrt{b^2 - 4z + 8}}{2} \tag{11}$$

The example below shows the curves for both surfaces, and is checking the pair (z,z-2) = (14,12).

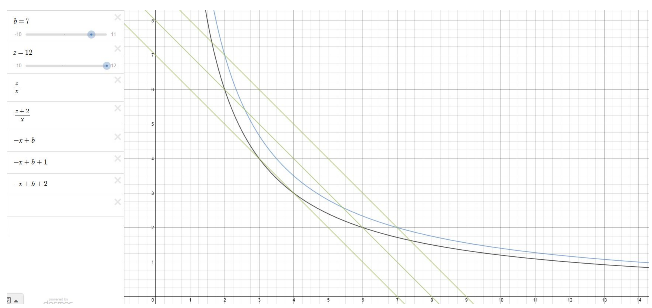

Figure 7: A pair of curves with solutions

For both surfaces, and according to equations 6 and 11, since the values are natural, x can only be natural, and therefore make z composite, when the square roots evaluate to a whole number. Just as the original curve, equation 2, was scanned with a line to look for natural solutions, the current square roots can be scanned with a new line. Before scanning, also note that since z is all natural numbers, and 4z is just a subset of those, that if the root does not evaluate to a whole number for all z, that it will certainly not evaluate to a whole number for any 4z. This allows a substitution. Let 4z = s. Then from eq.6, and a line in variable b with y-intercept c, write equation 12.

$$\sqrt{b^2 - s} \neq -b + c \quad c = \mathbb{N} \tag{12}$$

Solving for s for disallowed values yields:

$$s = -c^2 + 2bc \tag{13}$$

This is just another surface in s(b,c), which has translated a requirement of the surface, y-value curve, and line intersection to this new surface, however, the letters being used as variables at this point are not following a traditional z(x,y) convention. For familiarity, personal preference, and rather than talk about b-c space, I rewrite the current surface in terms of the familiar x and y, where b becomes x, and c becomes y. This is simply a syntactical rewrite of eq.13.

$$s = -y^2 + 2xy \tag{14}$$

Repeating steps 12-14 for the root of the Twin Prime surface, eq. 11, gives equation 15:

$$s = -y^2 + 2xy + 8 \tag{15}$$

The question is now, "what do these 2 new surfaces say about the original values of z being checked?"

3.3 A Recap of the Method So Far

So far we started to develop a method to plug values into the range of the field on the surface z = xy, using it to check whether those values are prime or composite. Initially, the domain of x and y are restricted to $1 < x < z$ and $1 < y < z$, otherwise products of 1 and the number contaminate the field. The left boundary can be restricted further by noting that eq. 2 has x-y symmetry, and that the first place the scanning line will touch the curve is along y = x. Solving for x gives $\sqrt{z}$. It was also noted that since there are no factors greater than half a number, when 1 and the number itself are ignored, that the right restriction can be cut in half. This gives, $\sqrt{z} \le x, y \le z/2$, and for the 2nd surface, $\sqrt{z-2} \le x, y \le (z-2)/2$. For the scanning line, those domain restrictions can be put in terms of the y-intercept, b, and this will be calculated in the next section.

Next, the value checking equations were rearranged to determine the value of x in terms of the value z being checked and the y-intercept, and it was noted that when there is no natural x for all y-intercepts in the appropriate domain, that the value being checked is prime. These were rearranged further, and a second scanning line was introduced based on conditions on the square roots. Finally, these were put in terms of conventional variables, x now representing the 1st scanning line's y-intercept, and y now representing the second scanning line's y-intercept. These were equations 14 and 15.

Equations 14 and 15 tell us that if s takes any natural value from these surfaces within the appropriate domains of y-intercepts, that the square roots will evaluate to whole numbers, leaving the original values with a chance to be composites. Likewise, if s is a value that does not appear on the surfaces within the appropriate domains, then the roots will not be whole numbers, and the original value is prime. The Twin conjecture can now be restated as, "are there an infinite number of individual values that do not appear on the new surfaces, thus making an infinite number of corresponding z and z-2 prime numbers?"

