Augmented non-Euclidean Geometry

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Abstract

A new approach in non-Euclidean geometry is obtained by adding types of deformation of the coordinate system. The concept of curvature is clarified taking into account the type of deformation. Such deformations can be stretching-compression, bending, shear, and torsion. The straightness of reference systems, traditional for non-Euclidean geometry, remains unchanged.

1. Introduction

As is known, Bolyai-Lobachevskian hyperbolic geometry appeared as a result of the adoption of an alternative formulation of the postulate of parallel lines, which states that through the same point many lines parallel to a given line can be drawn. In another non-Euclidean geometry, elliptic, there are no parallel lines at all. Both systems are characterized by a constant curvature that is not equal to zero, and it is implied that this curvature is an analogue of Gaussian curvature. But what is special about curvature in non-Euclidean geometry? Gaussian curvature has a clear visual representation: its constituent principal curvatures reflect the degree of bending of the surface at a given point in the corresponding directions. In non-Euclidean geometry, curvature is determined indirectly, by the axiomatics of the behavior of lines considered straight, and by the sum of the angles of triangles. However, direct visualization of curvature in non-Euclidean geometry is possible and can be achieved by refining the type of deformation of the coordinate system. These can be bending, stretching-compression, shear, and torsion. Let us consider the properties of non-Euclidean geometry corresponding to these types of deformation.

2. One-Dimensional Uniform Stretching-Compression

Let us take a straight line lying on the plane and indicate a linear scale on it. Let us consider the longitudinal stretching or compression of this line relative to the plane. The longitudinal dimensions of figures on the plane, measured along this line and along its linear scale, will change in inverse proportion to the stretching or compression of this line. What is the point of this reasoning?

As we know from the special theory of relativity, moving bodies from the point of view of a stationary observer undergo a kinematic effect of reduction in linear dimensions along the direction of motion in proportion to the value of the Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \tag{1}$$

where:

- v – relative speed between observer and moving object,
- c – speed of light.

Reducing the longitudinal dimensions of a moving body means exactly the same reduction of an imaginary linear scale moving with the body along the direction of movement. Accordingly, an observer moving with the body does not observe any longitudinal reduction in the size of the body next to him. At the same time, from the point of view of a moving observer, the sizes of the bodies relative to which he is moving and the distances between them are reduced along the direction of motion by the Lorentz factor. It turns out that the scale of a stationary reference system for a moving body becomes uniformly elongated along the direction of movement of the body, and this stretching takes place both from the point of view of a moving observer and a stationary one. The stretching of the scale along the direction of movement can be designated as “apparent”, since it is noted at the moment of flight and completely disappears when the moving body stops.

Thus, it can be stated that the uniform movement of a body is accompanied by a relativistic kinematic effect of the apparent stretching of space along the direction of movement. It is important to note that the effect of longitudinal stretching of space appears not only to the moving observer, but also to his clock, as a result of which it runs slower relative to the clock of a stationary observer.

3. Gradient Stretching-Compression

Let us take a plane and indicate on it a scale that is constantly increasing or decreasing relative to some selected point O (that is, the center) on this plane. In this case, the curvature will be due to the deformation of the coordinate system in the form of gradient stretching or compression, respectively, when moving away from the center of the plane. Increasing and decreasing the scale away from the center affects the length of the circle centered at point O as follows:

$$L = 2\pi \sin(r) \tag{2}$$

$$L = 2\pi \sinh(r) \tag{3}$$

where:

- r – length of the radius vector.

Let us denote a plane with increasing scale as we move away from the center as “pseudo-circular”, and a plane with decreasing scale as “pseudo-hyperbolic”.

On a pseudo-circular plane (PCP), the scale at distances from the center r = π becomes infinitely large. Straight lines on the PCP have a finite length ≤ 2π, but look exactly the same as on the Euclidean plane, namely as straight lines going to infinity and remaining externally equidistant. However, as they move away from the center of the PCP, the actual distance between them tends to zero.

