An algebraic proof Sedenions are not a division algebra and other consequences of Cayley-Dickson Algebra definition variation

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Abstract

The Cayley-Dickson dimension doubling algorithm nicely maps $\mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{H} \rightarrow \mathbb{O} \rightarrow \mathbb{S}$ and beyond, but without consideration of any possible definition variation. Quaternion Algebra $\mathbb{H}$ has two orientations, and they drive definition variation in all subsequent algebras, all of which have $\mathbb{H}$ as a subalgebra. Requiring Octonion Algebra $\mathbb{O}$ to be a normed composition algebra limits the possible orientation combinations of its seven $\mathbb{H}$ subalgebras to sixteen proper $\mathbb{O}$ orientations, which are itemized. Identification of the $\mathbb{O}$ subalgebras for Sedenion Algebra $\mathbb{S}$ and orientation limitations on these subalgebras provides a fully algebraic proof that all $\mathbb{O}$ subalgebras cannot be oriented as proper Octonion Algebras, verifying Sedenion Algebra is not generally a normed composition division algebra. A simple mnemonic form for validating proper $\mathbb{O}$ orientations is provided. The method to partition any number of $\mathbb{O}$ algebraic element products into product term sets with like responses to all possible proper $\mathbb{O}$ orientation changes; either to a single algebraic invariant set or to one of 15 different algebraic variant sets, is provided. Most important for $\mathbb{O}$ based mathematical physics, the stated Law of Octonion Algebraic Invariance requires observables to be algebraic invariants. Its converse, The Law of the Unobservable suggests homogeneous equations of algebraic constraint built from the algebraic variant sets. These equations of constraint are important to mathematical physics since they will limit the family of solutions for the differential equations describing reality and do not have their genesis in experimental observation. An alternative to the Cayley-Dickson doubling scheme which builds by variations is provided.

1.0 Fundamental considerations

The difference between two n dimensional algebras resides in their basis element multiplication rules determining the n^2 products of any two basis elements. Any n dimensional algebra is thus fully characterized by these rules. All Cayley-Dickson algebras can be generally classified as hypercomplex algebras. The product of any two basis elements is within sign one member of the full set, this may be represented generally as $e_a*e_b = \pm e_c$. The basis element set for an n dimensional hypercomplex algebra has one scalar basis element e_0 equivalent to the real number +1, and n – 1 non-scalar basis elements that will square to $-e_0$. We thus have fixed definitions for 2n – 1 real number product rules of the form (e_0*e_0), (e_0*e_m) and (e_m*e_0) for m = 1 to n – 1, and n – 1 product rules of the form $(e_m*e_m) = -e_0$. For all n dimensional hypercomplex algebras, we must additionally define $n^2 - 3n - 2 = (n-1)(n-2)$ products of unlike non-scalar basis elements. This is where definition flexibility will show up.

The first Cayley-Dickson algebra where products of unlike non-scalar basis elements comes into play is the first with more than one, Quaternion Algebra. The general form for these products that will be used throughout this document is called the *ordered permutation triplet multiplication rule*. It defined by the following mnemonic form simply defining six separate products.

$(e_a\ e_b\ e_c) :=$

cyclic left to right positive products: $e_a*e_b = +e_c$ $\quad e_b*e_c = +e_a$ $\quad e_c*e_a = +e_b$

cyclic right to left negative products $e_b*e_a = -e_c$ $\quad e_c*e_b = -e_a$ $\quad e_a*e_c = -e_b$

The forms $(e_a\ e_b\ e_c)$, $(e_b\ e_c\ e_a)$ and $(e_c\ e_a\ e_b)$ all describe the same rule set, meaning we may cyclically shift any ordered permutation triplet multiplication rule without changing its meaning.

If we exchange e_a and e_b we get the form $(e_b\ e_a\ e_c)$, which is distinct from $(e_a\ e_b\ e_c)$, and is equivalent to any odd number of exchanges of two basis elements on $(e_a\ e_b\ e_c)$ as can be seen with cyclic shift equivalence. The six product rules for $(e_b\ e_a\ e_c)$ are the negation of the six for $(e_a\ e_b\ e_c)$. The two rule forms $(e_a\ e_b\ e_c)$ and $(e_b\ e_a\ e_c)$ fully cover the definition variability for Quaternion Algebra. These two orientation choices are given the chiral names right-handed and left-handed because they cover the familiar 3D vector cross product definitions castable within a right-handed or left-handed system. The negation of an ordered permutation triplet multiplication rule is the move $(e_a\ e_b\ e_c) \rightarrow (e_b\ e_c\ e_a)$.

I will always use ( ) to specify an ordered triplet set, implying a given orientation. It will be convenient to also form sets of three basis elements or simply their integer indexes used to enumerate them, where no orientation is implied. I will consistently use { } to specify such an unordered set. This way, if we are given $\{e_a\ e_b\ e_c\}$, we can later orient it by assigning one of the choices $(e_a\ e_b\ e_c)$ or $(e_b\ e_a\ e_c)$.

One nice outcome from the Cayley-Dickson construction, if successive dimension doublings enumerate basis element integer subscripts sequentially, is for all Quaternion subalgebra ordered triplets their three basis element indexes represented as binary numbers exclusive or to zero. The logic operation "exclusive or", for short "xor", is commonly expressed in computer languages with the operator ^. It is defined bit-wise for binary number representations as 0^0 = 0, 0^1 = 1^0 = 1, and 1^1 = 0. The operation is fully commutative and associative. From the first and last ^ rule, it is easy to see for any binary number x, we have x^x = 0. If we are given a^b = c, we have a^b^c = c^c = 0. From $(e_a\ e_b\ e_c)$ we have $e_a{*}e_b = e_c$. The observation that the three basis element indexes will xor to zero implies a^b = c is a representation of $e_a{*}e_b = \pm e_c$. The xor operation cannot give us the particular orientation. However, we may make a representation of $\{e_a\ e_b\ e_c\}$ with a^b^c = 0 without issue, since the orientation is undeclared.

We can relax the definition of $\{e_a\ e_b\ e_c\}$ to allow a, b, or c to include the index number 0, representing the scalar basis element. There will be two possible forms this may take where the xor of all three will result in zero: $\{e_0\ e_0\ e_0\}$ and $\{e_m\ e_m\ e_0\}$. The former represents the algebra of real numbers. Combined with the latter, we have a representation of Complex number algebra, since the three cyclic shifts are representations the products $e_1 * e_1 = -e_0$, $e_1 * e_0 = e_1$, $e_0 * e_1 = e_1$. The exceptionally nice outcome of all of this is we can define for any given indexes a and b, $e_a * e_b = \pm e_{a\wedge b}$. The proper singular sign choice will depend on the particular indexes a and b and any orientation choice if there is one.

Since Quaternion Algebra is a subalgebra of all Cayley-Dickson algebras of higher dimension, it might be nice to determine if all can be built exclusively from the combination of the singularly defined rules involving the scalar basis element and like non-scalar products, and some whole number of Quaternion subalgebra triplets defining all products of unlike non-scalar basis elements. For dimension n, we will need an integer number of unique Quaternion subalgebra triplet rules to cover all $(n-1)(n-2)$ unlike non-scalar basis element products for the algebra. Each subalgebra triplet rule will cover six product combinations. Therefore, the number of required triplet rules is $T = (n-1)(n-2)/6$, which must be an integer. Each ordered triplet rule has three elements, so there will be $3T = (n-1)(n-2)/2$ total triplet positions. These need to be evenly distributed across all $(n-1)$ non-scalar basis elements, so each non-scalar basis element will appear in $3T/(n-1) = (n-2)/2$ Quaternion subalgebra triplets. The $(n-2)/2$ Quaternion subalgebra triplets that any given non-scalar basis element appears in will have $3(n-2)/2$ total positions and therefore $(n-2)$ positions not including the common basis element occupation, just the number required to have any single non-scalar basis element appear once with each of the other non-scalar basis elements, singularly defining product pairs. This should, and indeed does work for all

Cayley-Dickson algebras with $n \geq 4$.

Plugging in $n = 8$, we will have for Octonion Algebra $(8 - 1)(8 - 2) = 42$ unlike non-scalar basis element product pairs. The number of Quaternion subalgebra triplets is $(8 - 1)(8 - 2)/6 = 7$. There will be $(8 - 1)(8 - 2)/2 = 21$ total positions in these seven triplet rules, and each non-scalar basis element will appear in $(8 - 2) / 2 = 3$ triplet rules. The three triplets any given non-scalar basis element appears in have $(8 - 2) = 6$ additional positions, one slot for each of the remaining non-scalar basis elements, as required to define all product pairs once.

For Sedenion Algebra $n = 16$, we will need to determine $(16 - 1)(16 - 2) = 210$ unlike non-scalar basis element products pairs There must be $(16 - 1)(16 - 2)/6 = 35$ Quaternion subalgebra triplets. Each non-scalar basis element must appear in $(16 - 2)/2 = 7$ of them. The 7 triplets that any single non-scalar basis element appears in have $(16 - 2) = 14$ additional triplet positions, one slot for each of the remaining 14 non-scalar basis elements.

1.0 Enumerating Cayley-Dickson Algebra Quaternion subalgebra basis element triplets

Quaternion Algebra has three non-scalar basis elements, and we may trivially enumerate them abiding by the xor rule with the unordered triplet $\{e_1\ e_2\ e_3\}$. We have two possible orientation choices given by $(e_1\ e_2\ e_3)$ and $(e_2\ e_1\ e_3)$. For $n > 4$ where multiple Quaternion subalgebra oriented permutation triplet multiplication rules are required, we need a method to uniquely enumerate them from the full set of non-scalar basis elements, preferably in a general way that will work for all such hypercomplex algebras. All Cayley-Dickson algebras have dimension $n = 2^m$ where m is an integer ≥ 0. We enumerate the scalar basis element with index 0 and the non-scalar basis elements 1 through $2^m - 1$. For Quaternion Algebra, its triplet set is the sole triplet of three binary numbers in the range 01 to 11 that xor to zero. Since we know the xor rule applies for all possible Quaternion subalgebra triplets, for a 2^m dimension Cayley-Dickson algebra we might try forming all possible sets of three different binary index numbers that xor to 0 from the set of binary numbers in the range 1 through $2^m - 1$.

For Octonion Algebra n=8, the representation needs to use 3-bit binary numbers: 000 through 111 for basis element indexes. Applying our proposed triplet index identification rule, the following are all possible combinations of three different binary numbers in the range 001 to 111 that xor to zero. The proposed triplet representation rule comes up with the correct number of Quaternion subalgebra triplets for Octonion Algebra. The Quaternion Algebra representation above is the first below.

{001} e_1	{111} e_7	{101} e_5	{110} e_6	{101} e_5	{110} e_6	{111} e_7
{010} e_2	{110} e_6	{111} e_7	{101} e_5	{100} e_4	{100} e_4	{100} e_4
{011} e_3	{001} e_1	{010} e_2	{011} e_3	{001} e_1	{010} e_2	{011} e_3

Since we anticipate having Octonion subalgebras for Sedenion Algebra, the above Octonion Algebra seven will be part of the Sedenion set also. We must double the range for our binary numbers to 4-bit integers 0000 to 1111 to enumerate the basis element set. We can complete the representations for all Quaternion subalgebra triplets for Sedenion Algebra by filling out the remainder of combinations of three different 4-bit integer indexes in the range 0001 through 1111 that xor to 0.

