The Meaning of Uncertainty and the Geometry of the Wavefunction

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Abstract

Uncertainty may result from (1) an impossibility to measure what we want to measure, i.e. an impossibility to *observe* the system, (2) the limited precision of our measurement, (3) the measurement fundamentally disturbing the system and, as such, causing the information to be unreliable, (4) an uncertainty that is inherent to Nature. The latter position is referred to as the Copenhagen interpretation of quantum mechanics. We agree with Lorentz's and Einstein's viewpoint that there is no need to elevate indeterminism to a philosophical principle. The more important question is: how does quantum physics *model* it? How does it *deal* with it?

This paper offers some thoughts on that and, in the process, highlights some contradictions which support Lorentz's (and Einstein's) position: we only have statistical indeterminism in quantum physics and, as such, quantum physics is *not* a radical departure from classical physics. Statistical indeterminism is, effectively, the fifth interpretation of uncertainty which can be added to the list above, and we think it is the right one. We illustrate our position with a detailed discussion of the wavefunction(s) in the context of Schrödinger's wave equation for the hydrogen atom. The same example also further explores the question in regard to the (possible) physical dimension of the real and imaginary part of the wavefunction. To paraphrase Feynman, we wonder what could be 'sloshing back and forth' between the real and imaginary part of the wavefunction? We think it is kinetic and potential energy. We, therefore, briefly present our two-dimensional oscillator model again, but using the metaphor of a multi-piston radial engine as a metaphor this time, and augmented by an analysis of the quantum-mechanical energy operator.

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Introduction

Quantum mechanics combines Maxwell's equations and the Planck-Einstein relation. The Planck-Einstein relation gives us Planck's quantum of action, which models an elementary oscillation: an electron is an oscillating charge, a photon, a ring current in a superconductor is an oscillation too, an atomic or molecular orbital obeys the same law, an oscillation in a two-state system, etcetera. Understanding quantum is difficult because the mathematical formalism abstracts away from such specifics. We talk of quantum-mechanical states, but we abstract away from the *physical* reality underneath: we think of them as *energy* states only, but they must represent the system as a whole. The wavefunction must have all of the information on position and momentum (linear or angular): otherwise we would not be able to apply the relevant operators and get (average) values (or probabilities) for all of the *observables* (or *measurables*) out of it.[1]

The main difference between classical physics and quantum physics is that, in quantum physics, we have only limited knowledge of the state of the system: there is uncertainty. The exact *nature* of this uncertainty is the subject of philosophical discussion. Uncertainty may result from:

1. An impossibility to measure what we want to measure, or an impossibility to *observe* the system: we might, perhaps, refer to this as an *Ungewissheit*.[2]
2. The limited precision of our measurement: this is what Heisenberg originally referred to as an *Ungenauigkeit*, i.e. before it became some metaphysical or epistemological principle.
3. The measurement might fundamentally disturb the system and, as such, cause the information to be unreliable.
4. The uncertainty is, perhaps, inherent to Nature. This philosophical position is referred to as the Copenhagen interpretation of quantum mechanics, and Heisenberg referred to it as the *Unbestimmtheitsprinzip*.

Bell's theorem is supposed to prove the latter position but a theorem depends on its assumptions – and these assumptions may be challenged. We basically agree with the remarks of the Dutch physicist H.A. Lorentz at the occasion of the 1927 Solvay Conference: there is no need to elevate indeterminism to a philosophical principle.[3] The more important question is: how does quantum physics *model* it? How

[1] Physicists prefer the term observable: a physical quantity that can be measured. This definition shows we could also refer to it as a measurable. Both nouns have the same meaning.

[2] We did not check with the philosophers here, so our terminology suggestions are just what they are: suggestions. Words do not matter, but the distinctions might.

[3] The full quote is this : "Je pense que cette notion de probabilité [Heisenberg-Bohr] serait à mettre à la fin, et comme conclusion, des considérations théoriques, et non pas comme axiome a priori, quoique je veuille bien admettre que cette indétermination correspond aux possibilités expérimentales. Je pourrais toujours garder ma foi déterministe pour les phénomènes fondamentaux, dont je n'ai pas parlé. Est-ce qu'un esprit plus profond ne pourrait pas se rendre compte des mouvements de ces électrons. Ne pourrait-on pas garder le déterminisme en en faisant l'objet d'une croyance? Faut-il nécessairement ériger l' indéterminisme en principe?"

does it *deal* with it? This paper wants to offer some thoughts on that and, in the process, highlights some contradictions which support Lorentz's (and Einstein's) position: we only have statistical indeterminism here and, hence, quantum physics is *not* a radical departure from classical physics. Hence, we will argue that quantum-mechanical uncertainty is nothing but statistical indeterminism. This is, effectively, a fifth interpretation which can be added to the list above, and we think it is the right one.

The more interesting – but related – question is whether or not we can show that quantum-mechanical amplitudes and the wavefunction[4] – think of Schrödinger's equation and the solutions to it – have *physical* meaning. We think we can.

Functions and physical dimensions

A dimensional analysis is always a good place to start when trying to *understand* the equations describing a physical situation, but what equations should we use? Feynman's canonical examples include the maser (the ammonia molecule as a two-state system), an electron moving in a lattice (*n*-state system modeling position), electron orbitals (Schrödinger's equation in a central field[5]), and many others. So where *exactly* should we start? We will probably want to start from the simplest and let us, therefore, analyze the two-state system. In fact, our short list already triggers an obvious remark: the formalism of quantum mechanics talks about the states of system but, in practice, the state is often reduced to one *aspect* only: the position state, the momentum state, the energy state, etcetera. Using Dirac's bra-ket notation, we may formally write this as:

- $| x = n - 1\rangle$, $| x = n \rangle$, $| x = n + 1\rangle$, etc. (position states in an *n*-state system[6])
- $|\, \text{mom} = p \,\rangle$ (momentum state[7])
- $|\, E = -E_R/n^2 \,\rangle$ (energy states[8])

Hence, we should be cautious and, at each stage, clearly identify what *exactly* we are talking about. These states will all be represented by a complex-valued function (the wavefunction) or a complex number (a quantum-mechanical amplitude) but, *a priori*, we should expect that the *interpretation* of what the real and imaginary part of the wavefunction or amplitude might actually *be*, might depend on the situation at hand and, while developing the argument, we should carefully watch out to not widen or narrow the meaning of the symbols we are using.

