"Phase-Tube" Structure Associated with Quantum State Vector and the Born Rule

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According to non-dualistic interpretation of quantum mechanics, the initial/global/overall phase associated with a quantum state vector is related to a particular eigenstate of an observable. This phase gives raise to a "tube" like geometrical structure, associated with the state vector and the tube branches into several smaller tubes. The total number of smaller tubes is equal to the total number of eigenstates of the observable; each branch is associated with a particular eigenstate. The cross-sectional area of the initial tube is equal to the sum of cross-sectional areas of all tubes, resulting in the Born rule and also in the conservation of probability in quantum mechanics.

I. INTRODUCTION

Quantum mechanics is one of the most successful scientific description of Nature, especially in the context of microscopic systems. Yet, at the fundamental level, what kind of physical reality is being revealed by it seems to be mysterious. Various interpretations are proposed to make sense of quantum formalism and to uncover the quantum mysteries [1–14]. Recently, the present author also proposed a new 'wave-particle non-dualistic interpretation of quantum mechanics at a single-quantum level [15–23].

In Section-II, the Stern-Gerlach measurements on a spin-$\frac{1}{2}$ particle, in the context of non-duality, is briefly revised. In Section-III, the construction of "phase-tube" associated with any quantum state vector is explained with suitable diagrams and also shown how this construction naturally yields the Born rule in quantum mechanics. Section-IV contains the conclusions and discussions.

II. NON-DUALITY OF SPIN-$\frac{1}{2}$ PARTICLE IN THE STERN-GERLACH EXPERIMENT

Consider a spin-$\frac{1}{2}$ particle, initially filtered 'up along Y-axis', $|S_y :\uparrow>$, say by a filter F_{Uy}, and subjected to the Stern-Gerlach measurement along Z-axis [24, 25], SG_Z, as shown in the FIG. (1). The complex vector space of SG_Z is spanned by the eigenstates of the Z-component of spin operator, $\hat{S}_z$, and is having an unit operator $\hat{I}_z = |S_z;\uparrow>< S_z;\uparrow| + |S_z;\downarrow>< S_z;\downarrow|$. Representation of $|S_y :\uparrow>$ in SG_z's space results in the following superposition state:

$$\begin{aligned} |S_y;\uparrow> &= |S_z;\uparrow> .| < S_z;\uparrow |S_y;\uparrow> |.e^{i\alpha} + |S_z;\downarrow> .| < S_z;\downarrow |S_y;\uparrow> |.e^{i\beta} \\ &= |S_z;\uparrow> .\frac{1}{\sqrt{2}}.e^{i\alpha} + |S_z;\downarrow> .\frac{1}{\sqrt{2}}.e^{i\beta}, \end{aligned} \tag{1}$$

where, $< S_z;\uparrow |S_y;\uparrow>= \frac{1}{\sqrt{2}}e^{i\alpha}$, $< S_z;\downarrow |S_y;\uparrow>= \frac{1}{\sqrt{2}}e^{i\beta}$ and $| < S_z;\uparrow |S_y;\uparrow> | = | < S_z;\downarrow |S_y;\uparrow> | = \frac{1}{\sqrt{2}}$. Now, according to the principle of minimum phase introduced in non-dualistic interpretation, the particle in $|S_y;\uparrow>$ will enter into either $|S_z;\uparrow>$ or $|S_z;\downarrow>$, depending on whether $|\alpha| < |\beta|$ or $|\alpha| > |\beta|$, respectively; here, $|\alpha| + |\beta| = \pi$. For example, let $|\alpha| < |\beta|$, then the particle will be in $|S_z;\uparrow>$, while $|S_z;\downarrow>$ remains as an ontological empty state - see FIG. (1).