{0100} e_4	{0101} e_5	{0110} e_6	{0111} e_7	{0100} e_4	{0101} e_5	{0110} e_6
{1000} e_8	{1000} e_8	{1000} e_8	{1000} e_8	{1001} e_9	{1001} e_9	{1001} e_9
{1100} e_{12}	{1101} e_{13}	{1110} e_{14}	{1111} e_{15}	{1101} e_{13}	{1100} e_{12}	{1111} e_{15}

{0111} e_7	{0010} e_2	{0011} e_3	{0100} e_4	{0101} e_5	{0110} e_6	{0111} e_7

{1001} e_9	{1000} e_8	{1000} e_8	{1010} e_{10}	{1010} e_{10}	{1010} e_{10}	{1010} e_{10}
{1110} e_{14}	{1010} e_{10}	{1011} e_{11}	{1110} e_{14}	{1111} e_{15}	{1100} e_{12}	{1101} e_{13}

{0001} e_1	{0010} e_2	{0011} e_3	{0100} e_4	{0101} e_5	{0110} e_6	{0111} e_7
{1000} e_8	{1001} e_9	{1001} e_9	{1011} e_{11}	{1011} e_{11}	{1011} e_{11}	{1011} e_{11}
{1001} e_9	{1011} e_{11}	{1010} e_{10}	{1111} e_{15}	{1110} e_{14}	{1101} e_{13}	{1100} e_{12}

{0001} e_1	{0010} e_2	{0011} e_3	{0001} e_1	{0010} e_2	{0011} e_3	{0001} e_1
{1010} e_{10}	{1100} e_{12}	{1100} e_{12}	{1100} e_{12}	{1101} e_{13}	{1101} e_{13}	{1110} e_{14}
{1011} e_{11}	{1110} e_{14}	{1111} e_{15}	{1101} e_{13}	{1111} e_{15}	{1110} e_{14}	{1111} e_{15}

As required by our numeric analysis above, each of our 15 non-scalar Sedenion basis elements appear 7 times in these 35 different unordered basis element triplets. This representation process can be extended by continuing the dimension doubling to n = 32, 64, 128 ….

If we assign orientations for all Quaternion subalgebra basis element triplets, and include the fixed scalar and like non-scalar basis element products, all n^2 basis element products will be defined, giving a full description of the algebra with variations determined by the particular choice of Quaternion subalgebra triplet orientations.

2.0 Octonion Algebra limitations on Quaternion subalgebra triplet orientations

We may have characteristics expected from an algebra that put restrictions on the orientation choices. The first Cayley-Dickson algebra with a Quaternion subalgebra is Octonion Algebra. There are seven Quaternion subalgebra triplets each with two independent orientation choices, yielding $2^7 = 128$ possible variations. We expect Octonion Algebra to be a normed composition algebra. Define an algebraic element by $\mathbf{z} = z_0\, e_0 + \ldots + z_{n-1}\, e_{n-1}$ and N($\mathbf{z}$) as the norm of $\mathbf{z} = (\mathbf{z} * \underline{\mathbf{z}})^{1/2}$ where $\underline{\mathbf{z}}$ is the conjugate of **z** formed by negating all coefficients attached to non-scalar basis elements. Every normed composition algebra will have the relationship N(**x**)N(**y**) = N(**x*****y**) for any two algebraic elements **x** and **y**. Octonion Algebra is not generally associative for multiplication, meaning **x***(**y*****z**) will not generally equal (**x*****y**)***z**. Octonion Algebra is an alternative algebra which is associative if there are only two algebraic elements: **x***(**x*****y**) = (**x*****x**)***y** and other combinations. Both of these two expectations restrict the 128 possible orientation combinations to the same set of 16. Enumerating, we have:

Right Octonion Algebra

R0	**R1**	**R2**	**R3**	**R4**	**R5**	**R6**	**R7**
(e_1 e_2 e_3)	(e_1 e_2 e_3)	(e_1 e_2 e_3)	(e_1 e_2 e_3)	(e_3 e_2 e_1)	(e_3 e_2 e_1)	(e_3 e_2 e_1)	(e_3 e_2 e_1)
(e_7 e_6 e_1)	(e_7 e_6 e_1)	(e_1 e_6 e_7)	(e_1 e_6 e_7)	(e_1 e_6 e_7)	(e_1 e_6 e_7)	(e_7 e_6 e_1)	(e_7 e_6 e_1)
(e_5 e_7 e_2)	(e_2 e_7 e_5)	(e_5 e_7 e_2)	(e_2 e_7 e_5)	(e_2 e_7 e_5)	(e_5 e_7 e_2)	(e_2 e_7 e_5)	(e_5 e_7 e_2)
(e_6 e_5 e_3)	(e_3 e_5 e_6)	(e_3 e_5 e_6)	(e_6 e_5 e_3)	(e_3 e_5 e_6)	(e_6 e_5 e_3)	(e_6 e_5 e_3)	(e_3 e_5 e_6)
(e_5 e_4 e_1)	(e_5 e_4 e_1)	(e_1 e_4 e_5)	(e_1 e_4 e_5)	(e_5 e_4 e_1)	(e_5 e_4 e_1)	(e_1 e_4 e_5)	(e_1 e_4 e_5)
(e_6 e_4 e_2)	(e_2 e_4 e_6)	(e_6 e_4 e_2)	(e_2 e_4 e_6)	(e_6 e_4 e_2)	(e_2 e_4 e_6)	(e_6 e_4 e_2)	(e_2 e_4 e_6)
(e_7 e_4 e_3)	(e_3 e_4 e_7)	(e_3 e_4 e_7)	(e_7 e_4 e_3)	(e_7 e_4 e_3)	(e_3 e_4 e_7)	(e_3 e_4 e_7)	(e_7 e_4 e_3)

Left Octonion Algebra

L0	**L1**	**L2**	**L3**	**L4**	**L5**	**L6**	**L7**
(e_3 e_2 e_1)	(e_3 e_2 e_1)	(e_3 e_2 e_1)	(e_3 e_2 e_1)	(e_1 e_2 e_3)	(e_1 e_2 e_3)	(e_1 e_2 e_3)	(e_1 e_2 e_3)

$(e_1\ e_6\ e_7)$	$(e_1\ e_6\ e_7)$	$(e_7\ e_6\ e_1)$	$(e_7\ e_6\ e_1)$	$(e_7\ e_6\ e_1)$	$(e_7\ e_6\ e_1)$	$(e_1\ e_6\ e_7)$	$(e_1\ e_6\ e_7)$
$(e_2\ e_7\ e_5)$	$(e_5\ e_7\ e_2)$	$(e_2\ e_7\ e_5)$	$(e_5\ e_7\ e_2)$	$(e_5\ e_7\ e_2)$	$(e_2\ e_7\ e_5)$	$(e_5\ e_7\ e_2)$	$(e_2\ e_7\ e_5)$
$(e_3\ e_5\ e_6)$	$(e_6\ e_5\ e_3)$	$(e_6\ e_5\ e_3)$	$(e_3\ e_5\ e_6)$	$(e_6\ e_5\ e_3)$	$(e_3\ e_5\ e_6)$	$(e_3\ e_5\ e_6)$	$(e_6\ e_5\ e_3)$
$(e_1\ e_4\ e_5)$	$(e_1\ e_4\ e_5)$	$(e_5\ e_4\ e_1)$	$(e_5\ e_4\ e_1)$	$(e_1\ e_4\ e_5)$	$(e_1\ e_4\ e_5)$	$(e_5\ e_4\ e_1)$	$(e_5\ e_4\ e_1)$
$(e_2\ e_4\ e_6)$	$(e_6\ e_4\ e_2)$	$(e_2\ e_4\ e_6)$	$(e_6\ e_4\ e_2)$	$(e_2\ e_4\ e_6)$	$(e_6\ e_4\ e_2)$	$(e_2\ e_4\ e_6)$	$(e_6\ e_4\ e_2)$
$(e_3\ e_4\ e_7)$	$(e_7\ e_4\ e_3)$	$(e_7\ e_4\ e_3)$	$(e_3\ e_4\ e_7)$	$(e_3\ e_4\ e_7)$	$(e_7\ e_4\ e_3)$	$(e_7\ e_4\ e_3)$	$(e_3\ e_4\ e_7)$

The labels Right and Left come from two intrinsic properties within the sets of ordered triplets. If we gather up from within any **Rj** column the three ordered triplets that have any one of the basis elements in common, cyclically shift them until the common basis element is in the central position, the three basis elements on the right end of these three ordered triplets will correspond to one of the other four ordered triplets, those on the left end will not. Try **R3** and e_7

$(e_1\ e_6\ e_7) \rightarrow (e_6\ e_7\ e_1)$ $\{e_1\ e_5\ e_4\}$ are members of a valid ordered triplet, $\{e_6\ e_2\ e_3\}$ are not
$(e_2\ e_7\ e_5) \rightarrow (e_2\ e_7\ e_5)$
$(e_7\ e_4\ e_3) \rightarrow (e_3\ e_7\ e_4)$

If we now gather up the three ordered triplets that have any one of the basis elements in common within any **Lj** column, and again cyclically shift them until the chosen common basis element is in the central position, the three basis elements on the left end of the ordered triplets will correspond to one of the other four ordered triplets, and those on the right end will not. Next try **L5** and e_3

$(e_1\ e_2\ e_3) \rightarrow (e_2\ e_3\ e_1)$ $\{e_2\ e_6\ e_4\}$ are members of a valid ordered triplet, $\{e_1\ e_5\ e_7\}$ are not
$(e_3\ e_5\ e_6) \rightarrow (e_6\ e_3\ e_5)$
$(e_7\ e_4\ e_3) \rightarrow (e_4\ e_3\ e_7)$

Since each basis element in every Right Octonion Algebra exhibits the right end alignment and each basis element in every Left Octonion Algebra exhibits the left end alignment, it is easy to surmise there will be no possible bijective basis element exchange mapping capable of changing the chiral state of a given Octonion Algebra. Right and Left Octonion Algebras are therefore not equivalent algebras. They have distinctly different algebraic structure, but have the common trait of being 8-dimensional normed composition division algebras and thus are all legitimate Octonion Algebras.

Now for explaining the enumerations used above for both the Right and Left Octonion algebras. The choice for **R0** is totally arbitrary, any of the eight **Rn** could have been used. The choice for **R0** will however set the optimal n enumeration for the remaining **Rn** sets and the **L0** algebra.

One observation that can be made about the sixteen ordered triplet sets is if we negate the four ordered triplets that do not include one of the basis elements, we will reproduce one of the other. In fact, if we started with a Right Octonion we will end up with another Right Octonion, and if we started with a Left Octonion, we would end up with another Left Octonion. This mapping is thus an automorphism.

For Right Octonion Algebra **Rj** where j is not zero, we may produce it from our arbitrary **R0** choice by negating the four ordered triplet multiplication rules in **R0** that do not include the basis element e_j. Similarly, for Left Algebra **Lj** where j is not zero, we may produce it from **L0** by negating the four ordered triplet multiplication rules in **L0** that do not include the basis element e_j.