As we are talking terminology here, we should warn the reader for another potentially confusing thing: the term amplitude may refer to (i) the complex number as a whole (let us, as per the convention[9], write

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[4] The two are not necessarily the same, and their meaning may also depend on the situation that is being modelled.

[5] A central field depends on *r* only: the distance from the pointlike charge which, in the case of electron orbitals, is the nucleus (the proton inside of the hydrogen atom).

[6] Think of a lattice on a line (a linear array of atoms or molecules).

[7] The *mom* abbreviation is Feynman's, and the example here is *linear* momentum. If we are interested in the *direction*, we should probably write the momentum as a vector: ***p***. We could also have given an example of an *angular* momentum state, in which case we should also distinguish between the magnitude and the *direction* of spin. Linear momentum is a polar vector (aka a true vector). Angular momentum is an axial vector (aka a pseudovector). Both are equally real – in a *physical* sense, that is.

[8] The energies here are the energy levels of the n^{th} orbital. E_R is the Rydberg energy (ionization energy).

[9] The use of a plus or a minus sign for the *phase* ($+\theta$ or $-\theta$) in the complex exponential – hence, writing $e^{i\theta}$ or $e^{-i\theta}$ – is a matter of *mathematical convention*. In our papers, we have consistently argued the two mathematical

it as $r = a \cdot e^{-i\theta}$) or (ii) to the coefficient in front of it (a only). Because the reader may doubt this statement, we will immediately give an example out of one of the more advanced models[10]: the wavefunctions for the state with an angular dependence to Schrödinger's equation for the hydrogen atom. These wavefunctions are written as[11]:

$$\psi_{n,l,m} = \mathrm{Y}_{l,m}(\theta, \phi)\mathrm{F}_{n,l}(\rho)$$

$$\text{with:} \quad \rho\mathrm{F}_{n,l}(\rho) = e^{-\alpha\rho} \sum_{k=l+1}^{n} a_k \rho^l$$

$$\text{and:} \quad \mathrm{Y}_{l,m}(\theta, \phi) = \mathrm{P}_l^m(\cos\theta) e^{im\phi}$$

These wavefunctions are, in fact, only the *coefficient* of the actual wavefunction because the whole derivation is based on a separation of the time-dependent and the spatial part of the wavefunction. Somewhat confusingly, the same symbol (*psi*) is used to denote both, so the difference is only obvious when one writes the argument (independent variables) of the function in full:

$$\psi(\boldsymbol{r}, t) = e^{-i\frac{\mathrm{E}}{\hbar}t}\psi(\boldsymbol{r}) = e^{-\frac{\mathrm{E}}{\hbar}it}\psi_{n,l,m}(\rho, \theta, \phi)$$

This all looks rather monstrous – because it *is* ! – so let us break it down piece by piece. You should first note the switch from Cartesian coordinates $\boldsymbol{r} = (x, y, z)$ to polar (or spherical[12]) coordinates $\boldsymbol{r} = (\rho, \theta, \phi)$, because that is easier when talking circular or orbital motion.[13] In addition, the distance from the center (the *radial* coordinate r) is now measured in a natural unit that goes with the system – the Bohr radius r_B, to be precise[14]:

$$\rho = \frac{r}{r_B} = \frac{\alpha \mathrm{m_e} c}{\hbar} r$$

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possibilities may represent two *different* states: if, for some reason, the wavefunction would actually represent a physical rotation (of charge or whatever), then the two possibilities obviously represent opposite *spin* directions.

[10] So we will *not* start with the simplest of models (the two-state system), then. 😊 We think we have analyzed the two-state system – and why and how probabilities (and, therefore, amplitudes) 'slosh back and forth' (as Feynman puts it) between two states – *ad nauseam* already. See, for example, our [rewrite of Feynman's theory of probability amplitudes]().

[11] We follow the notation from Feynman's *Lectures*, from which we borrow a lot of the material. We trust that the reader will be able to look up the original *Lectures* and distinguish between Feynman's formulas and text and our presentation and interpretation of it.

[12] Polar coordinates usually refer to a two-dimensional coordinate system, so a spherical coordinate system is then its three-dimensional version.

[13] We still need to prove we are actually talking circular or orbital motion of some charge here, but we think the circumstantial evidence is fairly convincing.

[14] We wrote the Bohr radius as a fraction of the Compton radius here. The reader can verify the substitutions, including Feynman's use of e^2 (the squared charge of an electron divided by $4\pi\varepsilon_0$), by substituting the fine-structure constant (α) for its definition:

$$r_B = \frac{\hbar}{\alpha \mathrm{m_e} c} = \hbar \frac{2\varepsilon_0 hc}{\mathrm{q_e^2 m_e} c} = \frac{4\varepsilon_0 \hbar^2}{\mathrm{m_e q_e^2}} = \frac{\hbar^2}{\mathrm{m_e e^2}}$$

Talking natural units, as part of solving the (Schrödinger wave) equation(s), Feynman also writes energies E in terms of the Rydberg energy: $E = E_R \cdot \epsilon$, with $\mathrm{E}_R = \frac{\alpha^2 \mathrm{m_e} c^2}{2} = \frac{\mathrm{q_e^4 m_e} c^2}{8\varepsilon_0{}^2 h^2 c^2} = \frac{\mathrm{q_e^4 m_e}}{2 \cdot (4\pi)^2 \varepsilon_0{}^2 \hbar^2} = \frac{\mathrm{m_e e^4}}{2\hbar^2}$. Hence, ϵ is like ρ, but it is used to measure energy.

