Geometric algebra application on tensors, point groups, and electrical circuits

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This text is motivated by the desire to point out some more applications of geometric algebra in physics. The presentation is simplified, and the reader is referred to the literature.

Key Words:

geometric algebra, *inertia tensor*, *anisotropy*, *point groups*, *electrical circuits*, *dielectric tensor*, *isometry*, *symmetry group*

1. AC electrical circuits

In AC8, Steinmetz proposed a symbolic method to calculate power in AC electric circuits. Given a voltage $V = V_1 + iV_2$ and a current $I = I_1 + iI_2$ in complex form ($i = \sqrt{-1}$), he tried to get the power formula (see AC7)

$$P = V_1 I_1 + V_2 I_2 + i\left(V_1 I_2 - V_2 I_1\right).$$

Note that simple multiplication gives $VI = V_1 I_1 - V_2 I_2 + i\left(V_1 I_2 + V_2 I_1\right)$, while the usual VI^* gives $V_1 I_1 + V_2 I_2 - i\left(V_1 I_2 - V_2 I_1\right)$, the complex conjugate of the Steinmetz's P, that is $V^* I$ (such a product is called *geometric product*, see AC4). Anyhow, in order to get the desired multiplication rule, Steinmetz suggested an additional “imaginary unit” that squares to 1, as well as some kind of non-commutative multiplication.

With the knowledge of geometric algebra, it is obvious that the “imaginary unit that squares to 1” could be a unit vector, while the non-commutative multiplication is just the geometric product. Defining $V = V_1 e_1 + V_2 e_2$ and $I = I_1 e_1 + I_2 e_2$, we have

$$P = VI = V_1 I_1 + V_2 I_2 + \left(V_1 I_2 - V_2 I_1\right) e_1 e_2 .$$

This gives the possibility to eliminate complex numbers and phasors from the AC circuit theory. Now we can see why the product $V^* I$ works. Writing

$$V = e_1\left(V_1 + V_2 e_1 e_2\right),$$

we have

$$P = e_1\left(V_1 + V_2 e_1 e_2\right) e_1\left(I_1 + I_2 e_1 e_2\right) = e_1 e_1\left(V_1 - V_2 e_1 e_2\right)\left(I_1 + I_2 e_1 e_2\right) = V^\dagger I ,$$

where the "imaginary unit" is e_1e_2 and the complex conjugation is replaced by the reverse involution.

It should be remarked that power theory has not been completed to date, despite many attempts. Nonlinear AC circuits are a particular problem. There are attempts to solve this problem in GA (see AC1-8), but we leave it to the reader to judge for himself.

2. Tensors

2.1 Moment of inertia without tensors

Linear transformations in geometric algebra provide the possibility to replace matrices and tensors with transformations of k-vectors in a unified way.

We can write a triple cross product (see [1]) of two vectors $\mathbf{a}\times(\mathbf{b}\times\mathbf{a})$, where $\mathbf{a}\perp(\mathbf{b}\times\mathbf{a})$, as

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{a})=-j\mathbf{a}\wedge(\mathbf{b}\times\mathbf{a})=-j\mathbf{a}(\mathbf{b}\times\mathbf{a})=\mathbf{a}(\mathbf{a}\wedge\mathbf{b}).$$

In classical mechanics, introducing the angular velocity vector $\boldsymbol{\omega}$, the angular momentum is given by

$$\mathbf{L}=\sum_i^N m_i\mathbf{r}_i\times(\boldsymbol{\omega}\times\mathbf{r}_i)=\sum_i^N m_i\mathbf{r}_i(\mathbf{r}_i\wedge\boldsymbol{\omega}).$$

In the continuum limit, this becomes

$$\mathbf{L}=\int\mathbf{r}(\mathbf{r}\wedge\boldsymbol{\omega})\mathrm{d}m.$$

In the traditional form, we define the inertia tensor I as $\mathbf{L}=I\boldsymbol{\omega}$, which we can write in the coordinate form

$$\begin{pmatrix}L_1\\L_2\\L_3\end{pmatrix}=\begin{pmatrix}I_{11}&I_{12}&I_{13}\\I_{21}&I_{22}&I_{23}\\I_{31}&I_{32}&I_{33}\end{pmatrix}\begin{pmatrix}\omega_1\\\omega_2\\\omega_3\end{pmatrix}.$$