As for the mapping between Right and Left Octonion Algebras, we can observe the map for **Ri** to **Li** and also **Li** to **Ri** is the involution negating all seven ordered triplet multiplication rules. This involution is an anti-automorphism. There is a simpler morph between Right Octonion and Left Octonion of different indexes, which is the negation of three ordered triplet rules that share a common basis element. Define then the two basic morphs as negating the four ordered triplets that do not include one basis element or negating the three ordered triplets that include a common basis element. The **Lj** ↔ **Rj** morph may be considered the composition of both basic morphs involving the same basis element. I have encapsulated these two basic morphs in the following rule to be used later:

Octonion Algebra 3:4 Morph Rule:
All maps between two different Octonion Algebras using the same set of seven unordered Quaternion subalgebra triplets may be reduced to the negation of three ordered permutation triplet multiplication rules that share a common basis element, or the negation of four ordered permutation triplet multiplication rules that do not include one basis element, and compositions thereof. Any ordered permutation triplet multiplication rule negations not one of these basic morphs or compositions thereof will create an algebra that is not proper Octonion.

3.0 Right and Left Ordered 9-tuples

Thinking about this intrinsic, ever-present structure for all proper Octonion Algebras led me to shorthand mnemonics to define a particular Octonion Algebra I have called Right and Left Ordered 9-tuples. The chiral side of three triplets sharing a common basis element and its end side triplet orientation can be indicated by a down arrow placed on the left side for a Left Ordered 9-tuple and on the right side for a Right Ordered 9-tuple. The three centrally located common basis element ordered triplets can be stacked such that the left to right triplet orientation for the chiral side triplet is indicated by the arrow. Using the variables a, b, c, d, e, f and g for indexes, we could select e_d to be the common basis element and $(e_a\ e_b\ e_c)$ the chiral side triplet appearing one element per row, yielding the following Right and Left Ordered 9-tuple mnemonic definitions for Right and Left Octonion Algebra

$$\begin{matrix}(e_e & e_d & e_a)\\(e_f & e_d & e_b)\downarrow\\(e_g & e_d & e_c)\end{matrix}$$

A Right orientation: $(e_a\ e_b\ e_c)$ $(e_g\ e_f\ e_a)$ $(e_f\ e_e\ e_c)$ $(e_e\ e_g\ e_b)$ $(e_e\ e_d\ e_a)$ $(e_f\ e_d\ e_b)$ $(e_g\ e_d\ e_c)$

$$\begin{matrix}(e_a & e_d & e_e)\\\downarrow(e_b & e_d & e_f)\\(e_c & e_d & e_g)\end{matrix}$$

A Left orientation: $(e_a\ e_b\ e_c)$ $(e_g\ e_f\ e_a)$ $(e_f\ e_e\ e_c)$ $(e_e\ e_g\ e_b)$ $(e_a\ e_d\ e_e)$ $(e_b\ e_d\ e_f)$ $(e_c\ e_d\ e_g)$

Orientations for triplets including the common index d and the chiral end are clear cut. The left to right triplet orientation order for the remaining three ordered triplets are determined by first selecting one of three basis elements on the non-arrow side, then going cyclically up for the next, then popping over to the arrow side for the third on the next cyclic up row. This is as compact as it can be, far superior to the Directed Fano Plane mnemonic, although the connection to the Fano Plane is simple.

The common basis element here represents the Fano Plane central position element, and the chiral end side triplet members represent the Fano Plane triangle side mid-point elements. The down arrow sets the Fano Plane mid-point connection direction, and its clockwise or counter clockwise orientation is typically duplicated for the triangle side directions. The all clockwise or all counter clockwise choice is an automorphism. The three non-arrow side basis elements represent the Fano Plane triangle vertex elements. We can see for all Right Octonion Algebras the three Fano plane arrows through the mid-point element are directed out of the vertexes, and for all Left Octonion Algebras they are directed into the vertexes. This is not an automorphism.

When dealing with Ordered 9-tuples it will be convenient to refer specifically to the triplet oriented by the arrow, and the common basis element. I have labeled these the cardinal triplet and cardinal basis element respectively.

For any given proper Octonion Algebra, we can devise a representative Ordered 9-tuple using any one of its seven ordered triplets as the cardinal triplet, and specific to both that algebra and that cardinal triplet choice, a specific unique cardinal basis element, to properly build seven equivalent Ordered 9-tuple representations, thus spanning the set of seven non-scalar basis elements with the seven cardinal triplet choices. The converse of this is if we are given just the Right or Left Octonion orientation, the oriented cardinal triplet and cardinal basis element, the particular Octonion Algebra described is known. As an example, the following seven all represent Octonion Algebra **R0**.

$(e_4\ e_1\ e_5)$	$(e_1\ e_2\ e_3)$	$(e_5\ e_3\ e_6)$	$(e_6\ e_4\ e_2)$	$(e_2\ e_5\ e_7)$	$(e_7\ e_6\ e_1)$	$(e_3\ e_7\ e_4)$
$(e_6\ e_1\ e_7)\downarrow$	$(e_4\ e_2\ e_6)\downarrow$	$(e_2\ e_3\ e_1)\downarrow$	$(e_7\ e_4\ e_3)\downarrow$	$(e_1\ e_5\ e_4)\downarrow$	$(e_3\ e_6\ e_5)\downarrow$	$(e_5\ e_7\ e_2)\downarrow$
$(e_3\ e_1\ e_2)$	$(e_7\ e_2\ e_5)$	$(e_4\ e_3\ e_7)$	$(e_5\ e_4\ e_1)$	$(e_6\ e_5\ e_3)$	$(e_2\ e_6\ e_4)$	$(e_1\ e_7\ e_6)$

Looking closely at the same position in each of these equivalent representations for **R0**, there are no duplications of basis elements in any of the 9-tuple positions.

The companion to the cyclic shift equivalence for ordered permutation triplet multiplication rules is the fact that for Ordered 9-tuples we can cyclically shift the stacking order for the three ordered permutation triplets with a common basis element without changing the Octonion Algebra it represents, so we have not seven but 21 equivalent Right Ordered 9-tuples for any given Right Octonion Algebra, and 21 equivalent Left Ordered 9-tuples for any given Left Octonion Algebra. Since there are eight proper Right Octonion Algebras, and eight proper Left Octonion Algebras, there are 168 Right Ordered 9-tuples, and 168 Left Ordered 9-tuples. It is no coincidence 168 is the order of the group PSL(2,7), the automorphism group for the Fano plane. The basis element transpositions that are a group operation representation for PSL(2,7) map within Right Octonion Algebras, or within Left Octonion Algebras, there is no cross over.

If we are given the orientations for all seven unordered Quaternion subalgebra triplets appropriate for a given Octonion Algebra, if they cannot be inserted into a Right or Left Ordered 9-tuple, the set of orientations does not describe a proper Octonion Algebra. The validation process is to first pick any non-scalar basis element to be the cardinal basis element. Cyclically shift the three ordered triplets that include it, if necessary, to centrally locate the cardinal basis element. Observe if the orientation indicated is Right or Left Octonion, rearrange the row stacking such that the given cardinal triplet orientation matches the arrow for the indicated Right or Left Ordered 9-tuple. Finally compare the orientations for the remaining three triplets in the Ordered 9-tuple to see if they match their given orientation expectations. Any discrepancy will indicate the set of triplet orientations do not define a proper Octonion Algebra.

We can generate all 16 proper Octonion Algebras from a set of seven proper enumerated and unordered triplets of basis elements by first selecting the Right Ordered 9-tuple, then take any one of the given triplets as the cardinal triplet. Orient the cardinal triplet one way then cycle in one at a time each of the other four basis elements into the cardinal basis element spot. Complete the three non-chiral side basis elements using for indexes the xor of the two placed basis element indexes in that triplet. This will define the first four Right Octonion Algebras. Then use the other orientation for the cardinal triplet and repeat for the last four Right Octonion Algebras. Doing the same using the Left Ordered 9-tuple produces all eight Left Octonion Algebras.

From this we can see that if we are given the orientation of one Quaternion subalgebra triplet, we could pick it for our cardinal triplet, and not need to do its other orientation as done above, halving the number of proper Octonion Algebras so restricted as one might expect.

We can define all 16 Octonion Algebras with the following Right and Left Oriented 9-tuples, all with cardinal basis element e_4:

R0	**R1**	**R2**	**R3**
$(e_5\ e_4\ e_1)$	$(e_3\ e_4\ e_7)$	$(e_1\ e_4\ e_5)$	$(e_2\ e_4\ e_6)$
$(e_6\ e_4\ e_2)\downarrow$	$(e_2\ e_4\ e_6)\downarrow$	$(e_3\ e_4\ e_7)\downarrow$	$(e_1\ e_4\ e_5)\downarrow$
$(e_7\ e_4\ e_3)$	$(e_5\ e_4\ e_1)$	$(e_6\ e_4\ e_2)$	$(e_7\ e_4\ e_3)$

R4	**R5**	**R6**	**R7**
$(e_7\ e_4\ e_3)$	$(e_5\ e_4\ e_1)$	$(e_6\ e_4\ e_2)$	$(e_7\ e_4\ e_3)$
$(e_6\ e_4\ e_2)\downarrow$	$(e_2\ e_4\ e_6)\downarrow$	$(e_3\ e_4\ e_7)\downarrow$	$(e_1\ e_4\ e_5)\downarrow$
$(e_5\ e_4\ e_1)$	$(e_3\ e_4\ e_7)$	$(e_1\ e_4\ e_5)$	$(e_2\ e_4\ e_6)$

L0	**L1**	**L2**	**L3**
$(e_3\ e_4\ e_7)$	$(e_1\ e_4\ e_5)$	$(e_2\ e_4\ e_6)$	$(e_3\ e_4\ e_7)$
$\downarrow(e_2\ e_4\ e_6)$	$\downarrow(e_6\ e_4\ e_2)$	$\downarrow(e_7\ e_4\ e_3)$	$\downarrow(e_5\ e_4\ e_1)$
$(e_1\ e_4\ e_5)$	$(e_7\ e_4\ e_3)$	$(e_5\ e_4\ e_1)$	$(e_6\ e_4\ e_2)$

L4	**L5**	**L6**	**L7**
$(e_1\ e_4\ e_5)$	$(e_7\ e_4\ e_3)$	$(e_5\ e_4\ e_1)$	$(e_6\ e_4\ e_2)$
$\downarrow(e_2\ e_4\ e_6)$	$\downarrow(e_6\ e_4\ e_2)$	$\downarrow(e_7\ e_4\ e_3)$	$\downarrow(e_5\ e_4\ e_1)$
$(e_3\ e_4\ e_7)$	$(e_1\ e_4\ e_5)$	$(e_2\ e_4\ e_6)$	$(e_3\ e_4\ e_7)$

4.0 {basic quad : its unordered triplet} mnemonic for specifying unoriented Octonion Algebras

There is one more mnemonic that will be useful for describing Octonion subalgebras for Sedenion Algebra. We can more generally enumerate it with variables, anticipating multiple instances requiring identical rules and structures. All of our operations are on basis element indexes, so there is no loss in generality and a reduction of clutter, to simply use the basis element indexes themselves instead of repeating the e in e_m. Enumerate Octonion non-scalar basis indexes as a, b, c, d, e, f and g.