Another spin state passing through F_{Uy} will differ from the previous one only by an

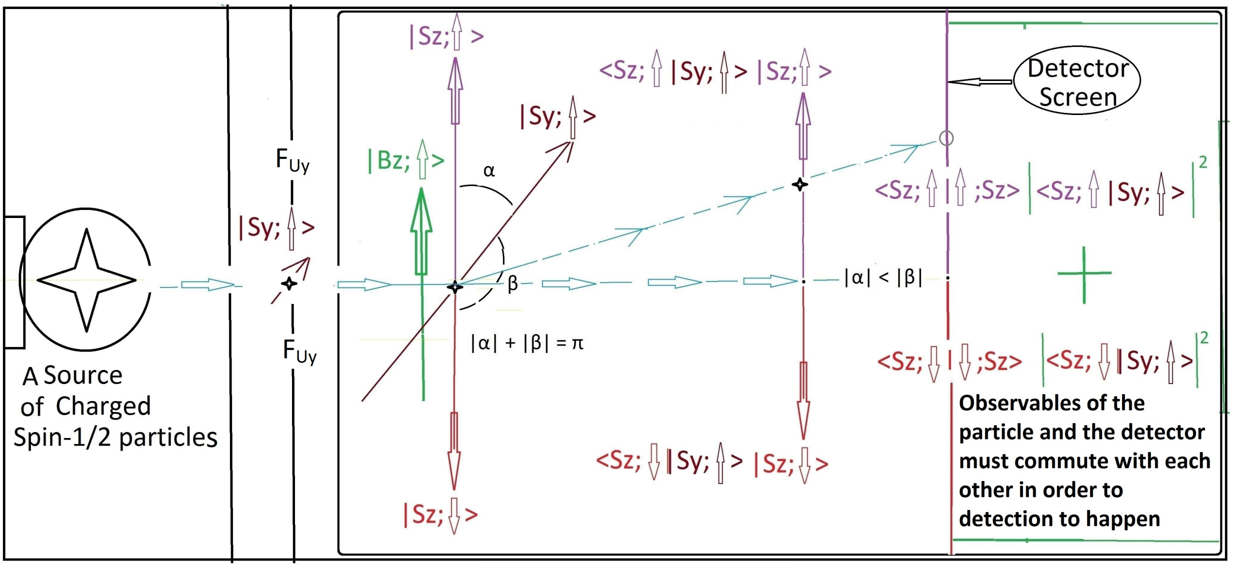

FIG. 1. **Schematic Diagram for the Stern-Gerlach Apparatus**: A source emits a charged spin-$\frac{1}{2}$ particle, whose initial state is filtered 'up along Y-axis', $|S_y;\uparrow>$, by a filter F_{Uy}. Then the particle is subjected to the Stern-Gerlach measurement along Z-axis. For the case of $|\alpha| < |\beta|$, the particle enters into $|S_z;\uparrow>$ and the state, $|S_z;\downarrow>$, remains without a particle. During the observation, the particle contributes a point to $| < S_z;\uparrow |S_y;\uparrow> |^2$, while the empty mode, $|S_z;\downarrow>$, contributes nothing.

overall phase as,

$$|S_y(\phi);\uparrow>= e^{i\phi}.|S_y;\uparrow> . \tag{2}$$

The SG_z feels $|S_y(\phi);\uparrow>$ as,

$$|S_y(\phi);\uparrow>= |S_z;\uparrow> .R.e^{i(\alpha+\phi)} + |S_z;\downarrow> .R.e^{i(\beta+\phi)}. \tag{3}$$

Depending on whether $|(\alpha+\phi)| < |(\beta+\phi)|$ or $|(\alpha+\phi)| > |(\beta+\phi)|$, the particle enters into either $|S_z;\uparrow>$ or $|S_z;\downarrow>$, respectively. Therefore, it's sufficient to notice in Eq. (1) that, the values of α and β are different for different 'up along Y' spin states.

III. PHASE-TUBE STRUCTURE OF QUANTUM STATE VECTOR

In this section, any quantum state state is shown to fall into a phase-hole, P_H, which sweeps a phase-tube, P_T, along the direction of particle's motion. If the quantum state becomes a superposition of, say, two orthogonal eigenstates of some observable, then the phase-tube branches into two smaller tubes as shown in FIG. (2). This kind of geometrical structures exist only in the complex vector space, where, the actual quantum phenomena happen.