In geometric algebra (see [1], Sect. 1.7), we are introducing a linear transformation I as

$$\mathbf{L}=\mathrm{I}(\boldsymbol{\omega})=\mathrm{I}\boldsymbol{\omega}=\int\mathbf{r}(\mathbf{r}\wedge\boldsymbol{\omega})\mathrm{d}m.$$

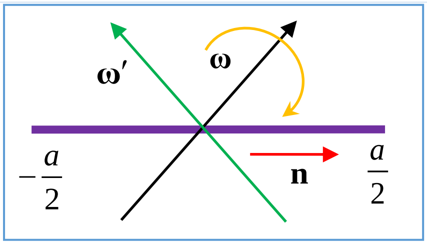

Fig. 1: Rotating rod

Consider a thin rod of length a extending from $-a/2$ to $a/2$ and rotating about an arbitrary axis passing through its center (see **Fig. 1**). Defining a new variable s by

$$\mathrm{d}m = m\mathrm{d}s/a\,,\ \mathbf{r} = s\mathbf{n}\,,\ \mathbf{n}^2 = 1\,,$$

we have

$$\mathbf{L} = \mathrm{I}(\boldsymbol{\omega}) = \frac{m}{a}\int_{-a/2}^{a/2} s\mathbf{n}(s\mathbf{n}\wedge\boldsymbol{\omega})\mathrm{d}s = \frac{m}{a}\mathbf{n}(\mathbf{n}\wedge\boldsymbol{\omega})\int_{-a/2}^{a/2} s^2\mathrm{d}s = \frac{ma^2}{12}\mathbf{n}(\mathbf{n}\wedge\boldsymbol{\omega})\,.$$

Using $\mathbf{n}\wedge\boldsymbol{\omega} = (\mathbf{n}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{n})/2$, we get

$$\mathrm{I}(\boldsymbol{\omega}) == \frac{ma^2}{24}(\boldsymbol{\omega} - \mathbf{n}\boldsymbol{\omega}\mathbf{n})\,.$$

Note that the vector $\boldsymbol{\omega}' = -\mathbf{n}\boldsymbol{\omega}\mathbf{n}$ is just reflected vector $\boldsymbol{\omega}$ with respect to the plane $j\mathbf{n}$ (see [1], Sect. 1.9.3).

Generally, we can define bivectors

$$L = j\mathbf{L}\,,\ \Omega = j\boldsymbol{\omega}\,,$$

and write

$$L = j\int \mathbf{r}(\mathbf{r}\wedge\boldsymbol{\omega})\mathrm{d}m = \frac{j}{2}\int\left(r^2\boldsymbol{\omega} - \mathbf{r}\boldsymbol{\omega}\mathbf{r}\right)\mathrm{d}m = \frac{1}{2}\int\left(r^2\Omega - \mathbf{r}\Omega\mathbf{r}\right)\mathrm{d}m\,.$$

The inertia tensor now becomes a *biform*, i.e., a bivector-valued linear transformation of bivectors

$$\mathrm{I}(\Omega) = \frac{1}{2}\int\left(r^2\Omega - \mathbf{r}\Omega\mathbf{r}\right)\mathrm{d}m\,.$$

Again, the geometric content of the problem, hidden in the tensor formulation (we need to find out what the tensor is doing), is clearly seen in the GA formulation (the geometric interpretation is immediately clear). For other applications (including *the Riemann curvature tensor*), see IT2.

2.2 Anisotropy without tensors

A similar procedure can be applied to other physical problems with tensors. In IT3, there is a nice explanation of application of geometric algebra in optics. The relation between the electric field $\mathbf{E}$ and the electric displacement $\mathbf{D}$ in anisotropic crystals is given by

$$\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E},$$

where ε_0 is the permittivity of vacuum and ε is a *dielectric tensor*. In *Cl*3, we define a linear transformation $\mathrm{\varepsilon}$

$$\mathbf{D} = \varepsilon_0 \mathrm{\varepsilon}(\mathbf{E}),$$

where the *principal dielectric axes* are given by the eigenvalue problem

$$\mathrm{\varepsilon}(\mathbf{a}) = \lambda \mathbf{a}, \ \lambda \in \mathbb{R}.$$

See IT3 for the details on specific crystal configurations.