Our standard Octonion Algebra non-scalar basis element set was enumerated using index integers 1 through 7. We have 1^2^3^4^5^6^7 = 0. Any differently enumerated Octonion Algebra with Quaternion subalgebra triplets enumerated with our xor method will also have all non-scalar basis element indexes xor to 0, so we have generally a^b^c^d^e^f^g = 0. Using this, if we have {a b c} for any unordered Quaternion subalgebra triplet index set, we have a^b^c = 0, implying d^e^f^g = 0. The quad of Octonion non-scalar basis elements excluding the three that are part of a particular Quaternion subalgebra triplet is commonly referred to as a basic quad. Each of the seven Quaternion subalgebra triplets is associated with a different basic quad. We will find it extremely useful to have all basic quad indexes xor to 0.

From $d \wedge e \wedge f \wedge g = 0$ we have the following equalities: $d \wedge e = f \wedge g$, $d \wedge f = e \wedge g$ $d \wedge g = e \wedge f$. Clearly the xor of any two basic quad indexes must be one of its associated Quaternion subalgebra triplet indexes. Since we are using variables, there is no loss of generality making the following not unique assignments giving the remaining six unordered triplet enumerations.

$d \wedge e = f \wedge g = a$	$a \wedge d \wedge e = a \wedge f \wedge g = 0$ so {a d e} and {a f g} are proper unordered triplets
$d \wedge f = e \wedge g = b$	$b \wedge d \wedge f = b \wedge e \wedge g = 0$ so {b d f} and {b e g} are proper unordered triplets
$d \wedge g = e \wedge f = c$	$c \wedge d \wedge g = c \wedge e \wedge f = 0$ so {c d g} and {c e f} are proper unordered triplets

Of course, if we were using integers not variables, the partitions would be singularly defined by xor operations, and all unordered triplets could then be defined uniquely from only knowledge of any basic quad set of index integers. This will be an important distinction later on.

We can then generally define unordered Quaternion subalgebra triplet index sets for an Octonion Algebra enumerated with non-scalar basis elements a, b, c, d, e, f and g with the following enumerations that will be used below, and popped out above in the Ordered 9-tuple definitions.

{a b c} {a d e} {a f g} {b d f} {b e g} {c d g} {c e f}

Given a non-scalar basis element index enumeration, a partition between one Quaternion subalgebra triplet and its basic quad, and one select basic quad member, we can uniquely determine all seven unordered Quaternion subalgebra triplets simply, due to our xor construction. To this end, define the following equivalent shorthand mnemonic representations for the same unordered triplet set just shown

{d e f g : a b c}	e f g order sets a b c order by choosing basic quad index d to set
{d d^a d^b d^c : a b c}	$d \wedge e = a$, $d \wedge f = b$, $d \wedge g = c$, uniquely determining all seven triplets
{d e f g : d^e d^f d^g}	since $d \wedge e = f \wedge g = a$, $d \wedge f = e \wedge g = b$, $d \wedge g = e \wedge f = c$
{e e^a e^b e^c : a b c}	
{e d g f : a b c}	
{f f^a f^b f^c : a b c}	
{f g d e : a b c}	
{g g^a g^b g^c : a b c}	
{g f e d : a b c}	

These have the advantage of indicating each basis index once, allowing comparison with other distinct enumerations at a glance, yet fully define the unordered Quaternion subalgebra basis element triplet enumerations without multiple instances of the basis elements in a larger structure.

5.0 Octonion subalgebra candidates for Sedenion Algebras

We can assume for any Sedenion Algebra all 35 Quaternion subalgebra triplets are defining structures of some number of Octonion subalgebras, that taken in isolation can be oriented in any of the 16 proper ways. Later on, we will find out any given Quaternion triplet will appear in multiple Octonion subalgebra candidates, so Octonion subalgebras of Sedenion Algebra cannot be taken in isolation from one another. We now need a method to pull out basis element sets and their Quaternion subalgebra unordered triplet enumerations for all Octonion subalgebra candidates, and demonstrate an integer number of Octonion subalgebras will cover any Sedenion Algebra without additional triplet definitions without an Octonion home.

As discussed above for Octonion Algebra, if we are given only integer indexes for one of seven basic quads, the index sets for all seven Octonion non-scalar basis elements and their unordered Quaternion subalgebra triplets are calculable. We can generate both sets equivalently using any of the other six basic quads. We identified all Quaternion subalgebra triplets by determining all possible unique sets of three different binary integers in a given range that xor to zero. Since basic quad indexes also xor to zero, it would make sense then to determine the full complement of basic quad indexes for Sedenion Algebra by picking out all possible unique sets of four different binary numbers in the range 0001 to 1111 that xor to zero. The count is 105. Since each Octonion subalgebra requires seven basic quads, there will be 105/7 = 15 different Octonion subalgebra candidates for Sedenion Algebra. Sorting out the seven different basic quads per single Octonion subalgebra redundancy we have the following set of Octonion subalgebra candidates by index numbers using our {basic quad : its triplet} mnemonic from which all unordered Quaternion subalgebra triplets can be determined:

{4 5 6 7 : 1 2 3}	O1
{8 9 10 11 : 1 2 3}	O2
{12 13 14 15 : 1 2 3}	O3
{8 15 14 9 : 7 6 1}	O4
{11 12 13 10 : 7 6 1}	O5
{8 13 15 10 : 5 7 2}	O6
{9 12 14 11 : 5 7 2}	O7
{8 14 13 11 : 6 5 3}	O8
{10 12 15 9 : 6 5 3}	O9
{8 13 12 9 : 5 4 1}	O10
{14 11 10 15 : 5 4 1}	O11
{8 14 12 10 : 6 4 2}	O12
{15 9 11 13 : 6 4 2}	O13
{8 15 12 11 : 7 4 3}	O14
{13 10 9 14 : 7 4 3}	O15

Itemizing our set of 35 Sedenion unordered Quaternion subalgebra basis triplets, we can see all triplets defined by the mnemonics for O1 through O15 are in this set. By the by, these are shown in the orientations produced by the Cayley-Dickson doubling definition of Sedenion Algebra.

$\{ e_1\ e_2\ e_3 \}$	$\{ e_1\ e_4\ e_5 \}$	$\{ e_1\ e_7\ e_6 \}$	$\{ e_1\ e_8\ e_9 \}$	$\{ e_1\ e_{11}\ e_{10} \}$	$\{ e_1\ e_{13}\ e_{12} \}$	$\{ e_1\ e_{14}\ e_{15} \}$
$\{ e_2\ e_4\ e_6 \}$	$\{ e_2\ e_5\ e_7 \}$	$\{ e_2\ e_8\ e_{10} \}$	$\{ e_2\ e_9\ e_{11} \}$	$\{ e_2\ e_{14}\ e_{12} \}$	$\{ e_2\ e_{15}\ e_{13} \}$	$\{ e_3\ e_4\ e_7 \}$
$\{ e_3\ e_6\ e_5 \}$	$\{ e_3\ e_8\ e_{11} \}$	$\{ e_3\ e_{10}\ e_9 \}$	$\{ e_3\ e_{15}\ e_{12} \}$	$\{ e_3\ e_{13}\ e_{14} \}$	$\{ e_4\ e_8\ e_{12} \}$	$\{ e_4\ e_9\ e_{13} \}$
$\{ e_4\ e_{10}\ e_{14} \}$	$\{ e_4\ e_{11}\ e_{15} \}$	$\{ e_5\ e_8\ e_{13} \}$	$\{ e_5\ e_{12}\ e_9 \}$	$\{ e_5\ e_{10}\ e_{15} \}$	$\{ e_5\ e_{14}\ e_{11} \}$	$\{ e_6\ e_8\ e_{14} \}$
$\{ e_6\ e_{15}\ e_9 \}$	$\{ e_6\ e_{12}\ e_{10} \}$	$\{ e_6\ e_{11}\ e_{13} \}$	$\{ e_7\ e_8\ e_{15} \}$	$\{ e_7\ e_9\ e_{14} \}$	$\{ e_7\ e_{13}\ e_{10} \}$	$\{ e_7\ e_{12}\ e_{11} \}$

The term Octonion subalgebra candidate requires some clarification. It was clear-cut when using the term Quaternion subalgebra “triplet” that the full Quaternion Algebra the subalgebra referred to was not produced solely from the “triplet”, only the product rules for pairs of unlike non-scalar basis elements are defined by the triplet. A given orientation for a Quaternion subalgebra triplet will however unambiguously define a particular full Quaternion Algebra. The same limitations are implied with the notion Octonion subalgebra candidate. The candidate structures referred to above are only covering the products of unlike non-scalar basis element products with seven deterministic unordered Quaternion subalgebra triplets, not the entire Octonion Algebra. Like the Quaternion Algebra case, once all seven Quaternion subalgebra triplets in the Octonion subalgebra candidate are properly oriented, the Octonion subalgebra candidate will unambiguously define one particular Octonion Algebra.

Before continuing, it is worthy to note the largest subalgebra for Sedenion Algebra is Octonion Algebra and the number of Octonion subalgebra candidates for Sedenion Algebra dimension n=16: (n – 1)=15. We have for the largest subalgebra for Octonions n=8: (8 – 1) = 7 Quaternion subalgebras, and for the largest subalgebra for Quaternions n=4: (4 – 1) = 3 Complex subalgebras, and for the one subalgebra for Complex Algebra n=2: (2 – 1) = 1 Real number subalgebra, and finally for Reals n=1: (1 – 1) = 0 indicating no subalgebras for Real number Algebra.

Each of the 35 Quaternion subalgebra triplets for Sedenion Algebra appear three times in O1 through O15, giving the required 3*35 = 105 copies for our 105 basic quads. The twelve basis elements in the three basic quads associated with the three appearances of any given Quaternion subalgebra triplet are unique and when appended to the triplet basis elements span all 15 non-scalar Sedenion basis elements. The basis element intersection between any two Octonion subalgebra candidates is always a Quaternion subalgebra triplet. This is a very important fact, since for any given Sedenion Algebra, all 35 triplets must be singularly oriented in each intersection and in the three Octonion subalgebras they appear in. We will find this to be the bone breaker for defining all Octonion subalgebra candidates as proper Octonion Algebras. It can’t be done. The single Quaternion subalgebra triplet intersection between two Octonion subalgebra candidates also means two different selected triplets will never be found in more than one Octonion subalgebra candidate.

We cannot effectively deal with 2^{35} different orientation combinations for each of the 35 Quaternion subalgebra triplets to determine which of O1 through O15 can end up proper Octonion Algebras, the sets are too numerous to grasp any general algebraic principles from. Even running through all proper Octonion Algebra assignment combinations for O1 through O15 has too many variations. A better approach will be to determine a minimal subset of O1 through O15 that prevents all from being valid Octonion Algebras. The minimum number is five, and they must be carefully chosen. We have now laid the foundation to form a fully algebraic proof that Sedenions are not a normed composition division algebra.