3.4 The Values of the Secondary Surfaces

In order to analyze the values that do not appear on the new surfaces, one needs to understand the practical restrictions on x and y in eqs.14 and 15. The current x, originally b, is the 1st scanning line's y-intercept, and using the domain restrictions put forth for that line in the recap section translates into a domain of $2\sqrt{z} \le x \le (z+4)/2$ for eq.14. Note, remember that s = 4z. For the second surface, eq.15, it is $2\sqrt{z-2} \leq x \leq (z+2)/2$. To find the domain for y, originally c, the 2nd line's intercept, insert those extremes of the domain for b, now x, into eq.12. This leads to $2\sqrt{z} \leq y \leq z$ for eq.14. Repeating the process for the second surface, using its analog of eq.12, not shown, results in $2\sqrt{z-2} \leq y \leq (z-2)$ for eq.15.

For the first translated surface, eq.14:

$$2\sqrt{z} \leq x \leq (z+4)/2 \quad \text{AND} \quad 2\sqrt{z} \leq y \leq z \tag{16}$$

For the second translated surface, eq.15:

$$2\sqrt{z-2} \leq x \leq (z+2)/2 \quad \text{AND} \quad 2\sqrt{z-2} \leq y \leq (z-2) \tag{17}$$

For these surfaces, a table of values along with a few examples helps make the technique clear and the needed information accessible. Charting the values of equation 14, with columns x and rows y, shows the squares down the diagonal when x = y. Equation 15, the lower table, has the same values plus 8.

	A	B	C	D	E	F	G	H	I	J
1	1	3	5	7	9	11	13	15	17	19
2	0	4	8	12	16	20	24	28	32	36
3	-3	3	9	15	21	27	33	39	45	51
4	-8	0	8	16	24	32	40	48	56	64
5	-15	-5	5	15	25	35	45	55	65	75
6	-24	-12	0	12	24	36	48	60	72	84
7	-35	-21	-7	7	21	35	49	63	77	91
8	-48	-32	-16	0	16	32	48	64	80	96
9	-63	-45	-27	-9	9	27	45	63	81	99
10	-80	-60	-40	-20	0	20	40	60	80	100
11										
12										
13										
14	9	11	13	15	17	19	21	23	25	27
15	8	12	16	20	24	28	32	36	40	44
16	5	11	17	23	29	35	41	47	53	59
17	0	8	16	24	32	40	48	56	64	72
18	-7	3	13	23	33	43	53	63	73	83
19	-16	-4	8	20	32	44	56	68	80	92
20	-27	-13	1	15	29	43	57	71	85	99
21	-40	-24	-8	8	24	40	56	72	88	104
22	-55	-37	-19	-1	17	35	53	71	89	107
23	-72	-52	-32	-12	8	28	48	68	88	108

Figure 8: Values of Equations 14 and 15 with columns x and rows y

3.4.1 5 Examples

Example 1: Use the top table to test if z = 4 is a prime number. Plug z into eqs.16 and 17. According to the domains, $2\sqrt{4} \leq x \leq (4+4)/2$, and $2\sqrt{4} \leq y \leq 4$. This means looking at the table in column 4 and row 4 for the value 4z, in this case 16. If the value appears then 4 is composite, if it does not, than 4 is prime. Indeed, 16 shows up at that location indicating the number is composite.

Example 2: Test if z = 11 is prime. According to the domains, $2\sqrt{11} \leq x \leq (11+4)/2$, and $2\sqrt{11} \leq y \leq 11$. Because we're dealing in natural numbers, this becomes $7 \leq x \leq 7$, and $7 \leq y \leq 11$. This means looking at the table in column 7, from rows 7 to 11, for the value 4z, in this case 44. In this example, we need to check up to and including row 11, but the table is only shown to row 10, however, this is a good time to point out that the table is diagonalized, and that the values of a given column below the diagonal simply mirror those above it. One can see, either by looking at the mirrored values, or by expanding the table, that the number 44 does not appear on those specific rows, and thus 11 is prime.