3. Point groups in crystollagraphy

In PG1, Hestenes shows that each of the 32 lattice point groups and 230 space groups in three dimensions is generated from a set of three symmetry vectors. Here we present just some basic ideas. As expected, the power of geometric algebra is seen here as well, bringing clarity and giving fresh insights.

An *isometry* that permutes parts of a rigid body, leaving it unchanged as a whole, is called a *symmetry*. The symmetries of an object form a group called the *symmetry group* of the object. Every symmetry S can be given the mathematical form

$$S : \mathbf{x} \to \mathbf{x}' = \underline{\mathrm{R}}\mathbf{x} + \mathbf{a}, \qquad (3.1)$$

where $\mathbf{x}$ designates a point in the object, $\underline{\mathrm{R}}$ is an orthogonal transformation with the origin as a *fixed point*, and the vector $\mathbf{a}$ designates a *translation*. In most applications, the operator $\underline{\mathrm{R}}$ is represented by a matrix. There is a problem with the standard representation for a symmetry by (3.1), namely, the orthogonal group is multiplicative while the translation group is additive, so combining the two destroys the simplicity of both. Geometric algebra provides an elegant solution of this problem (see PG1).

In geometric algebra, we use the coordinate-free canonical form

$$\underline{\mathrm{R}}\mathbf{x} = \pm R^{\dagger} \mathbf{x} R, \qquad (3.2)$$

where R is an invertible *versor*, with even (odd) parity corresponding to the plus (minus) sign, normalized to unity, so its reverse $R^{\dagger}$ is equal to its inverse R^{-1}. When R is even, equation (3.2) describes a rotation.

3.1 Point Groups in Two Dimensions

As an example, consider the *benzene molecule* shown in **Fig. 2**, with the *fixed point* condition, which eliminates translations, so all the symmetries are orthogonal transformations. This molecule has the structure of a *regular hexagon* with a carbon atom at each vertex. The simplest symmetry of this molecule is the rotation $\underline{\mathrm{R}}$ taking each vertex $\mathbf{x}_k$ into its neighbor $\mathbf{x}_{k+1}$, as described by

$$\mathbf{x}_{k+1} = \underline{\mathrm{R}}\,\mathbf{x}_k = R^{\dagger}\mathbf{x}_k R = \mathbf{x}_k R^2 . \qquad (3.3)$$

Obviously, $\underline{\mathrm{R}}$ satisfies the operator equation

$$\underline{\mathrm{R}}^6 = 1 \qquad (3.4)$$

This relation implies that we have a group with six distinct elements $\underline{\mathrm{R}}^k$, $k = 1, \ldots, 6$. This is the rotational symmetry group of a hexagon, called the *cyclic group* (of order 6) and commonly denoted by C_6. The element $\underline{\mathrm{R}}$ is a *generator* of C_6, while the condition (3.4) is called a *presentation* of the group. In GA, the relation (3.4) corresponds to the *versor* relation

$$S^6 = 1, \; S = R^2 \qquad (3.5)$$

(see (3.3)), which has the advantage of admitting the explicit solution

$$S = \exp(2\pi j/6) = \exp(j\pi/3).$$

The representation (3.5) shows explicitly that the generator of C_6 is a rotation through angle $\pi/3$.

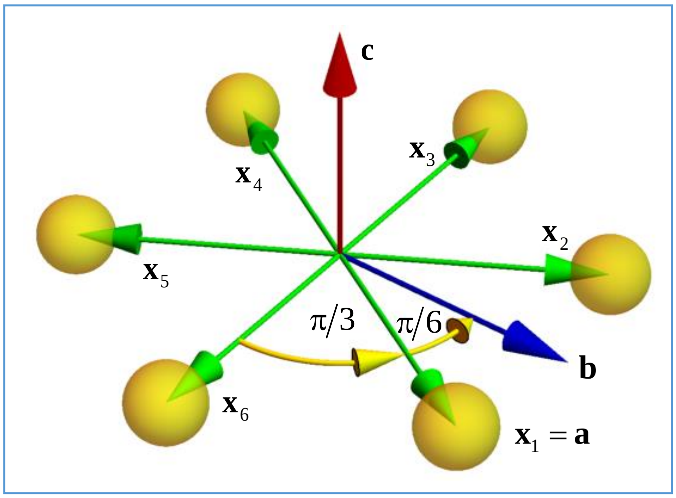

Fig. 2: Planar benzene (C_6H_6), showing generators of the symmetry group. (Hydrogen atoms are not shown.)