The pseudo-hyperbolic plane (PHP) has a finite size and looks like a circle whose radius is:

$$\int_0^\infty \frac{r}{\sinh(r)} dr = \frac{\pi^2}{4} \tag{4}$$

The Beltrami-Klein model corresponds exactly to the PHP. Straight lines on the PHP are chords inside the PHP circle. It is generally accepted that the Beltrami-Klein model illustrates a situation in which Euclid’s axiom on parallel lines does not hold: through a point A that does not lie on line *b*, there are arbitrarily many straight lines that do not intersect it (see Fig. 1). However, if we set the requirement that parallel lines must look exactly like parallel lines, that is, be visually equidistant, then Euclid’s axiom about parallel lines is satisfied and there is only one line passing through point A, parallel to this line, and the remaining lines simply do not intersect with the line *b*.

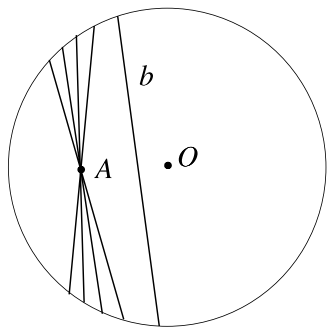

Figure 1. Straight lines on the PHP
(Beltrami-Klein model)

Thus, we state that the gradient stretching or compression of the plane, expressed through a corresponding change in the local scale, preserves the external (visual) straightness of the straight lines.

4. Flat Bend

Bending deformation on a plane has two types. The first case is when the radii of one or two bends lie on the normal to the plane, as a result of which we obtain a surface characterized at each point by two principal curvatures and the product between them is Gaussian curvature. The second case is when the bend radius lies in the plane, which results in a non-Euclidean plane with a flat bend.

A flat-bending plane (FBP) is characterized by a constant radius of curvature at any distance from some selected point O (that is, the center) of the FBP.

When we talk about the apparent (visible) or internal (intrinsic) curvature of a curved line, we mean that this is the curvature relative to the center of the FBP. A curved line with a concavity towards the center of the FBP will be considered to have a positive apparent curvature. Conversely, a curve that bulges toward the center of the FBP will be considered to have negative apparent curvature.

As for the sign of the internal curvature of the curve, there appears an internal convexity or concavity, which may not coincide with the visible one. The signs of the apparent and internal curvature of the curve may or may not coincide. Thus, on the FBP, the internal curvature of central circles with radius $r < R$ is positive, and with radius $r > R$ it is negative. A central circle with radius $r = R$ has zero internal curvature and is considered a straight line.

Consider a central circle of arbitrary radius r. The internal curvature of the central circle is:

$$k_C = \widetilde{k}_C - k_B \quad (5)$$

where:

- k_C – internal curvature of the central circle,
- $\widetilde{k}_C$ – apparent (visible) curvature of the central circle, $\widetilde{k}_C = 1/r$,
- k_B – curvature of the FBP, $k_B = 1/R$

How are the internal and apparent curvature of the curve related to each other at points where the angle between the tangent and the radius vector from the center of the NAPI is not straight? The answer lies in the following formula:

$$k = \widetilde{k} - k_B \sin^3(\beta) \tag{6}$$

where:

- k – internal curvature of the curve,
- $\tilde{k}_C$ – apparent (visible) curvature of the curve,
- k_B – curvature of the FBP,
- β – angle between tangent and radius vector.

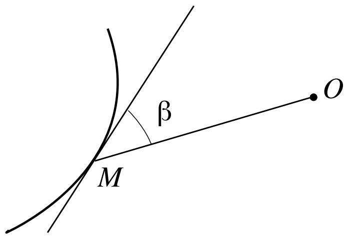

Figure 2. Angle between tangent and radius vector

Let us take a round cone and align its base with the FBP we are considering, so that the axis of the cone will pass through the center of the FBP at point O, and the radius of the base of the cone and the radius of the circle of zero internal curvature will be equal. Let us add an arbitrary plane P passing through the point O. The orthogonal projection onto the FBP of conic sections (ellipses, parabolas, hyperbolas), formed by the intersection of the plane P with the cone under consideration, form a family of conic sections, united by the fact that their focus lies at the point O, and the focal parameter is equal to the radius of curvature of the FBP. Ellipses, parabolas and hyperbolas included in this family on the considered FBP are characterized by the fact that their internal curvature at any point is zero. On the FBP with positive curvature, they can be considered as straight lines (excluding the second branches of hyperbolas deviating from the focus at point O).