Example 3: Test if z = 12 is prime. According to the domains, $2\sqrt{12} \leq x \leq (12+4)/2$, and $2\sqrt{12} \leq y \leq 12$. Because we're dealing in natural numbers, this becomes $7 \leq x \leq 8$, and $7 \leq y \leq 12$. Checking those columns and rows for the value 4z = 48, one sees the value appear both in col.7, row 8, and in col.8 row 12. Thus 12 is composite. Also note that each time a value appears in a given range, that it represents a pair of factors of the number being tested. In this case, the 48 in col.7 corresponds to the factors of 12 being (2,6), and the one in col.8 to the factors (3,4).

Example 4: Test if z = 13 and z = 11 are a twin prime pair. Note that instead of checking both numbers individually against the top surface, we instead check the larger number against both surfaces. For the top table, the domains are, $2\sqrt{13} \leq x \leq (13+4)/2$, and $2\sqrt{13} \leq y \leq 13$. Because we're dealing in natural numbers, this becomes $8 \leq x \leq 8$, and $8 \leq y \leq 13$. For the bottom table, the domains are, $2\sqrt{13-2} \leq x \leq (13+2)/2$, and $2\sqrt{13-2} \leq y \leq (13-2)$. This becomes $7 \leq x \leq 7$, and $7 \leq y \leq 11$. We are looking for the number 4z = 52. Because the number 52 does not show up in the proper locations, 13 and 11 are indeed a twin pair.

Example 5: Test if z = 17 and z = 15 are a twin prime pair. For the top table, the domains are, $9 \leq x \leq 10$, and $9 \leq y \leq 17$. For the bottom table, the domains are, $8 \leq x \leq 9$, and $8 \leq y \leq 15$. We are looking for the number 4z = 68. Notice that the number 68 does not show up on the top table, signifying that 17 is prime, however it does appear within our range in col.8, row 10 of the bottom table, signifying 15 is composite. Thus 17 and 15 do not make a pair.

Now we can return the focus to the question asked at the end of section 3.3 regarding if an infinite such number of locations can be found. The goal is then to find values not appearing on either surface over appropriate domains of columns and rows. If it is shown that there is always a new domain that includes a new number that does not appear on either surface, then there are infinitely many twin prime pairs. It may be considered preferred, more complete, or of more interest, to scan a given domain collecting all the missing values, however, only at least one regular missing value on neither surface per each new domain is needed.

3.5 Finding Non-Occurring Values

So how does one go about finding values that do not appear on either chart for a given domain of columns and rows? As it turns out, there is a sort of trick, with a good bit of logic, and a location in the tables where such new values repeatedly appear. First, notice that the minimum value for rows and columns, y and x, is the same per surface. That is, $2\sqrt{z}$ from eq.16, and $2\sqrt{z-2}$ from eq.17. This corresponds to starting on the diagonal of the upper table. Next, notice that the maximum value for the rows, y always becomes greater than for the columns, x, namely $z > (z+4)/2$ for $z > 4$ from eq.16, and $z-2 > (z+2)/2$ for $z > 6$ from eq.17. This means that for $z > 6$ one will always check a number of rows within any given range of columns greater than or equal to the number of columns in that range. However, since the tables are diagonalized, and all values below the diagonal in a column are also found above it, no matter how many rows are called on to be checked, checking all the rows above the diagonal in a column are the same as checking below it.

For small z, this may include values to be checked that one would normally skip, and thus lead to missing certain primes, however this only makes the criteria more stringent, and doesn't matter anyways once the number of rows being checked is greater than the greatest column number within a given domain. Specifically, this is when $z-2\sqrt{z} \geq (z+4)/2$, which is $z \geq 4(3+2\sqrt{2})$, or rather $z \geq 24$ for natural numbers.

Now notice that the lower limit of the domain for the columns x of the 2nd surface is slightly lower than that of the first, namely $2\sqrt{z-2}$ vs $2\sqrt{z}$, whereas the upper limit is slightly greater on the first, namely $(z+4)/2$ vs $(z+2)/2$. This means that by using the lesser lower limit from the second surface, and the larger upper limit from the first surface, that one is guaranteed to cover the domain fielded by both surfaces. Once again, this may rule out primes and twins that would otherwise be fine, when checking values that are found in the beginning or ending columns of a range of columns, however this only narrows and strengthens the criteria.