As we are talking natural units, we may also note that, as per the Planck-Einstein relation ($E = \hbar\cdot\omega \Leftrightarrow \omega = E/\hbar$), the time-dependent part of the wavefunction ($e^{-\omega\cdot t}$) may be thought of as a clock ticking at the natural frequency of this oscillation.[15] The (other) functions and symbols may be briefly explained as follows:

- The $F_{n,l}(\rho)$ function is a (finite) power series and is, obviously, just some real-valued function of the radial distance ρ.
- The $P_l^m(\cos\theta)$ functions are known as the 'associated Legendre polynomials' (or functions). They are usually written in terms of derivatives of ordinary Legendre polynomials. We must refer the reader to readily accessible material here[16]

The $Y_{l,m}(\theta, \Phi)$ functions as a whole are known as the *spherical harmonics* (beautiful name, isn't it?[17]) and they are a function of the polar and azimuthal angles θ and Φ.[18] You should note that the $\psi_{n,l,m}$ amplitude (the *coefficient* of the actual wavefunction, really) would be real-valued, *always*, if we would not have that $e^{im\Phi}$ factor, which is equal to 1 (and, therefore, equally real-valued) if $m = 0$. And, of course, if we would multiply it through with the time-dependent part of the wavefunction ($e^{-i\cdot(E/\hbar)\cdot t}$):

$$e^{-i\cdot(E/\hbar)\cdot t}\cdot e^{i\cdot m\cdot\Phi} = e^{-i\cdot[(E/\hbar)\cdot t + m\cdot\Phi]} = e^{-i\cdot(\omega\cdot t + m\cdot\Phi)}$$

Hence, this factor is just a phase *shift* and, therefore, should not matter at all in terms of the *physics* of the situation (it is just a matter of choosing our $t = 0$ point). So let us quickly look at that *quantum number*: what does it stand for? It is the *magnetic* quantum number, and it is usually denoted as m_z and referred to as the *z*-component of the angular momentum. This sounds very mysterious, and it is: *it is related to the weird 720-degree symmetry of the wavefunction of spin-1/2 particles which, in turn, results from mainstream academics not using the plus or minus sign of the imaginary unit to distinguish between the direction of spin.*

[…] You should read the latter phrase again, slowly. And because you may not understand what we are talking about here, we added an annex to this paper which briefly talks about spin and the mathematical convention(s) in regard to the sign of the imaginary unit in the wavefunction. So here we will only

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[15] We will let the reader think this through, and just remind him of the obvious formula for the cycle time (T): $\omega = 2\pi\cdot f \Leftrightarrow T = 1/f = 2\pi/\omega$. This shows the cycle time T is equal to $T = \omega/2\pi = E/2\pi\hbar = E/h$. The natural (angular) frequency is nothing but the natural time measured in *radians*: $\omega = 2\pi/T$. It is a somewhat weird idea to measure time in *radians* but, on the unit circle, the radian may be thought of as a natural distance as well as a natural time unit. It helps to literally think of an old-fashioned clock (with a hand for the seconds) ticking time away, with the tip of the hand doing the (circular) distance. Another, more abstract way, of thinking is this: we *count* the time in terms of the cycle of this oscillation (1, 2,…, *n*,…) but, if we would want to subdivide these cycles any further, we would divide them 2π (radians) rather than 12 (hours) or 60 (minutes or seconds).

[16] The superscript m is an order number here: it is *not* an exponential. It is *not* a power of P_l, in other words. We used the Wikipedia article on these mathematical functions for more detail.

[17] The remark is not cynical. One of my early blog pieces is titled Music and Math, and it is one of the blog pieces I still like: simple, logical and, therefore, beautiful.

[18] We use Feynman's notation here, and so he uses θ (*theta*) instead of some other letter (e.g. φ, *phi*) for the polar angle, which is slightly confusing because, in physics, θ is also used to denote the *phase* of the wavefunction, like in $\psi = e^{-\theta} = e^{-\omega t}$. Wikipedia says the *mathematical* convention is to use θ (*theta*) and φ (*phi*) for the polar and azimuthal angle respectively. Our *phi* (Φ) for the azimuthal angle is the capital letter *phi*. We may, therefore, use the lowercase *phi* (φ) if we would need to denote a phase, which is what we might do. As long as we know what we are talking about, it is all good, right?

remind you of what you know already: m is a number between $-l$ and $+l$ ($-l \leq m_z \leq +l$) and it gives us the (possible) orientations of the subshell. As a further reminder of the basics, we should quickly add that l is the quantum number that gives us the subshell within a given energy state n. This n is the principal quantum number, and l = 0, 1, 2,… $n - 1$. Hence, if we have one energy state only, then we have only state: l = 0.[19]

What is the point? The point is that, when thinking about the *physics* of the situation, we can forget about that that $e^{im\Phi}$ factor. Think of it as being part of the time-dependent part of the wavefunction: we just shift the origin of time. That amounts to looking at the system – the *oscillation*, that is – a tiny bit earlier or later, and that does not matter because it is a perfectly regular oscillation. What we are interested in the *shape* of the *physical* orbitals, their energies, and other *physical* variables. Hence, for all practical purposes, we should think of the coefficient of our wavefunction – or the amplitude *sensu stricto*, or the spatial (position-dependent) part of the wavefunction, or whatever you want to call it – as a real number !

Is that important? Yes, it is. Knowing that a wavefunction – *any* wavefunction, really – can always be written as the product of a time-dependent and a spatial or time-independent function is huge, and it is equally huge to know that the time-dependent part will always look like $e^{-i\cdot\omega\cdot t + \varphi}$, and that the φ here is just some random phase shift which does not matter because we can always shift the t = 0 point however we would want to shift it: the physics of the situation won't change ! This is reflected in the fact that the absolute square[20] of a complex exponential (when its coefficient a is 1, of course) is always equal to 1[21]:

$$\left|e^{i\theta}\right|^2 = |\cos\theta + i\sin\theta|^2 = \sqrt{\cos^2\theta + \sin^2\theta}^2 = \sqrt{1}^2 = 1$$

Let us continue our search of some *physical meaning* of the real and imaginary parts of the wavefunction by continuing our example.

What does it all mean?

Below we copy table 19.1 out of Feynman's Lectures, which gives us the *functional form* of those spherical harmonics: they combine sine and cosine functions. Now, we are interested in the *probability* to find the electron at point $\boldsymbol{x}$ = (x, y, z)[22], and quantum mechanics tells us we can calculate these probabilities by taking the absolute square of the $\psi(\boldsymbol{x})$ wavefunction. To be precise, the theory of

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[19] We should refer to standard textbooks here, but we think our own presentation in our classical explanation of the Lamb shift has the advantage of (1) being succinct and (2) relating it to what we said on these weird 720-degree symmetries vanishing if one would use the ± sign in front of the imaginary unit to incorporate the two possible spin directions in the analysis straight from the start.

[20] This term is a (slightly confusing, perhaps) shorthand for the square of the absolute value of a (complex- or real-valued) number. It is also referred to as the square of the *modulus* of the complex sum (sum of the real and imaginary part of the number).

[21] We apologize for writing such simple things but it is, perhaps, good to remind ourselves of what a complex number really is (the vector sum of a sine and a cosine) and, hence, that they are nothing but just one of the many logical expressions of Pythagoras's Theorem.