We know that to every rotation there correspond two rotors differing only by a sign. Consequently, to every finite rotation group there corresponds a rotor group with twice as many elements. In the present case, the generator R of the rotor group is related to the generator S of the cyclic group by $S = R^2$. Taking the negative square root of the relation $S^6 = \left(R^2\right)^6 = \left(R^6\right)^2 = 1$, we get the presentation for the *dicyclic group* of order 12 generated by R

$$R^6 = -1,$$

and we denote it as $2C_6$. Note that the pair of rotors $\pm R$ distinguishes equivalent rotations of opposite senses (the cyclic group does not assign a sense to rotations).

A hexagon has reflectional as well as rotational symmetries. It is evident (see **Fig. 2**) that the hexagon is invariant under reflection along any diagonal through a vertex or the midpoint of a side. For example, with $\mathbf{a} = \mathbf{x}_1$, the reflections

$$\underline{A}\mathbf{x} = -\mathbf{a}^{-1}\mathbf{x}\mathbf{a} \quad \text{and} \quad \underline{B}\mathbf{x} = -\mathbf{b}^{-1}\mathbf{x}\mathbf{b}$$

are symmetries of the regular hexagon. These reflections generate a symmetry group of the hexagon, which we denote by H_6 (sometimes called the *dihedral group*). Note that the product of two reflections $\underline{A}$ and $\underline{B}$ is a rotation

$$\underline{B}\underline{A}\mathbf{x} = \left(\mathbf{ab}\right)^{-1}\mathbf{x}\mathbf{ab},$$

which means that C_6 is a subgroup of H_6. We can normalize the vectors $\mathbf{a}$ and $\mathbf{b}$ to unity to get $R = \mathbf{ab}$ and the presentation of the group $2H_6$ (for the details, see PG1)

$$\mathbf{a}^2 = \mathbf{b}^2 = 1,\ \left(\mathbf{ab}\right)^6 = -1.$$

How to find and describe all the fixed-point symmetry groups of all two-dimensional figures? Consider the multiplicative group $2H_p$, $0 < p \in \mathbb{N}$, generated by two unit vectors $\mathbf{a}$ and $\mathbf{b}$ related by the *dicyclic condition*

$$\left(\mathbf{ab}\right)^p = -1 = \exp\left(j\pi\right),$$

which means $\mathbf{ab} = \exp\left(j\pi/p\right)$. ♦ What if $p = 1,\ 2,\ \infty$? ♦

3.2 Point Groups in Three Dimensions

If three unit vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are to be generators of a finite multiplicative group, then each pair of vectors must generate a finite subgroup, which means that they must satisfy the *dicycle conditions*

$$\left(\mathbf{ab}\right)^p = \left(\mathbf{bc}\right)^q = \left(\mathbf{ac}\right)^r = -1, \qquad (3.6)$$

$p,\ q,\ r > 1$ (why?). From

$$(\mathbf{ab})(\mathbf{bc}) = (\mathbf{ac}), \qquad (3.7)$$

(this equation relates the sides of a spherical triangle with vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, see PG2) we see that the three generators of rotations in (3.6) are not independent. In fact, it can be shown (*spherical excess* formula, see PG1 and PG2) that (3.7) leads to the relation

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1,$$

which has no solution for $p = q = r = 3$. Choosing $r = 2$, we get

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2}.$$

For $q = 2$, any value of p is allowed, while for $q = 3$, $p \in \{3,\ 4,\ 5\}$. This exhausts the possibilities and (3.6) reduces to

$$(\mathbf{ab})^p = (\mathbf{bc})^q = (\mathbf{ac})^2 = -1.$$

The tables of the possible point groups for the possible values of p and q can be found in PG1.

♦ Prove the relation (35) from PG1. ♦

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