We're almost ready to find a source of values that never appear, but a bit more logic is needed first. Looking at the upper bound $(z+4)/2$, which determines the greatest column to be checked, shows that the column increases by 1 each time z increases by 2. Increasing z by 1 each time looking for primes, one may have to go through many numbers, and thus many columns, until they find the next prime, however, we know as a given that there are an infinite number of primes, so we know that eventually z will hit the next prime and correspond to a number 4z that does not appear in the appropriate domain of columns on the first surface. Since the 4z will not be in any of the corresponding columns, it is known that this includes the rightmost column of the set.

Thinking about treating each column as the rightmost column of a set corresponding to some z to be checked, leads to the question, "are there values in a column that do not appear in any previous columns?" It turns out there are, and this occurs among the "trailing" values of the greatest column within a given range, which are the values in that column greater than the value of the previous column's diagonal. That is, greater than the square of the previous column number.

As an example, compare columns 7 and 8 on the first chart. The diagonal square value in column 7 is 49, and all the other values in that column and previous columns are less than 49. Column 8 has a square of 64. Therefore all values greater than 49 and less than 64, which do not appear in the 8th column, are also guaranteed to not show up in any previous column. This concept can also be applied to the 2nd surface by still using the diagonal, and by simply adding 8 to all the square values since it is literally the first surface plus 8. These are what were referred to as the ”trailing” values of a column.

Lastly, one can restrict the search to numbers not appearing within the range of trailing values for the columns on the 2nd surface. That is for a given column x, to the values between x^2 and $x^2 + 8$. They won't appear on the first surface since they are strictly larger than the values in that range, and they won't show up in previous columns of the 2nd surface due to the diagonalization and squares as was just described. Technically, it should be noted this is not true for the first 4 columns, after which it is always true. Restricting the search for values that do not appear on either surface to this subset of values greatly increases the chances of finding primes and twin pairs. However, now there is a new problem. Sure it's great to be able to only check among the trailing values of columns, but as was mentioned, there is no guarantee that a given column corresponds to a maximum domain of a prime. Another way to think about it, is that while a number may not appear in the trailing range for a given column, that it may end up appearing in a greater column, but lesser row, as the 4z check for some other z.

A good example of this is column 9 and the value 84. Using the method, in col.9, one is looking for values greater than 81 and less than 89 that are not on chart 2. While 84 meets the criteria, it then appears later in col.10, row 6, chart 1 as the check for z = 21.

Yet as also mentioned, it is known that eventually some larger column will correspond to a greater prime, and that this happens infinitely often. It is in those cases where we want to verify and sort out all of the values that now do not correspond to primes. It turns out that this can be done using a parabola and a few other filters.

3.6 Defining some Parabolas

At this point, a range of 8 values is being checked due to the x^2, $x^2 + 8$, and the fact that the surfaces are 8 apart. Those values are checking each 4z. By returning the surfaces to be directly in terms of z, that range of 8 shrinks to only 2 values. Substitute back in for s in eqs.14 and 15.

$$z = \frac{-y^2 + 2xy}{4} \quad \text{AND} \quad z = \frac{-y^2 + 2xy + 8}{4} \tag{18}$$

These 2 surfaces have a difference of 2, as they represent the original prime and pair surfaces. When they evaluate to whole numbers, the difference of 2 lands on those whole numbers, and therefore contains only 1 whole number value in between those 2 that does not show up on the surfaces in that range. When the surfaces evaluate to mixed number fourths, the difference of 2 contains 2 whole number values in between those 2 that do not appear on the surfaces. This is shown plotted with y along the horizontal axis.