[22] We have a bad habit of switching from ***r*** to ***x***, or vice versa, for no reason whatsoever – except that you will find ***x*** is more common than ***r*** in the literature. A bold letter is a vector, in any case, and you may think ***r*** suggests we are working in polar rather than Cartesian coordinates, and vice versa.

operators – and of the position operator, in particular – tells us the probability *density* P(***x***) will be equal to P(***x***) = $|\psi(\boldsymbol{x})|^2 = \psi(\boldsymbol{x})\cdot\psi^*(\boldsymbol{x}) = \psi^*(\boldsymbol{x})\cdot\psi(\boldsymbol{x})$, with $\psi^*(\boldsymbol{x})$ the complex conjugate of $\psi(\boldsymbol{x})$.[23]

Orbital angular momentum, l	z-component, m	Angular dependence of amplitudes	Name
0	0	1	s
1	$+1$	$-\frac{1}{\sqrt{2}}\sin\theta e^{i\phi}$	p
	0	$\cos\theta$	
	-1	$\frac{1}{\sqrt{2}}\sin\theta e^{-i\phi}$	
2	$+2$	$\frac{\sqrt{6}}{4}\sin^2\theta e^{2i\phi}$	d
	$+1$	$-\frac{\sqrt{6}}{2}\sin\theta\cos\theta e^{i\phi}$	
	0	$\frac{1}{2}(3\cos^2\theta - 1)$	
	-1	$\frac{\sqrt{6}}{2}\sin\theta\cos\theta e^{-i\phi}$	
	-2	$\frac{\sqrt{6}}{4}\sin^2\theta e^{-2i\phi}$	
3		$\langle l,0\,|\,R_y(\theta)R_z(\phi)\,|\,l,m\rangle$	f
4		$\propto Y_{l,m}(\theta,\phi)$	g
5		$\propto P_l^m(\cos\theta)e^{im\phi}$	h
⋮			⋮

Figure 1: Spherical harmonics (source: Feynman III-19-3)

That gives us these wonderful *polar* graphs which, literally, depict the *shape* of those electron orbitals.[24] We may note here that we are taking the square of the absolute value of a *real-valued* amplitude here. Hence, what matters is the magnitude only: positive or negative amplitudes give the same probability. Take, for example, the *p*-orbital (*l* = 1) for m = 0. The spherical harmonic is a simple cosθ function and, yes, $|\cos\theta|^2 = \cos^2\theta = |\cos(-\theta)|^2 = \cos^2(-\theta)$.

So, yes, interpreting the math is not all that difficult. We are effectively talking the *physical* orbitals of the pointlike electron charge here, and the uncertainty is a mere statistical indeterminism. So it is really just like the propeller of that airplane: we do not know where it is, *exactly*, but we know it is always *somewhere*, at any given point in time. Please note this is *not* your usual crackpot interpretation of quantum physics. We may usefully quote Richard Feynman here:

> "The wave function ψ(***r***) for an electron in an atom does not describe a smeared-out electron with a smooth charge density. The electron is either here, or there, or somewhere else, but wherever it is, it is a point charge." (Feynman's Lectures, III-21-4)

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[23] The extension of quantum-mechanical ideas and formulas from one-dimensional space (a line) to three dimensions is not always as straightforward (Feynman, III-20-4) but, in this case, it surely is!

[24] To show we do google other textbooks from time to time, we refer the reader to [a chapter of a course (in physical chemistry) at the University of Michigan](): instructive, no hocus-pocus and good graphs.

Figure 2: Where is the propeller, *exactly*?[25]

The point is this: because of the high velocity[26], we are not able to precisely define the position (x, y, z) at time t, because we are not able to precisely define the initial (x_0, y_0, z_0, t_0) condition(s). Hence, we can only talk about cycles, averages, and probabilities. This rather primitive comparison with the physics of an airplane propeller triggers two more useful associations. One is the metaphor of an old-fashioned radial airplane engine, in which linear and circular motion come together (we will come back to this). The other is an analogy with the synchronization gear that was used in WW I for machineguns firing their bullets through the propeller: if there was no synchronization gear, some of the bullets would actually hit and considerably damage the propeller: the analogy with light (consisting of photons) going through a three-dimensional lattice with electrons in all kinds of orbitals readily comes to mind. We invite readers to also google scatter plots of electron position measurements for hydrogen and other atoms.[27]

However, these reflections do not solve the question we started out with: what is the *physical meaning* of the real and imaginary parts of the wavefunction? Would they have a physical dimension, like a field – something like newton per coulomb (N/C), like the electric field, for example? In addition, we should, perhaps also raise some other interpretational issues: Schrödinger's orbitals imply the electron spends most of its time right on top of the proton, so how should we think of that? We could, perhaps, imagine some short-range repulsive force here but such solution would inject entirely new dynamics and, therefore, looks pretty unacceptable: assuming the electron, somehow, does go straight through the center or, else, bounces back – fully elastically, because momentum and energy should be conserved – is the only solution but raises other questions (which we will try to examine later[28]). Back to the question of a physical dimension for the wavefunction.

[25] I downloaded this image from a website selling Christmas presents long time ago, and I have not been able to trace back from where I have got it. If someone recognizes this as *their* picture, please let us know and we will acknowledge the source or remove it.

[26] The classical analysis (Bohr orbitals) tells us the velocity of the electron in an atomic orbital is of the order of $v = \alpha \cdot c$, with α the fine-structure constant (approximately 1/137 or 0.73%) and c the speed of light. Velocities further decrease as a fraction of this velocity in outer orbitals. To be precise, $v = (\alpha/n) \cdot c$ with n = 1, 2, 3,… the (main) orbital number.

[27] The above-mentioned [basic physical chemistry course of the University of Michigan]() offers one, but here is [another one from Chemistry LibreTexts]().

[28] Our more speculative papers – such as the one on [what protons and neutrons might actually *be*]() – made a start in exploring these, and we also added a few notes on this in the Annex to this paper as part of our discussion of spin versus orbital angular momentum.