6.0 A fully algebraic proof Sedenion Algebra is not a normed composition division algebra.

Five Octonion subalgebra candidates for Sedenion Algebra will have a total of ten intersecting Quaternion subalgebra triplets. Our goal will be to determine a set of five separate Octonion subalgebra candidates from O1 through O15 above where all ten intersecting triplets are unique, and the intersections between any given candidate and the other four will be four Quaternion subalgebra ordered triplets from that candidate algebra that do not include one of its basis elements. Call any set of five Octonion candidates that satisfy this a *K5 set*. The K is for killer, for any K5 set will kill the chances for Sedenion Algebra to be oriented as a normed composition division algebra. We will use variables for basis element indexes as done above to leave the particular Octonion subalgebra choices undetermined, coming up with a general requirement that will provide the path to all K5 sets.

Define the first of five K5 set Octonion subalgebra candidates O with non-scalar basis indexes a, b, c, d, e, f and g as we have done above. Process wise, since this particular choice is first up, we may use it to represent a free choice of one of the Octonion subalgebra candidates O1 through O15. We are free to start with a Right Ordered 9-tuple with the cardinal triplet indexes (a b c) and cardinal basis element index d, to set up a valid Octonion Algebra defined by the indicated triplet orientations:

O
(e d a)
(f d b)↓ (a b c) (g f a) (f e c) (e g b) (e d a) (f d b) (g d c)
(g d c)

If instead we made this a Left Octonion Algebra but kept the same cardinal triplet and cardinal basis element, only the orientations of the Quaternion subalgebra ordered triplets including the basis index d will be negated. Since we are looking for four triplets that do not include one basis element for our intersections between O the other four, we can be Right/Left agnostic for their orientations by selecting the four Quaternion subalgebra triplets in O that do not include basis index d. The intersections between O and the other four Octonion subalgebra candidates in the K5 set will then be the oriented permutation triplet multiplication rules (a b c), (g f a), (f e c) and (e g b). These orientations will restrict the orientation choices for the other four K5 set candidates.

From the perspective of any one of the other four Octonion subalgebra candidates in the K5 set, their intersection with O, one of (a b c), (g f a), (f e c) and (e g b), is defined for it as one of four ordered triplets not including one of its basis elements. We are free to place the Quaternion subalgebra triplet that intersects with O in the cardinal triplet positions of the 9-tuples we use to define the remaining K5 set Octonion subalgebra candidates since we can reach any proper Octonion Algebra with any of its Quaternion subalgebra triplets in the cardinal triplet position. With any cardinal triplet choice, one of its basic quad members must be in the cardinal basis element position. For the four remaining K5 set members then, define cardinal triplets uniquely from the four defined O intersections, and their cardinal basis element indexes with the variables h, i, j and k taken from the basic quads defined by the cardinal triplet choice. Each of h, i, j and k will then be used to specify the omitted basis element index selecting their set of four intersections with the other members of the K5 set. Enumerate the remaining members as Op, Oq, Or and Os with Right Ordered 9-tuples since all intersections will be Right/Left agnostic.

Op
(a^h h a)
(b^h h b)↓ (a b c) (c^h b^h a) (b^h a^h c) (a^h c^h b) do not contain h
(c^h h c)

Oq
(g^i i g)
(f^i i f)↓ (g f a) (a^i f^i g) (f^i g^i a) (g^i a^i f) do not contain i
(a^i i a)

Or
(f^j j f)
(e^j j e)↓ (f e c) (c^j e^j f) (e^j f^j c) (f^j c^j e) do not contain j
(c^j j c)

Os
(e^k k e)
(g^k k g)↓ (e g b) (b^k g^k e) (g^k e^k b) (e^k b^k g) do not contain k
(b^k k b)

Excluding the given intersections with O, we can see the enumerations for the remaining intersecting ordered triplets do not directly correspond. We need to force the issue by insisting on an equivalence for the enumerations in the remaining intersection pairs. The pairings can be determined immediately by equating triplets with the same bare index for Op, Oq, Or and Os. Our six intersection pairings to be added to the four from O are then

Op:Oq intersection (c^h b^h a) is equivalent to (f^i g^i a)
Op:Or intersection (b^h a^h c) is equivalent to (e^j f^j c)

Op:Os intersection (a^h c^h b) is equivalent to (g^k e^k b)
Oq:Or intersection (g^i a^i f) is equivalent to (c^j e^j f)
Oq:Os intersection (a^i f^i g) is equivalent to (e^k b^k g)
Or:Os intersection (f^j c^j e) is equivalent to (b^k g^k e)

Now all of the intersecting ordered permutation triplet multiplication rules were so ordered assuming all five Octonion subalgebras were individually proper Octonion Algebras since they were pulled out of Ordered 9-tuples that will enforce validity. With the bare indexes in the same positions in equivalent ordered permutation triplet multiplication rules, we should be able to equate their compound indexes in the same positions in both equivalences, meaning if we xor both, the result will be zero. Doing this just for the first three we have the following restrictions, reduced using the O triplet rules repeated for convenience.

O: (a b c) (g f a) (f e c) (e g b) (e d a) (f d b) (g d c)

c^h^f^i = b^h^g^i = 0 = e^h^i   therefore i = e^h
b^h^e^j = a^h^f^j = 0 = g^h^j   therefore j = g^h
a^h^g^k = c^h^e^k = 0 = f^h^k   therefore k = f^h

Accepting the definitions for i, j and k in terms of h and another known index, we can insert them into our Ordered 9-tuples for Op, Oq, Or and Os

Op
(a^h   h   a)
(b^h   h   b)↓   intersections (a b c) (c^h b^h a) (b^h a^h c) (a^h c^h b)
(c^h   h   c)

Oq
(b^h   e^h g)
(c^h   e^h   f)↓   intersections (g f a) (d^h c^h g) (c^h b^h a) (b^h d^h f)
(d^h   e^h   a)

Or
(a^h   g^h   f)
(b^h   g^h   e)↓   intersections (f e c) (d^h b^h f) (b^h a^h c) (a^h d^h e)
(d^h   g^h   c)

Os
(c^h   f^h   e)
(a^h   f^h   g)↓   intersections (e g b) (d^h a^h e) (a^h c^h b) (c^h d^h g)
(d^h   f^h   b)

Replacing indexes i, j and k with e^h, g^h and f^h respectively now show all intersections with identical triplet sets. Intersections including basis element indexes a, b and c show consistent orientations. Intersections including basis element indexes e, f and g show opposite orientations. With the restrictions we applied to get to this place, three intersections having issues is what we get. They do however determine the general rule from which we can itemize any K5 set.

Restricted K5 Set General Rule:
O = {d e f g : a b c}

Op = {h a^h b^h c^h : a b c}
Oq = {e^h   b^h c^h d^h : g f a}
Or = {g^h   b^h   h   e^h : e g b}
Os = {f^h   h   c^h   e^h : f e c}

If we are to secure a proof Sedenions cannot have all 35 Quaternion subalgebra triplets oriented such that all resultant Octonion subalgebras are proper Octonion Algebras, we must relax the restriction that the cardinal basis elements in Ordered 9-tuples for Op, Oq, Or and Os are their omitted basis elements we used to define their intersections. We will keep the cardinal triplets as their intersections with O. Alternate choices for the cardinal basis element are any one of the other three basic quad members defined by the O intersection cardinal triplet. Let's see what that does to Os when we move the cardinal basis element index from f^h to each of the other three basic quad indexes.

Original Os
(c^h   f^h   e)
(a^h   f^h   g)↓   intersections (e g b) (d^h a^h e) (a^h c^h b) (c^h d^h g)
(d^h   f^h   b)

Modified Os 1
(f^h   c^h   e)
(d^h   c^h   g)↓   intersections (e g b) (a^h d^h e) (a^h c^h b) (d^h c^h g)
(a^h   c^h   b)

Modified Os 2
(d^h   a^h   e)
(f^h   a^h   g)↓   intersections (e g b) (d^h a^h e) (c^h a^h b) (d^h c^h g)
(c^h   a^h   b)

Modified Os 3
(a^h   d^h   e)
(c^h   d^h   g)↓   intersections (e g b) (a^h d^h e) (c^h a^h b) (c^h d^h g)
(f^h   d^h   b)

We see each of these moves has the effect of negating two intersections. The problem with Modified Os 2 and Modified Os 3 is that they both break the previously consistent orientation (a^h c^h b) while fixing another for no net improvement. For Modified Os 1 we make two orientation changes that are both currently in conflict: (d^h a^h e) to (a^h d^h e) and (c^h d^h g) to (d^h c^h g). We now have repaired the Or:Os intersection mismatch with (a^h d^h e) now in Os, and the Oq:Os intersection mismatch with (d^h c^h g) now in Os, without breaking any other intersection. The result for Modified Os 1 is we now have only one Quaternion subalgebra triplet intersection with conflict. The 3:4 morph rule defined above tells us we cannot fix this. Rather than taking it on faith, lets go through the possible changes.

Looking at the non-intersection ordered triplets in both Original Os and Modified Os 1, we find that in addition to the two intersecting ordered triplet changes, we also find triplet (c^h f^h e) was changed to (f^h c^h e) and triplet (a^h f^h g) was changed to (f^h a^h g). These four modified triplets are the four that do not include index b. This is the :4 Morph side of the 3:4 Morph rule defined above. The Modified Os 2 move negates all four ordered triplets that do not include index e, and the Modified Os 3 move negates all four ordered triplets that do not include the index g. These three changes to the original Os definition are the complete set of :4 morphs consistent with the assigned orientation for the cardinal triplet. This may be verified by observing any oriented triplet appears in four Right or four Left Octonion forms.

The 3: morph we have not tried yet negates the three ordered triplets that include a common basis element. The intersection of the two ordered triplets in Os needing negation is d^h, so we could have achieved the same success for Os ordered triplet conflicts with the 3: morph on Os negating all ordered triplets including the index d^h since the other two intersections are untouched. Every other choice of common basis index for 3: negation will either fix one intersection conflict at the cost of breaking another, have no impact, or violate the assumed orientations for O intersections.

Our single intersection conflict we cannot fix must be resolved by negating either (b^h d^h f) in Oq or (d^h b^h f) in Or. Both Oq and Or, taken in isolation of the other, are both proper Octonion Algebras since we pulled them out of Ordered 9-tuples that enforce it. Any single triplet orientation change in any proper Octonion Algebra is certain to break it. The bottom line is for any K5 set, we can only properly orient four of five Octonion subalgebra candidates. The remaining Octonion subalgebra candidate is a broken Octonion Algebra.

The nice thing about subalgebras is if we limit the algebraic elements we work with to those with zero coefficients attached to all basis elements not part of the subalgebra basis element set, working in the full algebra is no different than working exclusively in the subalgebra. So, if we limit two Sedenion algebraic elements **x** and **y** to the basis set defined by any of the 15 Octonion subalgebra candidates plus the scalar basis element, manipulating these algebraic elements in the full Sedenion Algebra will be no different than manipulating them in the particular Octonion Algebra defined by the Octonion subalgebra candidate. If the Octonion subalgebra candidate is a proper Octonion Algebra, we will have $N(\mathbf{x})N(\mathbf{y}) = N(\mathbf{x}*\mathbf{y})$. If the Octonion subalgebra candidate is determined to be a broken Octonion Algebra, and **x** and **y** have no additional zero coefficients on any Octonion subalgebra candidate basis elements restricting them to an even smaller subalgebra, we will have $N(\mathbf{x})N(\mathbf{y}) \neq N(\mathbf{x}*\mathbf{y})$. We have just shown it is impossible to define all 15 Octonion subalgebra candidates as proper Octonion Algebras. Therefore, we will always be able to build Sedenion Algebra algebraic elements **x** and **y** such that $N(\mathbf{x})N(\mathbf{y}) \neq N(\mathbf{x}*\mathbf{y})$.