A. Phase-Hole Representation of Quantum State Vector

As considered in Eq. (1), various spin states, filtered through F_{Uy}, can be written as given below:

$$|S_y(\alpha);\uparrow>= |S_z;\uparrow> .| < S_z;\uparrow |S_y;\uparrow> |.e^{i\alpha} + |S_z;\downarrow> .| < S_z;\downarrow |S_y;\uparrow> |.e^{i\beta}, \tag{4}$$

where, α is a discrete and random variable depending on the nature of source. Notice that, different $|S_y;\uparrow>$ states can be characterized either by α or β, because, α and β are always related as shown in section-II; here, α is chosen. The following set of vectors,

$$P_H = \{|S_y(\alpha);\uparrow> |\alpha \in [0, 2\pi]\}, \tag{5}$$

can be plotted on a complex-plane as shown in FIG. 2(a). The tips of all vectors lie on the circumference of a circle of unit radius, since, $|S_y(\alpha);\uparrow>$ is normalized to unity. Therefore, any vector belonging to P_H always passes through the F_{Uy}. In other words, in the perspective of quantum particle, our perspective of single direction in F_{Uy} appears as hole (P_H). In a nutshell, the unit vector $|S_y;\uparrow>$ is actually a phase-hole, P_H, for the quantum particle. In reality, there is nothing special about the vector $|S_y;\uparrow>$. Hence, any arbitrary state vector encountered by a quantum particle can always be regarded as a corresponding phase-hole associated with that vector.

B. Superposition of Eigenstates with Equal Amplitudes

Consider $|S_y(\alpha);\uparrow>$ in Eq. (4) as a superposition of $\hat{S}_z$'s eigenstates with equal amplitudes as given below:

$$|S_y(\alpha);\uparrow>= \left(\frac{1}{2}\right)^{\frac{1}{2}} e^{i\alpha}|S_z;\uparrow> + \left(\frac{1}{2}\right)^{\frac{1}{2}} e^{i\beta}|S_z;\downarrow> . \tag{6}$$

However, according to non-duality, as already shown earlier, $|S_y(\alpha);\uparrow>$ lies in a circular phase-hole, P_H, of unit radius. By the same token, $|S_z;\uparrow>$ and $|S_z;\downarrow>$ can also be said to lie in the corresponding circular phase-holes, say P_{UH} and P_{DH}, each with $(\frac{1}{2})^{\frac{1}{2}}$ as radius; here,

P_{UH} and P_{DH} correspond to up-hole and down-hole as shown in the FIG. 2(b), respectively. Notice that, $P_{UH} \cap P_{DH} = \{\}$, because, any vector from P_{UH} is orthogonal to any vector in P_{DH}. As the particle moves, P_H sweeps a tube, say P_T, which branches into P_{UT} and P_{DT}; here, P_{UT} and P_{DT} are phase-tubes generated by P_{UH} and P_{DH}, respectively. Also notice that, every particle state in P_{UT} has a corresponding empty state in P_{DT} and vice versa (see FIG. 2(b) & 2(c)).

When a huge number of particles, say N, enters P_T, then some of them, say N_U, moves through P_{UT} and the remaining, say N_D, through P_{DT}. Obviously, one has $N = N_U + N_D$. Also, $N_U = (A_U/A)N$ and $N_D = (A_D/A)N$; here, A, A_U and A_D are the areas of cross-section of P_T, P_{UT} and P_{DT}, respectively. Therefore, one has,

$$\frac{N_U}{N} + \frac{N_D}{N} = \frac{A_U}{A} + \frac{A_D}{A} = 1 = R_U + R_D, \tag{7}$$

where, $R_i = N_i/N = A_i/A$, corresponds to the relative frequency of detection or Born's probability; here, $i = U, D$. Therefore, it's clear that, the conservation of total number of particles implies the conservation of the total of area of cross-sections of the phase-tubes, which yields the Born rule in Eq. (7). Hence, one has,

$$A = A_U + A_D \implies \pi = \frac{\pi}{2} + \frac{\pi}{2}. \tag{8}$$

The above equation implies the spitting of the interval, $[0, \pi]$, as,

$$[0, \pi] = [0, \pi/2] \cup [\pi/2, \pi], \tag{9}$$

and the physical phenomenon in the interval, $[\pi, 2\pi]$, is exactly identical to the one in $[0, \pi]$. Therefore, depending on whether $|\alpha| \in [0, \pi/2]$ or $|\alpha| \in [\pi/2, \pi]$, the quantum particle enters into either P_{UT} or P_{DT}, respectively.