Should it have one? The argument is time and position – simple *numbers*, right? – so the wavefunction might just project these numbers onto a two-dimensional *mathematical* space only, right? Maybe. Maybe not. Perhaps the operators can give us a clue? Unfortunately not. Their physical dimensions are OK already[29]:

- The energy operator $\mathrm{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r})$ comes with the $\frac{\mathrm{N}^2\mathrm{m}^2\mathrm{s}^2}{\mathrm{m}^2\mathrm{kg}} + \mathrm{N}\cdot\mathrm{m} = \frac{\mathrm{N}^2\mathrm{m}^2\mathrm{s}^2}{\mathrm{m}^2\frac{\mathrm{Ns}^2}{\mathrm{m}}} + \mathrm{N}\cdot\mathrm{m} = \mathrm{N}\cdot\mathrm{m}$ dimension, so that is the physical dimension of energy alright.
- Likewise, the position operator *x* or and the momentum operator $\mathrm{P}_x = -i\hbar\frac{\partial}{\partial x}$ come with the physical dimension of distance (m) and momentum ($\frac{\mathrm{N\cdot m\cdot s}}{\mathrm{m}} = \mathrm{N}\cdot\mathrm{s}$) respectively.
- Finally, the angular momentum operator $\mathrm{L}_z = x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial z}$ comes with the $\mathrm{m}\frac{\mathrm{N\cdot m\cdot s}}{\mathrm{m}} - \mathrm{m}\frac{\mathrm{N\cdot m\cdot s}}{\mathrm{m}} = \mathrm{N}\cdot\mathrm{m}\cdot\mathrm{s}$ dimension, so that is, effectively the same as that of Plank's quantum of action itself (in reduced or non-reduced form).

So there is nothing lacking here: there seems to be no *need* to associate a physical dimension with the real and imaginary part of the wavefunction. However, we need to be able *explain* these probabilities in terms of the physics, right? Right. So let us soldier on. Can we think of a physical dimension that would suit the P($\boldsymbol{x}$) = $|\psi(\boldsymbol{x})|^2$ equation? Thinking of our airplane propeller again, we may think probabilities or – to be precise – probability *densities* – should match energy or mass densities, right? Hence, we are talking kg/m^3 or N·m/m^3 = N/m^2, and we can now take a square root or something, right?[30]

Correct, but note that the wavefunction here does *not* have the time-dependent part.[31] In fact, this wavefunction – the wavefunction for Schrödinger's electron orbitals – is a *real-valued* wavefunction: it is the amplitude *sensu stricto* and, hence, talking of the meaning of the real or imaginary part of this wavefunction makes no sense: there is only a real part to it. If we want to talk about the whole thing, then we should put the time-dependent part (the complex-valued function that gives the whole its real and imaginary mathematical dimension) back in.

So, again, what are we talking about, *really*?

[29] The energy operator – and the others as well, perhaps – depend on the problem at hand. The one here is derived from Schrödinger's wave equation for electron orbitals, so we basically continue the analysis for the very same problem at hand. Note that the symbols used for operators vary (with or without hat or special script). Ours are probably too simple.

[30] Note that we can often switch from energy to mass units and vice versa without too much trouble, but units matter here, and kg/m^3 or J/m^3 are different units. The physical dimension of the c^2 in the mass-energy equivalence relation (E = mc^2) matters here. It is *not* just some constant. Converting kg to N·s^2/m units yields the kg/m^3 = N·s^2/m^4 unit. We have no idea what we could possibly do with that. In contrast, the N/m^2 is much more natural: force per unit surface. Easy, right?

[31] The reader should also carefully check on what the listed operators are operating on: as mentioned, physicists often conveniently forget about the time-dependent when doing their math. It is usually *not* a problem but when trying to carefully *interpret* what is what – as we are trying to do here – it is.

The oscillator model

We have been thinking about these things for a while now, and we have no definite answer. However, the interplay between the real and imaginary part of the wavefunction does remind one of these probabilities '*sloshing back and forth*', as Feynman would say, as a function of time in a simple two-state system. So what would slosh back and forth between the real and imaginary part of the wavefunction in an *n*-state system, or in a more complicated analysis such as this one (Schrödinger orbitals)? We see only one obvious candidate and that is kinetic and potential energy.[32] Here we need to revive, perhaps, our two-dimensional oscillator model, but extend it from circular orbitals to orbitals with fancier geometric shapes, such as those in Schrödinger's model of an atom, indeed!

Let us briefly recap the metaphorical idea.[33] If we combine *two* oscillators in a 90-degree angle – think of two springs or two pistons attached to some crankshaft[34] – then we get some *perpetuum mobile* which stores *twice* the energy of a single oscillator, and the motion of the pistons will reflect that of a mass on a spring: it is described by a sinusoidal function, with the zero point at the center of each cylinder. We detailed the math elsewhere[35] and only note the model is relativistically correct. Indeed, the relativistically correct force equation for *one* oscillator is:

$$\mathrm{F} = \mathrm{d}p/\mathrm{d}t = \mathrm{F} = -\mathrm{k}x \text{ with } p = \mathrm{m}_v v = \gamma \mathrm{m}_0 v$$

The energy conservation equation can be derived from multiplying both sides with v = dx/dt. One can then verify the following[36]:

$$v\frac{\mathrm{d}(\gamma \mathrm{m}_0 v)}{\mathrm{dt}} = -\mathrm{k}xv \Leftrightarrow \frac{\mathrm{d}(\mathrm{m}c^2)}{\mathrm{dt}} = -\frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{1}{2}\mathrm{k}x^2\right] \Leftrightarrow \frac{\mathrm{dE}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{1}{2}\mathrm{k}x^2 + \mathrm{m}c^2\right] = 0$$

For the potential energy, one gets the same $kx^2/2$ formula one gets for the non-relativistic oscillator. That is no surprise: potential energy depends on position only, not on *velocity*, and there is nothing relative about position. However, the $(½)m_0v^2$ term that we would get when using the non-relativistic formulation of Newton's Law is now replaced by the $mc^2 = \gamma m_0 c^2$ term. Both energies vary – with position and with velocity respectively – but the equation above tells us their *sum* is some constant. Hence, the game with two oscillators working in tandem should work here too.[37] In addition, the analogy can be extended to include two *pairs* of springs or pistons, in which case the springs or pistons

[32] To be truthful: we do not see any other candidates. The reader may suggest other suitable *complementary* variables (our use of complementary here has nothing to do with Bohr's concept of complementary or conjugate pairs, of course), but so we do not have any.

[33] These ideas will probably intrigue us for the rest of our life, and we are not sure if we will ever get beyond metaphorical ideas only in regard to these deep questions.