In summary, we can select any Sedenion Algebra K5 set of Octonion subalgebra candidates from the 15 possible for Sedenion Algebra given by O1 through O15 above, using the Restricted K5 Set General Form. This is done by first assigning one of O1 through O15 to be O in the general form, setting integer index values for a, b, c, d, e, f and g. Assigning any index number not one of these to the general form h variable will define choices for remaining four K5 set general form Octonion subalgebra candidates Op, Oq, Or and Os from the remaining members of O1 through O15, completing the K5 set. Every possible choice of proper Octonion Algebra for Octonion subalgebra candidates O, Op, Oq, Or and Os will lead to at least one orientation conflict for their 10 Quaternion subalgebra intersections. Since all 35 Quaternion subalgebra ordered triplets must be singularly defined, one side of the conflict will need its Quaternion subalgebra ordered triplet negated, breaking that Octonion subalgebra candidate. Therefore, we can maximally assign proper Octonion Algebras to four of five Octonion subalgebra candidates in any K5 set. This makes it impossible to consistently orient all 35 Quaternion subalgebras for a Sedenion Algebra such that every resultant Octonion subalgebra candidate specifies a valid Octonion Algebra. We can restrict any two Sedenion algebraic elements **x** and **y** to the scalar basis element - basis element set specified of any invalid Octonion subalgebra candidate by assigning zero valued coefficients to basis elements outside this basis set. If we further restrict the remaining basis element coefficient zero values to those that do not leave the algebraic elements exclusive to any smaller subalgebra, it will force the result $N(\mathbf{x})N(\mathbf{y}) \neq N(\mathbf{x}*\mathbf{y})$. Sedenions are therefore not generally a normed composition algebra and thus are open to divisors of zero, preventing them from generally being a division algebra. QED.

Let's do a specific example. Assume our choice for {d e f g : a b c} is O1: {4 5 6 7 : 1 2 3}. We may choose h to be any basic quad index for (1 2 3) in either O2 or O3. Choose h=8 from O2. Then we have

i = 5^8 = 13 j = 7^8 = 15 k = 6^8 = 14

O = {4 5 6 7 : 1 2 3} = O1
Op = {h h^1 h^2 h^3 : 1 2 3} = {8 9 10 11 : 1 2 3} = O2
Oq = {i i^7 i^6 i^1 : 7 6 1} = {13 10 11 12 : 7 6 1} = O5
Or = {j j^5 j^7 j^2 : 5 7 2} = {15 10 8 13 : 5 7 2} = O6
Os = {k k^6 k^5 k^3 : 6 5 3} = {14 8 11 13 : 6 5 3} = O8

With suitable symbolic algebra software, we can go through all possible combinations of proper Right and Left Octonion Algebras for all five, and for each combination determine if their Quaternion subalgebra triplet intersections are consistent orientations, in a matter of seconds. The best that can be done is verified to be one intersection orientation conflict.

It will be informative to tabulate the selected K5 sets for all h choices using {4 5 6 7 : 1 2 3} for O.

h = 8:	O1 O2 O5 O6 O8
h = 9:	O1 O2 O5 O7 O9
h = 10:	O1 O2 O4 O6 O9
h = 11:	O1 O2 O4 O7 O8
h = 12:	O1 O3 O4 O7 O9
h = 13:	O1 O3 O4 O6 O8
h = 14:	O1 O3 O5 O7 O8
h = 15:	O1 O3 O5 O6 O9

None of these groups of five Octonion subalgebra candidates can be oriented with proper Octonion Algebras without one or more intersecting Quaternion subalgebras indicating opposite orientations. We could have used any of the 35 Quaternion subalgebra triplets for (a b c) above, select one of three Octonion subalgebra candidates the triplet choice appears in to define a, b, c, d, e, f, and g then chosen one of 8 possible values for h to select a K5 set that cannot all be proper Octonion Algebras.

7.0 How many Octonion subalgebras for Sedenion Algebra can be proper Octonion Algebras?

This all begs the question: how many of the 15 Octonion subalgebra candidates *can* be properly defined ignoring the fact that all of the remaining candidates cannot be proper Octonion Algebras? The answer is the largest set that avoids any possible K5 set combination. Our restricted K5 set prototype again is

O = {d e f g : a b c}
Op = {h a^h b^h c^h : a b c}
Oq = {e^h b^h c^h d^h : g f a}
Or = {g^h b^h h e^h : e g b}
Os = {f^h h c^h e^h : f e c}

We see that all 15 non-scalar basis indexes are represented, and that no single basis index appears more than three times. This immediately tells us that we cannot create a K5 set from any set of seven Octonion subalgebra candidates sharing a common basis element, so they may be structured in numerous ways with all seven being valid Octonion Algebras. This is easily verified with suitable symbolic algebra software. With any of these choices we will most certainly have remaining Octonion subalgebra candidates that are not proper Octonion Algebras. Our task here is to determine how many and which of the additional Octonion subalgebra candidates may be added without coming up with a set of candidates that cannot all be valid Octonion Algebras.

First, in a general way with our a – h variable set used above, itemize all Octonion subalgebra candidates including index a and those excluding index a

{ d e f g : a b c}	O`1
{ h a^h b^h c^h : a b c}	O`2
{d^h e^h f^h g^h : a b c}	O`3
{ h g^h f^h a^h : g f a}	O`4
{c^h d^h e^h b^h : g f a}	O`5
{ h e^h d^h a^h : e d a}	O`10
{f^h c^h b^h g^h : e d a}	O`11

{ h e^h g^h b^h : e g b}	O`6
{a^h d^h f^h c^h : e g b}	O`7
{ h f^h e^h c^h : f e c}	O`8
{b^h d^h g^h a^h : f e c}	O`9
{ h f^h d^h b^h : f d b}	O`12
{g^h a^h c^h e^h : f d b}	O`13
{ h g^h d^h c^h : g d c}	O`14
{e^h b^h a^h f^h : g d c}	O`15

The intersection of any Octonion candidate not including index a with the full set of seven Octonion candidates that do include index a will be the full set of Quaternion triplets within the candidate not including index a, there will be no duplications. We mentioned above that no basis element appears in a K5 set more than three times. This means we may add any one of the Octonion subalgebra candidates not including index a to our set of seven including index a without being able to build a K5 set from any combination of five of these eight candidates. This implies these eight Octonion subalgebra candidates can be assigned proper Octonion Algebra structure in multiple ways. This also can be verified with suitable symbolic algebra software to be true.

Adding in one more Octonion subalgebra candidate from the set not including index a, we now only need three from the set of candidates that include index a to complete a possible K5 set, so we might now be able to build one. The two Octonion subalgebra candidates that do not include index a will have a single Quaternion subalgebra triplet intersection. For both, this triplet will be in the set of four Quaternion subalgebra triplets that do not include any one of its respective basic quad members, and is accounted for in our requirement for unique intersections. This leaves three remaining Quaternion subalgebra triplet intersections from each of the two candidates without basis index a. We can use them to select which of the Octonion subalgebra candidates including basis element index a will intersect with our two selected candidates that do not include index a. We are only looking for three such Octonion subalgebra candidate intersection sources, so hopefully the two sets of three unique triplets will intersect with the same three Octonion subalgebra candidates including index a. We are assured this process will not pick up the Octonion subalgebra candidate including index a that also includes the intersection triplet between our two choices not including index a since two defined Quaternion triplets will never be seen in more than one Octonion subalgebra candidate.

Let's see where this takes us. Arbitrarily select O`6 and O`7 from above for our two not including index a.

{ h   e^h   g^h   b^h : e g b}   O`6
{a^h   d^h   f^h   c^h : e g b}   O`7

Clearly their intersection is the triplet (e g b), and this triplet's basic quad for O`6 is {h  e^h  g^h  b^h}, and for O`7 is {a^h   d^h   f^h   c^h}. Selecting basic quad indexes one at a time to define the set of four triplets not including it for both, expecting (e g b) to be one of the four intersections already, we may itemize the other three triplets, and each of the intersections they have with the Octonion subalgebra set including basis index a. We have

O`6 without h:    {g^h   b^h  e}  {e^h   b^h  g}  {e^h   g^h  b}    intersection  O`11 O`5  O`3
O`6 without e^h:  {g^h   b^h  e}  {h   g^h  g}   {h   b^h  b}    intersection  O`11 O`4  O`2
O`6 without g^h: { h   e^h   e}   { e^h  b^h  g}   { h  b^h  b}    intersection  O`10 O`5  O`2
O`6 without b^h:  {h  e^h  e }   {h   g^h  g}   { e^h   g^h  b}    intersection  O`10 O`4  O`3
O`7 without a^h:  {f^h  c^h  e}   {d^h  c^h  g}   {d^h  f^h  b}    intersection  O`11 O`5  O`3
O`7 without d^h:  {f^h  c^h  e}   {a^h  f^h  g}   {a^h  c^h  b}    intersection  O`11 O`4  O`2
O`7 without f^h:   {a^h  d^h  e}   {d^h  c^h  g}   {a^h  c^h  b}    intersection  O`10 O`5  O`2
O`7 without c^h:   {a^h  d^h  e}   {a^h  f^h  g}  {d^h  f^h  b}    intersection  O`10 O`4  O`3

As we can see, the intersection of our three O`6 and O`7 Quaternion subalgebra triplets excluding the common triplet (g d c) intersect with the same four sets of three Octonion subalgebra candidates that include basis element index a. So far, so good. Including O`6 and O`7 we have four separate possible K5 sets:

K5-1 = O`3  O`5 O`6  O`7 O`11
K5-2 = O`2  O`4 O`6  O`7 O`11
K5-3 = O`2  O`5 O`6  O`7 O`10
K5-4 = O`3  O`4 O`6  O`7 O`10

To be a legitimate K5 set, we must not be able to assign proper Octonion Algebra configurations to all five in any set without causing at least one intersecting triplet orientation conflict. Our trusty symbolic algebra program shows this is indeed the case. We do still have an outstanding question as to whether or not these groups of five fit our K5 general form, or if they are something different. Redefine the definition of a restricted K5 set using primed indexes since we have already used unprimed here.