The sine of the angle between the tangent to the conic section and the radius vector from the focus at point O is determined by the formula:

$$\sin(\beta) = \frac{1-\epsilon\cos(\phi)}{\sqrt{1+\epsilon^2-2\epsilon\cos(\phi)}} \tag{7}$$

where:

- ε – eccentricity of conic section,
- φ – polar angle in the equation of conic section.

On the FBP with positive curvature, any straight line, that is, a line of zero internal curvature (ellipse, parabola, hyperbola) serves as a projection of a geodesic line on some imaginary surface of rotation, the axis of rotation of which is perpendicular to the geodesic and passes through point O. The z coordinate for a given geodesic line is determined by the polar equation of a conic section:

$$\rho = \frac{p}{1-\epsilon\cos(\phi)}, \quad z = \rho \tag{8}$$

where:
- p – focal parameter of the conic section, p = R,

The angular parameter of the geodesic on the surface of revolution coincides with the polar angle φ, and the distance r from the z axis to the surface of revolution is defined as:

$$r = \frac{p}{\sin(\beta)} \tag{9}$$

For a geodesic on a surface of revolution, the Clairaut relation holds:

$$r\ \sin(\beta) = const \tag{10}$$

On the FBP with negative curvature, the second branches of hyperbolas deviating from the focus at point O act as straight lines. For these straight-hyperbolas, the relations 6 и 7 are valid.

On the FBP with negative curvature, the Euclidean axiom on parallel lines does not hold and through point A, which does not lie on the straight-hyperbola *b*, there are arbitrarily many straight-hyperbolas that do not intersect it.

5. Circular Shear

Let us consider a plane endowed with a shear of all points relative to the center at point O to the right or left side with a constant value at any distance from the center. Let us denote such a plane as a circular plane of left and right shear (CPLS, CPRS). Curvature due to shear deformation leads to the fact that the movement of point M away from the center describes a right (on the CPRS) or left (on the CPLS) logarithmic spiral. This spiral is a straight line on the circular shear plane.

6. Torsion

Let us consider a three-dimensional space with a selected z-axis, around which the normal plane undergoes a rotation through a certain angle when moving along this axis. In this case, points of adjacent normal planes experience a shear relative to each other in proportion to their distance from the z axis. Let us denote such a space as left and right torsion around the z axis (LTz, RTz). Left torsion occurs when, when moving away from us along the z-axis, the planes rotate counterclockwise, and vice versa, right torsion occurs when, when moving away from us along the z-axis, the planes rotate clockwise. The movement of a point M that does not lie on the z axis along the z axis describes a helical line (left or right). The movement of point M along the normal plane away from the z axis describes a straight line. The projection of the trajectory of the point M from the z axis at an oblique angle to the z axis onto the normal plane is an Archimedean spiral. All these trajectories of the point M are straight lines in the space of torsion.

7. Remarks

Straight lines in our overview of the varieties of non-Euclidean geometry are lines of zero internal curvature, but these lines are not lines of the shortest length. Therefore, they cannot be called “geodesic”.

The traditional two-sided elliptical geometry model has no analogues among the models we have considered.

The Bolyai-Lobachevskian geometry can be considered as a non-Euclidean plane model combining gradient compression and negative flat bend. However, this combination greatly complicates the calculations. The description of the properties of such a model is achieved mainly from intuitive conclusions based on an alternative interpretation of Euclid’s postulate about parallel lines.

8. Conclusion

The traditional description of non-Euclidean geometry is inaccurate, since it does not take into account the influence of the type of deformation of the coordinate system. Adding a type of deformation introduces the necessary clarification into the concept of curvature. Studying several of the simplest varieties of non-Euclidean geometry on the plane, as well as in space, allows us to achieve a more complete understanding of the relationship between deformation and curvature.

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