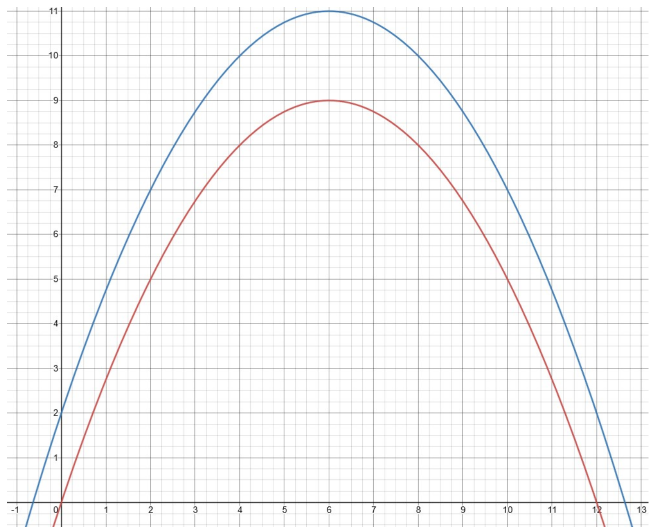

Figure 9: Whole number solutions with a single value in between at x=6

The value ranges being worked with were based on the diagonal squares when x = y, so they are both either even or odd in this instance. From eq.18 one can show that when the values are both even then the numerators are multiples of 4, and thus the surfaces evaluate to whole numbers. The single value sandwiched between them then alternates between even and odd. Whenever that number is composite, it will appear in at least 1 later column and fail as a number not on the surfaces. (This may not be immediately apparent as to why, and it is discussed in a section at the end of the paper. Furthermore, even if it were not the case, one could simply choose to only work with the mixed number fourth values anyways, as is done in the next paragraph, since it doesn't matter which source gives the generating values so long as it gives an infinite number of them.)

This prompts us to restrict the search to only odd values of x, and more specifically, the odd value contained between the surfaces when they take values of mixed number fourths. This now narrows the search to only 1 recurring location on both charts which will produce a value not on the surfaces for some range of columns. This number is found by adding 3/4 to the surface with lower values, or by subtracting 5/4 from the greater. This gives:

$$z = \frac{-y^2 + 2xy}{4} + \frac{3}{4} \tag{19}$$

Since the value is coming from the surface where $y = x$, this simplifies the equation into the parabola for odd x. This is shown below as the new upward parabola selects the odd value between mixed number fourths for an odd x.

$$z = \frac{x^2 + 3}{4} \quad x = \text{odd} \tag{20}$$

Figure 10: A missing number from both surfaces for a given range

Therefore for any odd x, $\frac{x^2+3}{4}$ will create a value that is not on either surface for some appropriate domain of parameters. Because there are infinite odd numbers, the parabola $\frac{x^2+3}{4}$ generates infinite such values. It's important to remember that there are many other locations among the surfaces where values can be found that do not appear on either surface. It's also important to remember that not all the values from this location will work, since we have no idea yet if we are stopping on a column that corresponds to a prime. All that remains is to remove any composite values from this location, as well as any non-twin primes, and to show that an infinite number of generators still remain.

3.7 Filtering the Generating Parabola

Being odd, these values contain only primes and some remaining odd composites for odd x. The next step is to filter those composites by writing them as the product of 2 odd numbers.

$$(2n+1)(2m+1) = 4nm + 2n + 2m + 1 \quad m, n = \mathbb{N} \tag{21}$$

$$\frac{x^2+3}{4} \neq 4nm + 2n + 2m + 1 \tag{22}$$

Which leads to the condition that:

$$x \neq \sqrt{16nm + 8n + 8m + 1} \tag{23}$$

Removing those values as input for x leaves only primes as the output. The final step is to filter the regular primes from the twins. If a prime is an upper twin prime, then 2 less than it will also be prime, however if it is not a twin, then 2 less than it will be an odd composite. That is:

$$\frac{x^2+3}{4} - 2 \neq 4pq + 2p + 2q + 1 \quad p, q = \mathbb{N} \tag{24}$$

Which gives the condition that:

$$x \neq \sqrt{16pq + 8p + 8q + 9} \tag{25}$$

Therefore, when x takes an odd value for which there are no natural solutions to equations 23 and 25, then $\frac{x^2+3}{4}$ will generate a value that is prime, and which does not appear on either surface. Because it does not appear on either surface, both square roots will not be whole numbers, and the original value being checked, and 2 less than it, will both be prime. Therefore, since there are an infinite number of odd numbers with no natural solutions to $x = \sqrt{16nm + 8n + 8m + 1}$ and $x = \sqrt{16nm + 8n + 8m + 9}$, then there are infinite generating values that create twin prime pairs.