O = {d` e` f` g` : a` b` c`}
Op = {h`  a`^h`  b`^h`  c`^h` : a` b`  c`}
Oq = {e`^h`   b`^h` c`^h` d`^h` : g`  f` a`}
Or = {g`^h`   b`^h`   h`   e`^h` : e` g`  b`}
Os = {f`^h`   h`   c`^h`   e`^h` : f` e`  c`}

Now use O`7 and O`6 inserted into O and Op respectively here. We must be careful on the remaining three since we must have basis index a show up in Oq Or and Os. Their common basis index is e`^h` which must equal a. If we pick h` = h then we must have e` = a^h. A consistent mapping with this consideration is

a\` = e	b\` = g	c\` = b	d\` = d^h	e\` = a^h	f\` = c^h	g\` = f^h
h\` = h	a\`^h\` = e^h	b\`^h\` = g^h	c\`^h\` = b^h			
d\`^h\` = d	e\`^h\` = a	f\`^h\` = c	g\`^h\` = f			

Inserting these into our primed K5 set prototype we have

O = {d^h a^h c^h f^h : e g b} equivalent to O\`7
Op = {h e^h g^h b^h : e g b} equivalent to O\`6
Oq = {a g^h b^h d : f^h c^h e} equivalent to O\`11
Or = {f g^h h a : a^h f^h g} equivalent to O\`4
Os = {c h b^h a : c^h a^h b} equivalent to O\`2

This is K5-2. Now do a shift on d\` e\` f\` g\` by making e\` = d^h. We must again have e\`^h\` = a. To get this we must shift h\` to e^h. Try the following

a\` = e	b\` = g	c\` = b	d\` = a^h	e\` = d^h	f\` = f^h	g\` = c^h
h\` = e^h	a\`^h\` = h	b\`^h\` = b^h	c\`^h\` = g^h			
d\`^h\` = d	e\`^h\` = a	f\`^h\` = c	g\`^h\` = f			

O = {a^h d^h f^h c^h : e g b} equivalent to O\`7
Op = {e^h h b^h g^h : e g b} equivalent to O\`6
Oq = {a b^h g^h d : c^h f^h e} equivalent to O\`11
Or = {f b^h e^h a : d^h c^h g} equivalent to O\`5
Os = {c e^h g^h a : f^h d^h b} equivalent to O\`3

This is K5-1. Next shift e\` to c^h. Then h\` must equal b^h to make e\`^h\` = a. Our mapping becomes

a\` = e	b\` = g	c\` = b	d\` = f^h	e\` = c^h	f\` = a^h	g\` = d^h
h\` = b^h	a\`^h\` = g^h	b\`^h\` = e^h	c\`^h\` = h			
d\`^h\` = d	e\`^h\` = a	f\`^h\` = c^h	g\`^h\` = f			

Substituting once again

O = {f^h c^h a^h d^h : e g b} equivalent to O\`7
Op = {b^h g^h e^h h : e g b} equivalent to O\`6
Oq = {a e^h h d : d^h a^h e} equivalent to O\`10
Or = {f e^h b^h a : c^h d^h g} equivalent to O\`5
Os = {c b^h h a : a^h c^h b} equivalent to O\`2

This is K5-3. Finally shift e\` to f^h, then h\` = g^h. Our mapping becomes

a\` = e	b\` = g	c\` = b	d\` = c^h	e\` = f^h	f\` = d^h	g\` = a^h
h\` = g^h	a\`^h\` = b^h	b\`^h\` = h	c\`^h\` = e^h			
d\`^h\` = d	e\`^h\` = a	f\`^h\` = c	g\`^h\` = f			

Substituting we have

O = {c^h f^h d^h a^h : e g b} equivalent to O\`7
Op = {g^h b^h h e^h : e g b} equivalent to O\`6
Oq = {a h e^h d : a^h d^h e} equivalent to O\`10

Or = {f   h   g^h   a : f^h a^h g}   equivalent to O`4
Os = {c   g^h   e^h   a : d^h f^h b}   equivalent to O`3

This is the final one, K5-4. Therefore, K5-1, K5-2, K5-3 and K5-4 are all representations of a K5 set.

Any two Octonion subalgebra candidates without index a could have been used here, and we can select any of O1 through O15 to be our {d e f g : a b c} Octonion subalgebra candidate, thus showing the largest possible number of Octonion subalgebra candidates for Sedenion Algebra defining proper Octonion Algebras without intersection conflict is eight: seven that share a common basis element, and one more from any of the remaining Octonion subalgebra candidates.

In section 6.0 above the 35 unordered Quaternion subalgebra triplets for Sedenion Algebra were enumerated with unordered triplets that actually indicated the oriented triplets produced by the Cayley-Dickson doubling process. Examining their oriented Octonion subalgebra candidates, they correspond to a conflict free set of proper oriented Octonion subalgebra candidates including the Octonion Algebra that builds the Sedenions in the doubling process using basis indexes 0-7, and proper orientations for all Octonion subalgebra candidates that include basis element index 8, the next in the sequence. All of the remaining seven Octonion subalgebra candidates are broken with either one or two intersection conflicts.

8.0 A Cayley-Dickson algebra dimension doubling scheme that builds all definition variations

In section 1.0 above, relaxing the requirements in an unordered triplet to allow scalar basis elements while keeping the xor rule allowed us to associate $\{e_0\ e_0\ e_0\}$ as a representation of Real number algebra, where the product of two scalar basis elements is another scalar basis element. We then added a new basis element to the mix, and created an additional unordered triplet that placed the new basis element in the central position and the known single basis element on the right, then completed the triplet with the basis element indexed by the xor of the two placed entries, yielding and $\{e_1\ e_1\ e_0\}$. Cyclic shifts of triplet $\{e_1\ e_1\ e_0\}$ represent the three Complex Algebra basis element products given by $e_1 * e_1 = -e_0$, $e_1 * e_0 = e_1$, and $e_0 * e_1 = e_1$. The triplets here are singularly oriented, indicating Real and Complex Algebras are singularly defined. They technically are part of the fixed definitions for scalar basis products and like non-scalar basis products singularly defined for any hypercomplex algebra.

We can continue this scheme to build higher dimension algebras. Setting aside the fixed definition basis element products, we can double the dimension by doing exactly what we did for adding the new basis element e_1 to the already established set, the single e_0.

For the next step we have known basis elements $[e_0\ ,\ e_1]$ from Complex Algebra, so we have one new triplet leaving e_0 for the fixed and known definitions for scalar and like non scalar products. Add in the next sequential index for the central basis element of a new unordered triplet with previously established index 1 then complete the triplet with the xor of indexes for the two set basis elements.

$\{e_{2\wedge 1}\ e_2\ e_1\} == \{e_3\ e_2\ e_1\}$ two orientations

This is the Quaternion Algebra unordered triplet with two orientations. With the fixed product definitions and orientation options, all Quaternion Algebra variations are specified.

For the next step, we have the established set of basis elements for Quaternion Algebra $[e_0, e_1, e_2, e_3]$ so we have three known non-scalar triplets to build with our new basis element e_4 putting aside e_0, as done above. These unordered triplets will be

$\{e_{4\text{^}1}\ e_4\ e_1\} == \{e_5\ e_4\ e_1\}$ two orientations
$\{e_{4\text{^}2}\ e_4\ e_2\} == \{e_6\ e_4\ e_2\}$ two orientations
$\{e_{4\text{^}3}\ e_4\ e_3\} == \{e_7\ e_4\ e_3\}$ two orientations

We can recognize these as triplets that can be used to form Ordered 9-tuples from which we can build Octonion Algebra. The variation for triplet $\{e_3\ e_2\ e_1\}$ is still in play as it shows up in the Ordered 9-tuple cardinal triplet position, with the new basis element in the cardinal basis element position. So, we have 16 possibilities from the two independent orientation choices for four triplets, just the number we expect for Octonion Algebra. We can start with $(e_1\ e_2\ e_3)$, do all eight orientation combinations on the Ordered 9-tuple triplets sharing the new basis element, and then move to $(e_2\ e_1\ e_3)$ then repeat the eight orientations for the other three. We must then determine their Left/Right Ordered 9-tuple orientations case by case, placing the arrow on the appropriate side. We can identify which Octonion Algebra is specified by comparison with the list at the end of section 4.0.

Remember that all Ordered 9-tuples have cyclic equivalence for the stacking order for permutation triplet multiplication rules sharing the cardinal basis element. Each of the following are within cyclic shifts of the presentation in section 4.0.

Using $(e_1\ e_2\ e_3)$

R0
$(e_5\ e_4\ e_1)$
$(e_6\ e_4\ e_2)\downarrow$
$(e_7\ e_4\ e_3)$

L1	**L2**	**R7**	**L3**	**R6**	**R5**	**L4**
$(e_1\ e_4\ e_5)$	$(e_5\ e_4\ e_1)$	$(e_1\ e_4\ e_5)$	$(e_5\ e_4\ e_1)$	$(e_1\ e_4\ e_5)$	$(e_5\ e_4\ e_1)$	$(e_1\ e_4\ e_5)$
$\downarrow(e_6\ e_4\ e_2)$	$\downarrow(e_2\ e_4\ e_6)$	$(e_2\ e_4\ e_6)\downarrow$	$\downarrow(e_6\ e_4\ e_2)$	$(e_6\ e_4\ e_2)\downarrow$	$(e_2\ e_4\ e_6)\downarrow$	$\downarrow(e_2\ e_4\ e_6)$
$(e_7\ e_4\ e_3)$	$(e_7\ e_4\ e_3)$	$(e_7\ e_4\ e_3)$	$(e_3\ e_4\ e_7)$	$(e_3\ e_4\ e_7)$	$(e_3\ e_4\ e_7)$	$(e_3\ e_4\ e_7)$

Using $(e_3\ e_2\ e_1)$

R4
$(e_7\ e_4\ e_3)$
$(e_6\ e_4\ e_2)\downarrow$
$(e_5\ e_4\ e_1)$

L7	**L6**	**R1**	**L5**	**R2**	**R3**	**L0**
$(e_3\ e_4\ e_7)$	$(e_7\ e_4\ e_3)$	$(e_3\ e_4\ e_7)$	$(e_7\ e_4\ e_3)$	$(e_3\ e_4\ e_7)$	$(e_7\ e_4\ e_3)$	$(e_3\ e_4\ e_7)$
$\downarrow(e_6\ e_4\ e_2)$	$\downarrow(e_2\ e_4\ e_6)$	$(e_2\ e_4\ e_6)\downarrow$	$\downarrow(e_6\ e_4\ e_2)$	$(e_6\ e_4\ e_2)\downarrow$	$(e_2\ e_4\ e_6)\downarrow$	$\downarrow(e_2\ e_4\ e_6)$
$(e_5\ e_4\ e_1)$	$(e_5\ e_4\ e_1)$	$(e_5\ e_4\ e_1)$	$(e_1\ e_4\ e_5)$	$(e_1\ e_4\ e_5)$	$(e_1\ e_4\ e_5)$	$(e_1\ e_4\ e_5)$

All 16 Octonion Algebras are correctly indicated. No surprise since we got here by restricting the orientations of the additional Quaternion triplets including e_5, e_6 and e_7 with the Ordered 9-tuple structure. This was required to only allow proper Octonion Algebras.