C. Superposition of Eigenstates with Unequal Amplitudes

Consider $|S_y(\alpha); \uparrow>$ in Eq. (4) as a superposition of $\hat{S}_z$'s eigenstates with unequal amplitudes as given below:

$$|S_y(\alpha); \uparrow> = \left(\frac{1}{4}\right)^{\frac{1}{2}} e^{i\alpha} |S_z; \uparrow> + \left(\frac{3}{4}\right)^{\frac{1}{2}} e^{i\beta} |S_z; \downarrow> . \tag{10}$$

All phase-tube details of the above equation is identical to the one given in section-II(b) for Eq. (6), except for how the interval, $[0, \pi]$, splits. Notice that, the phase-tube structure

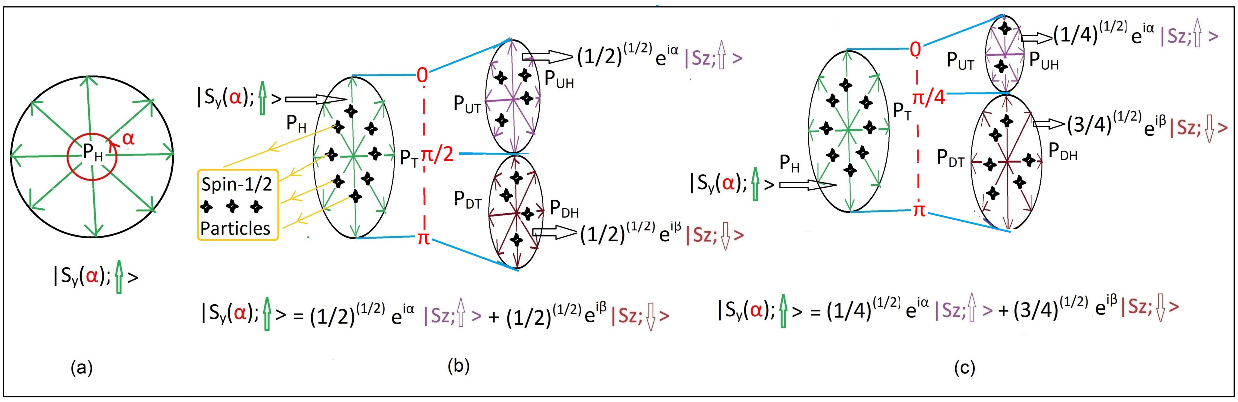

FIG. 2. **Schematic Diagram of Phase-Tubes**: **(a)** All initial states, $|S_y(\alpha);\uparrow>$, are plotted with a common origin on a complex plane. The tips of all vectors lie on a circle of unit radius, which is named as 'Phase-Hole', P_H; here, α occurs discretely and randomly. **(b) & (c)** P_H sweeps a 'Phase-Tube', P_T, in the direction of particle's motion. P_T branches into 'up-phase-tube', P_{UT}, and 'down-phase-tube', P_{DT}, because, any vector from P_{UH} is orthogonal to any vector in P_{DH}; here, P_{UH} and P_{DH} are up-phase-hole and down-phase-hole, respectively. For convenience, the state vectors are drawn symmetrically, which is not true in reality due to the nature of α. See main text for the details of equations.

given in FIG. 2(c) can be obtained from the one in FIG. 2(b) by uniformly shrinking and stretching the P_{UT} and P_{DT}, respectively.