At this point the proof concluded in a bit of a hurry, and one might notice or still be left wondering a few things about the last few sections. Primarily, what happened to worrying about whether a specific column represented the maximum range for a prime, how do we know the value generated won't be on either surface within a domain field, and what about the 2 square roots at the end not being a basis for the odds?

3.8 Quasi Circular Logic, Trivial Basis for the Odds, and the Inspiration for the 1st Proof

So what is really happening with this proof when understood as a whole, can it really serve to prove infinite twins, and what about the questions at the end of the previous section? A key to this is understanding how the surfaces in eq.18 are being used, and how it relates to the original number being checked. The surfaces are used such that one is checking to see if the number being queried appears on the surfaces within a field; if it does not, than the number is prime. In the case of primes, this means that the prime does not appear on the field, so when the process is reversed, that is looking for values not on the field to generate primes, it turns out one is looking for primes not on the field.

This is why it was referred to as quasi circular logic. It's not that one actually has to know the prime to find the prime, but rather that if a number happens to be prime, then it won't appear on a specific field of the surface, which in turn means the square root won't be whole, which in turn means the number is prime. Note, it doesn't mean that primes don't appear on those surfaces, they do, just not in the domains that would make them not prime.

We also know that there are infinite primes generating infinite values, ie the primes themselves, not appearing in specific fields. Since each number being checked only corresponds to one domain field, when a prime is found to not appear in a trailing location it means that column is the maximum of a domain; since it can't be found in any lesser row and column. This is true for $x^2 > (x-1)^2 + 8$, or in other words when $x \geq 5$. Again, there are other primes that are confirmed by not being found in their corresponding fields in a portion of that field other than the trailing locations, but those aren't used in this case. This is why worrying about a column being a maximum range is not a concern, since by demanding the value found in the trailing location to be prime, eq.22, it's ensured that it is.

These last facts, along with the statement about the basis for the odds, are what lead to the need and desire for a better proof. That is, we go all the way through this entire process just to arrive at eqs.22 to 25. At that point one is using the reasoned parabola with odd inputs and outputs, and then simply removing all the composites with eq.23, and all the twins with eq.25. We've basically checked for twin primality right there, only to say that those numbers mean that square roots aren't whole and thus create twin primes. It's ok, because we needed to, and did this to find and prove an infinite generator, but yet it seems redundant. It also relies on the statement that eqs.23 and 25 do not remove so many odds such that there are only a finite number of odds remaining. While a plot of those surfaces, or a table of their values, seems sparsely populated, and that they don't remove too many odd numbers, it's better to show that explicitly.

It was some time later, and from those understandings, that I realized why not just start directly from there; with all the odd numbers and remove all the composites. I also realized some surfaces are easier than others to show for what they are or are not a basis. This eventually lead to the second proof, presented in the first half of this paper.

3.9 Conclusion

In conclusion, it was shown that using an odd x, which was properly filtered for $\frac{x^2+3}{4}$, generates an infinite number of values that do not appear within a certain field on 2 surfaces, and that those values always create twin prime pairs when used as input in the original equations. It should follow, that one could replace the $z-2$ with $z-2n$ in equation 7, and show that the method can be used to extend the proof to any even sized prime gap.

4 Summary

So there you have it, 2 different methods to show that there are infinitely many twin prime pairs. Admittedly, I think the newer proof presented first is much stronger, more efficient, and more clear than the original proof presented second. However, it was the original proof that inspired the other, and I think it's still useful for expanding on the topic, showing another approach to the problem, and thinking about it in general.

If you enjoyed the proofs, found errors, or wish to offer other critiques, revisions, or to discuss the work further, please contact me.