Next, we now have the set of basis elements for Octonion Algebra $[e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7]$ so we have seven known non-scalar triplets to build seven new unordered triplets with our new basis element e_8 putting aside e_0, as done above. These unordered triplets will be

$\{e_{8^\wedge 1}\ e_8\ e_1\} == \{e_9\ e_8\ e_1\}$ two orientations
$\{e_{8^\wedge 2}\ e_8\ e_2\} == \{e_{10}\ e_8\ e_2\}$ two orientations
$\{e_{8^\wedge 3}\ e_8\ e_3\} == \{e_{11}\ e_8\ e_3\}$ two orientations
$\{e_{8^\wedge 4}\ e_8\ e_4\} == \{e_{12}\ e_8\ e_4\}$ two orientations
$\{e_{8^\wedge 5}\ e_8\ e_5\} == \{e_{13}\ e_8\ e_5\}$ two orientations
$\{e_{8^\wedge 6}\ e_8\ e_6\} == \{e_{14}\ e_8\ e_6\}$ two orientations
$\{e_{8^\wedge 7}\ e_8\ e_7\} == \{e_{15}\ e_8\ e_7\}$ two orientations

As with Octonion Algebra, we have restrictions on the combinations of orientations for Sedenion Algebra. The 16 different original proper Octonion orientations are still in play but are independent since they do not include any of the seven new unordered triplets including e_8. The seven ordered triplets in the original Octonion Algebra will individually intersect with the seven Octonion subalgebra candidates that include e_8. All seven Octonion subalgebras including e_8 will have three triplets including e_8 taken from the list above, each with two orientations, indicating the proper count of 16 different Octonion Algebras for each of the seven including e_8 if taken in isolation of the others. The three can be placed into an Ordered 9-tuple using e_8 as the cardinal basis element and the original Octonion Algebra ordered triplet intersection in the cardinal triplet position to enforce proper Octonion subalgebra structure.

We can then expect $16*2^7 = 2048$ different combinations of proper Octonion subalgebras for the doubled Sedenion Algebra just done, where this maximal set includes the seven Octonion subalgebras with the common basis element e_8 plus one more subalgebra given by the Octonion Algebra used to double to Sedenion Algebra.

We have 15 free choices for the common basis element in seven Octonion subalgebras, and once chosen we have eight more free choices for which one of the remaining Octonion subalgebras excluding the common basis element will be added to form the particular maximal set. So, in total we have 2048*15*8 = 245,760 combinations of maximal sets for Sedenion Algebra defining eight proper Octonion subalgebras and seven broken Octonion subalgebras. The Cayley-Dickson dimension doubling algorithm only provides one of them.

9.0 Octonion algebraic invariance and variance: why we should we bother with algebraic variability

Physical phenomenon exist that require an orientable algebraic structure, like the 3D vector cross product. The requirement a right-handed system and a left-handed system must lead to the same observable deflection direction for a charged particle moving through a magnetic field leads to the realization the magnetic field itself must also be an oriented vector. Orientation implies a choice within the variability of the algebra employed, so it is important to fully understand this variability. Physical observables must have no variability. They must be algebraic invariants, meaning the mathematical structure of a theory of observables must account for all algebraic variations in a way that the final results are unchanged for all possible definition variation for the algebra applied. This is the short definition of what I call *The Law of Algebraic Invariance*. It stands on the position no full understanding of the variability within the definition of an algebra gives us cause to find one definition preferable to any other. Any algebra based theory must have the math done within a single definition choice, and this is a free choice.

A general Octonion based theory for an observable will use some number of products of algebraic elements, likely with differential equation forms for the coefficients attached to basis elements. This mathematical structure must yield the same result for all 16 proper Octonion Algebras. If a candidate theory seems to produce the desired results, yet results are not algebraic invariants, this should be used as motivation to suspect the model is incorrect, and retooling is called for. To make this call, we must fully understand how algebraically invariant and variant product terms come about.

A simple example of an Octonion algebraic invariant form is the double product $e_5 * (e_7 * e_5)$. The product inside ( ) result is governed by the orientation of the triplet $\{e_2\ e_5\ e_7\}$, yielding $\pm e_2$. The second product will involve e_5 and e_2 and this product is also governed by the orientation of $\{e_2\ e_5\ e_7\}$. Any algebra change will either do no negations or two negations on the final result, resulting in no change in both cases. So, we have $e_5 * (e_7 * e_5) = - e_7$ for every Octonion Algebra.

This concept can be extended to any number of Octonion algebraic element products, resulting in product terms that are either algebraic invariants or algebraic variants. Observe the following table

	R0/(L0)	R1/(L1)	R2/(L2)	R3/(L3)	R4/(L4)	R5/(L5)	R6/(L6)	R7/(L7)
I	1	1	1	1	1	1	1	1
$\{e_1\ e_2\ e_3\}$	1	1	1	1	–1	–1	–1	–1
$\{e_7\ e_6\ e_1\}$	1	1	–1	–1	–1	–1	1	1
$\{e_5\ e_7\ e_2\}$	1	–1	1	–1	–1	1	–1	1
$\{e_6\ e_5\ e_3\}$	1	–1	–1	1	–1	1	1	–1
$\{e_5\ e_4\ e_1\}$	1	1	–1	–1	1	1	–1	–1
$\{e_6\ e_4\ e_2\}$	1	–1	1	–1	1	–1	1	–1
$\{e_7\ e_4\ e_3\}$	1	–1	–1	1	1	–1	–1	1

The first column of this table has row labels assigning the row context as I to represent product terms not governed by an ordered triplet rule, followed by each of the seven unordered triplet rules for Octonion Algebra. The second column is labelled **R0/(L0)** and all entries are +1 to set the context for other column ±1 table entries as follows. If the orientation for the row label triplet as defined within the column Octonion Algebra **Rn(Ln)** is the same as its orientation in **R0(L0)** the row/column intersection is +1. If the two have negated orientations, the row/column intersection is –1. The :4 side of the 3:4 morph rule is readily apparent.

Using this table, let's work out a specific double product with an algebraic variant result: $e_3 * (e_7 * e_5)$. The first product is $\pm e_2$ set by the orientation for $\{e_5\ e_7\ e_2\}$ for the particular algebra used. The second product will be governed by the orientation for $\{e_1\ e_2\ e_3\}$ with a final result $\pm e_1$. The result relative to **R0/(L0)** in any column **Rn/(Ln)** algebra will now be dependent on the product of the two table entries in column **Rn/(Ln)** row$\{e_5\ e_7\ e_2\}$ and column **Rn/(Ln)** row$\{e_1\ e_2\ e_3\}$. If the result is +1, $e_3 * (e_7 * e_5)$ evaluated in **Rn/(Ln)** will be the same as in **R0/(L0)** and if the result is –1, the result in **Rn/(Ln)** will be the opposite sign as in **R0/(L0)**. This suggests a row composition where like columns are multiplied.

The matrix of ±1 table entries is a Hadamard Matrix. As such, the composition of any two rows formed by multiplying common column values as just described, are assured to result in one of the matrix rows. That is, this row composition is closed for the table. Clearly the I row is the identity composition, and the composition of a row with itself always results in the I row. The basis element intersection for any two unordered triplets will be a single basis element which will always be found in a third unordered triplet. For our example just done, the intersection of {e_5 e_7 e_2} and {e_1 e_2 e_3} is the result of the first product; e_2. The third unordered triplet including e_2 is {e_6 e_4 e_2} Doing the row composition on {e_5 e_7 e_2} and {e_1 e_2 e_3} yields row {e_6 e_4 e_2}, and its content sets the results for e_3 * (e_7 * e_5) relative to **R0/(L0)** in any other Right(Left) algebra. The composition any of two unlike non-I rows will always be the third row sharing the intersection basis element between the two argument rows for the composition.

If before any products are performed we start out on the I row, the first product rule row and the I row composition will park the result on the first product rule row since I is the identity. This infers a single-product based algorithm where the notion of the current row and current basis element define the current variance state and result basis element of the in-progress product term. The product rule of the product between the next sequential basis element and the current basis element sets one of the row composition arguments, and the current row sets the other. The result of this composition becomes the next current row, and the basis product result becomes the next current basis element. Starting out, the current row will typically be the I row and the first basis element in the product history will be the initial current basis element. If a particular algebraic element subelement is defined fundamentally as an algebraic variant product term without any specific product history from the I row starting point, that will define its starting row defining the results of subsequent products.

Since the row composition is closed, we can do this on any number of consecutive product term products arising from products of algebraic elements, and at the end of the product history, finish up on a particular row with a particular final basis element and attached coefficient string. We must now bring into consideration the move between Right and Left Octonion Algebras. The **Rn ↔ Ln** involution is the negation of all seven ordered triplets. If we had an odd number of triplet rule compositions in the product history, the involution would do an odd number of negations netting out to a negation of the product term. An even number of triplet rule compositions would net out to no negation by the involution. So, we must track the odd/even triplet rule parity of the product history as well as determining which row we end up on and final basis element.

At the end of the product history for a given product term final result, if we end up on the I row through an even number of triplet rule compositions, that product term is an algebraic invariant. The other 15 possibilities define set partitions for separate algebraic variant classifications. Ending up on the I row through an odd number of triplet rules will require at least five products and those five are a very specific combination. This variant indicates the product term is a Right Octonion algebraic invariant and a Left Octonion algebraic invariant, but changes sign moving between Right and Left Octonion Algebras. Most physics done with Octonion Algebra will need four or less products or will not duplicate the specific five products, so in these typical situations only 14 variant classifications are reached.

We can enumerate the invariant/variant classifications, using I or the index set for the final row, and + for even parity and – for odd parity. My definitions follow

I	I–	V+{1 2 3}	V–{1 2 3}	V+{7 6 1}	V–{7 6 1}	V+{5 7 2}	V–{5 7 2}
V+{6 5 3}	V–{6 5 3}	V+{5 4 1}	V–{5 4 1}	V+{6 4 2}	V–{6 4 2}		
V+{7 4 3}	V–{7 4 3}						

Product sequences of Octonion algebraic elements can result in summed product terms having any possible final basis element result in each of these sets. Every member within a given classification will either change sign or not change sign in unison for every possible change in selected Octonion Algebra. Every algebraic invariant product term will maintain its sign for all possible Octonion Algebra changes.

Getting back to The Law of Algebraic Invariance, I state this as Law without apology. It is intuitively obvious and is actually born out in reality. Using Octonion Algebra for a potential theory unifying Electrodynamics and Gravitation leads to algebraic invariant forms for the 8-current density, all forces, work, energy density, the Octonion equivalent Poynting vector. Using the full complement of invariant algebraic products in the general algebraic form representing the Octonion equivalent of the classical stress-energy-momentum tensor divergence significantly simplifies the non-trivial task of coming up with an equivalent of the Octonion force-work equations, but with an outside differentiation on all terms, enabling construction of the conservation equations.

The corollary of the Law of Algebraic Invariance might be called The Law of the Unobservable. Observables must be algebraic invariant terms, so the algebraic variant terms could describe unobservable features. Just because they are not observable does not mean they are not important, but certainly the traditional path of observation by experimentation leading to theoretical modeling and analysis is cut off.

The standard theoretical approach of forming differential equations, but done within the structure of Octonion Algebra, will lead to specific and unique summed differential forms in select algebraic variant sets. Each of the individual terms in a given set will collectively either change sign or will not change sign when all possible changes in proper Octonion Algebra are used for their construction. If we were to assign a 0 result for their sum in each algebraic variant set, the ensemble form would then be fully invariant since $+0 = -0$. I call these algebraic variant differential equations *homogeneous equations of algebraic constraint*. They are important since they will limit the family of solutions for the differential equations describing reality.

As we have seen, we may generally classify an algebra by some characteristic, such as an eight-dimensional normed composition division algebra, yet have definition variability underneath. This variability defines the symmetries of the algebra, imposing symmetries on the differential equations formed on top of their foundation. In this way, the definition variability symmetries become the voice of the algebra. We should listen.

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