[34] Academics seem to prefer springs, but I like engines. In fact, the metaphor was inspired by a discussion with my son on the efficiency of a Ducati engine, which effectively has a 90-degree bank angle. The 90° angle of the V-2 makes it possible to perfectly balance the counterweight and the pistons, ensuring smooth travel always. With permanently closed valves, the air inside the cylinder compresses and decompresses as the pistons move up and down. It provides, therefore, a restoring force. As such, it will store potential energy, just like a spring.

[35] See: *The Wavefunction as an Energy Propagation Mechanism*.

[36] I am grateful to an unknown undergraduate student for posting this solution. Unfortunately, I lost the reference. Whomever recognizes this, please do email as I would like to properly credit the good work.

[37] The analogy can be extended to include two *pairs* of springs or pistons, in which case the springs or pistons in each pair would help drive each other.

in each pair would help drive each other. Even more interestingly, we may imagine a multi-piston *radial* engine (Figure 3)[38].

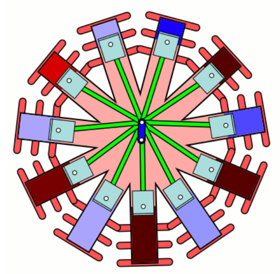

Figure 3: The metaphor of the radial engine (source: [Wikipedia](https://en.wikipedia.org/wiki/Radial_engine))

The point is this: somehow, in this beautiful interplay between linear and circular motion, energy is borrowed from one place and then returns to the other, cycle after cycle. While transferring kinetic energy from one piston to the other(s), the crankshaft will rotate with a constant angular velocity: linear motion becomes circular motion, and vice versa. Most importantly, we can *add* the total energy of all of the oscillators to get the total energy of the whole system to get the E = m$a^2\omega^2$ formula. The only thing that remains to be done then, is to substitute for the tangential velocity $v_t = a\omega$. In fact, substituting $a\omega$ for $c = a\omega$ gives us Einstein's mass-energy equivalence relation (E = mc^2) is what inspired our mass without mass model of an electron.

This is obvious and not so-obvious. Before we move on, let us briefly consider the one question that puzzles us too and that we have not satisfactorily answered as yet: we were switching between a classical (non-relativistic) oscillator and a relativistic energy formula. We should make a choice, right? Maybe. Maybe not. The classical oscillator gives us an energy (kinetic and potential) that adds up to E = m·a^2·ω^2/2. Because of the 1/2 factor, we have to think of *two* oscillators. In contrast, the relativistically correct calculations for *one* oscillator gives us a total energy of E = m·a^2·ω^2. No 1/2 factor here.

$$\frac{\mathrm{dE}}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{1}{2}\mathrm{k}x^2 + \mathrm{m}c^2\right] = 0 \Leftrightarrow \mathrm{E} = P.E. + K.E. = \frac{1}{2}\mathrm{k}x^2 + \mathrm{m}c^2$$

The m in this function is a variable, which depends on the velocity. In a one-dimensional oscillator, this velocity is equal to c – the maximum velocity – at the x = 0 point. The potential energy at this point is zero. Hence, the total energy adds up to mc^2. How can we reconcile the two formulas? We think the key lies in interpreting x and c in the equation for the relativistic oscillator as (1) the distance along an orbital, and (2) c as the tangential velocity of the pointlike charge along this orbital. However, as mentioned, this answer is puzzling us, still![39]

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[38] We did not google references here, but the [Wikipedia article on radial engines](https://en.wikipedia.org/wiki/Radial_engine) looks like a good start.
[39] We readily admit that the interpretation of x as a distance measured along some orbital (as opposed to the linear concept we are used to) and, consequently, thinking of c as some kind of tangential velocity along such orbital, looks pretty random. However, it is a way out of a logical paradox: the formula for the relativistic oscillator assumes a pointlike charge with *zero* rest mass oscillating between v = 0 and v = c. However, something with zero rest mass will *always* be associated with lightspeed: no variable velocity here! Think of a photon here: how would you slow it down? And you may think we could, perhaps, slow down a pointlike electric charge with zero rest mass in some electromagnetic field but, no! The slightest force on it will give it infinite acceleration according to Newton's force law. [Admittedly, we would need to distinguish here between its relativistic expression (F = dp/dt)

One thing stands out for us: Einstein's mass-energy equivalence relation only gets some *physical* meaning, in combination with the Planck-Einstein relation, if we write it like this:

$$E = mc^2 = m \cdot a^2 \cdot \omega^2 \Leftrightarrow c = a \cdot \omega$$

This tells us the energy of the elementary oscillation is proportional to the square of its amplitude and the square of its (angular) frequency, and m is the proportionality coefficient. The Planck-Einstein relation then tells us the argument of the wavefunction (in its own frame of reference) $\theta = \omega \cdot t$ is equal to:

$$\theta = \omega \cdot t = \frac{E}{\hbar} \cdot t = \frac{m \cdot a^2 \cdot \omega^2}{\hbar} \cdot t = \frac{m \cdot c^2}{\hbar} \cdot t$$

This gives us, for the electron itself (which we think of as a pointlike charge with zero *rest* mass[40]), its Compton radius:

$$a = \frac{c}{\omega} = \frac{\hbar c}{mc^2} = \frac{\hbar}{mc}$$

OK. This was a long digression. Back to the question: how can the oscillator metaphor shed any light on it?

The meaning of the wavefunction

So we have this general idea that the oscillations of the real and imaginary part of the wavefunction, somehow, incorporate the energy conservation law. This interpretation is quite consistent with Feynman's characterization of the wave *equation* as an energy diffusion equation, of course. Let us quote him once more:

> "We can think of Schrödinger's equation as describing the diffusion of the probability amplitude from one point to the next. […] But the imaginary coefficient in front of the derivative makes the behavior completely different from the ordinary diffusion such as you would have for a gas spreading out along a thin tube. Ordinary diffusion gives rise to real exponential solutions, whereas the solutions of Schrödinger's equation are complex waves."[41] (Feynman, III-16-1)

So, yes, we *get* this, sort of: the 'complex waves' are just local cyclical things – like circular or elliptical or other regular non-linear waves. Stuff that goes around and around or, when it starts moving linearly,

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and its non-relativistic expression ($F = m_0 \cdot a$) when further dissecting this statement, but you get the idea.] Hence, one may argue the relativistic oscillator cannot be linear: the velocity must be some tangential velocity, *always*, and it must equal lightspeed, *always*. This line of reasoning might well the conceptual *locus* where the one-dimensional relativistic oscillator ($E = m \cdot a^2 \cdot \omega^2$) and the two-dimensional non-relativistic oscillator ($E = 2 \cdot m \cdot a^2 \cdot \omega^2/2 = m \cdot a^2 \cdot \omega^2$) could meet. Of course, we welcome the view of any reader here! In fact, if there is a true mystery in quantum physics (we do not think so, but we know people – academics included – like mysterious things), then it is here!