By making use of the conservation of total cross-sectional area, one has from Eq. (10),

$$A = A_U + A_D \implies \pi = \frac{1}{4}\pi + \frac{3}{4}\pi, \tag{11}$$

implying the spitting of $[0, \pi]$ as,

$$[0, \pi] = [0, \pi/4] \cup [\pi/4, \pi], \tag{12}$$

Therefore, depending on whether $|\alpha| \in [0, \pi/4]$ or $|\alpha| \in [\pi/4, \pi]$, a quantum particle enters into either P_{UT} or P_{DT}, respectively.

D. General Case of Superposition of $\hat{S}_z$'s Eigenstates

The above analysis can be straightforwardly applied to the generic case given in Eq. (4),

$$|S_y(\alpha);\uparrow>=|S_z;\uparrow> .| < S_z;\uparrow|S_y;\uparrow> |.e^{i\alpha} + |S_z;\downarrow> .| < S_z;\downarrow|S_y;\uparrow> |.e^{i\beta}, \tag{13}$$

as follows:

By making use of the conservation of total cross-sectional area, one has,

$$A = A_U + A_D \implies \pi = |< S_z;\uparrow |S_y;\uparrow> |^2\pi + |< S_z;\downarrow |S_y;\uparrow> |^2\pi = R_U\pi + R_D\pi, \quad (14)$$

where, $R_U = |< S_z;\uparrow |S_y;\uparrow> |^2$ and $R_D = |< S_z;\downarrow |S_y;\uparrow> |^2$, implying the spitting of $[0,\pi]$ as,

$$[0,\pi] = [0, R_U\pi] \cup [R_U\pi, \pi], \quad (15)$$

Hence, depending on whether $|\alpha| \in [0, R_U\pi]$ or $|\alpha| \in [R_U\pi, \pi]$, the quantum particle enters into either P_{UT} or P_{DT}, respectively.

Consider the detection of a single particle in the SG_z apparatus for the case $|\alpha| \in [0, R_U\pi]$. According to non-duality, the state $|S_y(\alpha);\uparrow>$ induces its dual-state and interacts at the detector screen according to the inner-product:

$$\begin{aligned} < S_y(\alpha);\uparrow |S_y(\alpha);\uparrow> &= |< S_z;\uparrow |S_y;\uparrow> |^2 + |< S_z;\downarrow |S_y;\uparrow> |^2 \\ &\xrightarrow[|\alpha|\in[0,R_U\pi]]{\text{Detection}} |< S_z;\uparrow |S_y;\uparrow> |^2, \end{aligned} \quad (16)$$

resulting in the detection of eigenvalue, $+\frac{1}{2}$; the particle itself contributes a point to $|< S_z;\uparrow |S_y;\uparrow> |^2$, while $|< S_z;\downarrow |S_y;\uparrow> |^2$ receives zero contribution [see Fig. 1]. When a large number of particles are sent through F_{Uy}, either one at a time or all at once, then the particles from both intervals in Eq. (15) contribute:

$$< S_y(\alpha);\uparrow |S_y(\alpha);\uparrow> = |< S_z;\uparrow |S_y;\uparrow> |^2 + |< S_z;\downarrow |S_y;\uparrow> |^2, \quad (17)$$

which is the same result as in Eq. (14) modulo π.

IV. CONCLUSIONS AND DISCUSSIONS

According to the non-dualistic interpretation of quantum mechanics, the initial/ global/ overall phase associated with a quantum state vector is related to a particular eigenstate of an observable. It's shown that this initial phase gives raise to a "phase-tube" associate with the state vector. This phase-tube branches into smaller phase-tubes corresponding to the eigenstates in the superposition representing the original state vector. The total number of smaller phase-tubes is exactly equal to the number of eigenstates of an observable. The total area of cross-section of the phase-tube at any location is always equal to the the total area of cross-section at any other location, implying the conservation of the total of cross-sectional areas, which immediately results in the Born rule and the conservation of probability in quantum mechanics. Though the present analysis is done for the case of an observable with two eigenstates, the same can also be straightforwardly extended to the general case of an observable with infinite number of eigenstate, both discrete and continuous. The results obtained in the present article may become useful in realizing the deeper mysteries of Nature at the fundamental level.

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