[40] The mass of the electron as a whole is the *equivalent* mass of the inertia of the energy in this oscillation: we have a very practical interpretation of Wheelers' 'mass without mass' model.

[41] Feynman further formalizes this in his Lecture on Superconductivity (Feynman, III-21-2), in which he refers to Schrödinger's equation as the "equation for continuity of probabilities". However, the analysis here is really centered on the local conservation of energy, which confirms the interpretation of Schrödinger's equation as an energy diffusion equation.

combines linear and circular motion.[42] For linear waves – think of sound waves, water waves, radio waves or whatever wave that moves from here to there in space – we have real-valued wave equations, but for this circular stuff we have complex-valued wave equations because… Well… Because Euler invented complex numbers and they magically fit the bill when trying to model all of this. So that is clear and obvious enough, but is this interpretation compatible with all of the formalism of quantum mechanics, and with operator theory in particular? It should be: if we know the potential and kinetic energy – *at any point in time* – we should be able to derive position, momentum, and all other relevant physical observables from it, isn't it?

Of course, we admit we should formally show this by reexamining the textbook derivations of operators so as to prove the point. So how can we proceed then? We know we can extract the real and imaginary part using the general $Re(z) = (z+z^*)/2$ and $Im(z) = (z-z^*)/2i$ for a complex-valued number (and, hence, for a function as well) and, hence, we could use this operators and then try to see whether we find anything more interesting than what we already wrote above. Let us quickly do the first step, continuing the electron orbital example. The $\psi(\boldsymbol{r})$ function is the $\psi_{n,l,m}(\rho, \theta, \Phi)$ function without the complex exponential and is, therefore, the real-valued spatial (time-independent) part of the wavefunction. We, therefore, just get the obvious result that we started out with[43]:

$$Re[\psi(\boldsymbol{r},t)] = Re\left[e^{-i(\frac{\mathrm{E}}{\hbar}t+\,\varphi)}\cdot\psi(\boldsymbol{r})\right] = \frac{e^{-i(\frac{\mathrm{E}}{\hbar}t+\,\varphi)}\cdot\psi(\boldsymbol{r}) + e^{i(\frac{\mathrm{E}}{\hbar}t+\,\varphi)}\cdot\psi(\boldsymbol{r})}{2}$$

$$= \frac{e^{-i(\frac{\mathrm{E}}{\hbar}t+\,\varphi)}\cdot\psi(\boldsymbol{r}) + e^{i(\frac{\mathrm{E}}{\hbar}t+\,\varphi)}}{2}\cdot\psi(\boldsymbol{r}) = \cos\left(\frac{\mathrm{E}}{\hbar}t +\ \varphi\right)\cdot\psi(\boldsymbol{r}) = \cos(\omega t +\ \varphi)\cdot\psi(\boldsymbol{r})$$

$$Im[\psi(\boldsymbol{r},t)] = Im\left[e^{-i(\frac{\mathrm{E}}{\hbar}t+\,\varphi)}\cdot\psi(\boldsymbol{r})\right] = \frac{e^{-i(\frac{\mathrm{E}}{\hbar}t+\,\varphi)}\cdot\psi(\boldsymbol{r}) - e^{i(\frac{\mathrm{E}}{\hbar}t+\,\varphi)}\cdot\psi(\boldsymbol{r})}{2i}$$

$$= \frac{e^{-i(\frac{\mathrm{E}}{\hbar}t+\,\varphi)}\cdot\psi(\boldsymbol{r}) - e^{i(\frac{\mathrm{E}}{\hbar}t+\,\varphi)}}{2i}\cdot\psi(\boldsymbol{r}) = \sin\left(\frac{\mathrm{E}}{\hbar}t +\ \varphi\right)\cdot\psi(\boldsymbol{r}) = \sin(\omega t +\ \varphi)\cdot\psi(\boldsymbol{r})$$

This just shows, once again, that the real and imaginary part of our wavefunction (r, t) – yes, we are talking this very complicated functional form which combines power series and derivatives of Legendre polynomials! – varies as a simple sine and cosine *at any point **r** in space*. A sine and cosine function of what? Time. So what do we have here? It is a clock, once more, but this time it is a clock with a hand whose *length* varies as a function of the position. An elliptical clock, perhaps?[44] What is the formula for an ellipse again?

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[42] If there is one other paper of ours that we would recommend reading, it is the one that attracts the most attention on ResearchGate – for the right reasons, we think: *[De Broglie's Matter-Wave: Concept and Issues](#)*. We describe the (possible) *geometry* of the matter-wave in full detail there, including a geometric interpretation of the de Broglie wavelength.

[43] We took the $e^{im\Phi}$ factor out of the and replaced the $m\cdot\Phi$ term by an arbitrary phase shift φ.

[44] We are talking the shape of the clock as carved out in space by the tip of the *hand* of the clock, of course.

$$\frac{x^2}{a} + \frac{y^2}{b} = 1$$

That is too simple, obviously! This equation is not going to get us anywhere: our *x* and *y* here, so to speak, are the cos(ω·t + φ)·ψ(***r***) and sin(ω·t + φ)·ψ(***r***) functions and they are very different beasts! *Real-valued* functions, yes, but complicated functions: just look at those polar graphs once more, or the wonderful shapes of those subshells in 3D illustrations![45] However, jotting the functional form for an ellipse down usefully reminds of what a function actually *is*: a (mathematical) constraint on a set of variables. So what constraints do we have here?

Well… The wavefunction is a solution for a definite energy state, right? Hence, we should get the energy out the wavefunction and then we get an equation E = E_n with ψ(*r*) in it, and then… Well… Then what? We should just apply our energy operator $\mathrm{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r})$, right? And we should just get Schrödinger's wave equation again – which we started out with, right? Let us check, though, just to make sure we are not finding anything new or doing something wrong here. In fact, let us recap where those formulas for the energy operator come from, so we know what is what – not approximately, but *exactly*? These operators are actually used to calculate *average* or *expected* values[46] . So we are not assuming anything about the value for the energy, and we just take the value for the *average* energy of the system. This means we are going to start off by *not* assuming that the system (read: the state of our electron in its orbital – whatever that may be) should be in a *definite* energy state. The formula[47] is an integral, taken over the whole volume of the atom:

$$\langle \mathrm{E} \rangle_{average} = \int \langle \psi|\mathrm{H}|\psi \rangle \, d\mathrm{Vol} = \int \psi^* \mathrm{H}\psi \, d\mathrm{Vol} = \int \mathrm{H}\psi^*\psi \, d\mathrm{Vol} = \int \mathrm{H}|\psi|^2 d\mathrm{Vol}$$

$$\text{with:} \quad \mathrm{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r})$$

Are we allowed to write that Hamiltonian in front of the H expression? Good question. It is, but you should double-check: note that ψ is, once again, the ψ(***r***) function only: it does *not* include the time-dependent part. How should we think of this? You will want to think we have averaged the energy over a *cycle* of the oscillation. Sorry for mixing high-class math with simple illustrations once again, but inserting an easy reminder of how potential and kinetic energy vary and add up over a full cycle of an oscillation might help here (Figure 4).

[45] That is the reason why we keep putting the φ factor in: it is just a phase shift, but we need the quantum number m also for our derivatives (as an order number) of the Legendre polynomials: we can neatly separate out the time-dependent part but – for the time being, at least – we cannot simply forget about it!

[46] We use both as synonyms. To be precise, the expected value is the *average* value which a variable will take when an experiment (so that is a *measurement*) is repeated a large or (theoretically) an infinite number of times even, and so the mean (or weighted average) of all the values is calculated along the way.

[47] In case the reader would like to check the formulas we are using (or our consistency in terms of definitions), we refer to Feynman's treatment of operators and more in particular, Feynman's Lectures, section III-20-3 (average energy of an atom).

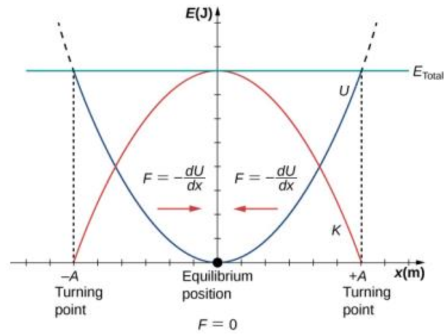

Figure 4: Kinetic (K) and potential energy (U) of an oscillator[48]

So what do we have here? The absolute square of the wavefunction is the probability of finding our electron at ***x***, so when integrating $|\psi(\boldsymbol{r})|^2$ over the volume, we get 1, right? All probabilities add up to 1: $\int|\psi(\boldsymbol{r})|^2 d\text{Vol} = 1$. Yes. For normalized wavefunctions. If we do *not* normalize our wavefunction, we should use this formula for the energy:

$$\langle \mathrm{E} \rangle_{average} = \frac{\int \psi^* \mathrm{H}\psi \; d\mathrm{Vol}}{\int \psi^* \psi \; d\mathrm{Vol}}$$

But what are we talking about here? How would we go about normalizing, anyway? The $\int|\psi(\boldsymbol{r})|^2 d\text{Vol} = 1$ condition amounts to:

$$\int \psi^* \psi \; d\mathrm{Vol} = 1 \iff \frac{\int \psi^* \mathrm{H}\psi \; d\mathrm{Vol}}{\langle \mathrm{E} \rangle_{average}} = 1 \iff \int \psi^* \mathrm{H}\psi \; d\mathrm{Vol} = \int \mathrm{H}|\psi|^2 d\mathrm{Vol} = \langle \mathrm{E} \rangle_{average}$$

And what is all that talk about averaging energy if we are talking *definite* energy states and we know we are averaging energy *over a full cycle of the oscillation*? Because that is what we are doing when separating out the time-dependent part of the wavefunction, right? Right. So we can just write this:

$$\mathrm{E} = \int \mathrm{H}|\psi|^2 d\mathrm{Vol} = \int \mathrm{H}|\psi|^2 d\mathrm{Vol} = \int -\frac{\hbar^2}{2m}\nabla^2|\psi|^2 + V(\boldsymbol{r})|\psi|^2 \; d\mathrm{Vol} = \int \mathrm{T}|\psi|^2 + \mathrm{V}|\psi|^2 \; d\mathrm{Vol}$$

So what are we doing here? We are applying our energy operators – total energy (H), kinetic energy (T) and potential energy (V)[49] – to the probability $\mathrm{P}(\boldsymbol{x}) = |\psi(\boldsymbol{x})|^2$. But the energy of what? It must be the energy of our pointlike electron if and when it would happen to be at ***x***, right? And then we multiply that value with the *probability* of the electron being there. So what we are doing is this: we do sum all of the energy *densities* – a sum of an infinite number of infinitesimally small volume elements (I am just reminding you of the definition of a 3D integral) – and, no surprise, we get the total energy E which – in turn – is used to *normalize* the probabilities that we are using. We can illustrate this *physical* normalization condition by writing:

[48] You will find this diagram in many texts, but we took this one from the https://phys.libretexts.org/ site—excellent hub for open-access textbooks.
[49] We apologize once again for not using fancier hat or script notation. We think it is not necessary: the meaning of the symbols is clear from the context.

$$\int \psi^{*}\psi\, d\mathrm{Vol} = 1 \Leftrightarrow \int \mathrm{H}|\psi|^{2} d\mathrm{Vol} = \int \mathrm{H}\,\mathrm{P}(\boldsymbol{r})\, d\mathrm{Vol} = \mathrm{E}$$

Is this a circular argument? It is, but we think it is a useful one (in the sense that it helps us understanding what is what here[50]). So where are we now? We now understand how potential and kinetic energy slosh back and forth in this system, always adding up to some constant, but we forgot about the original question: the real and imaginary parts of the wavefunction. We abstracted away from that by looking at the spatial part of the wavefunction only. So let us look at the whole thing by plugging the time-dependence back in. So we have a wavefunction which we can not only split into a time bit and a space bit – a simple scalar product of both, to be precise:

$$\psi(\boldsymbol{r}, \mathrm{t}) = \psi(\boldsymbol{r})\cdot e^{-i\cdot[(\mathrm{E}/\hbar)\cdot \mathrm{t} + \varphi]}$$

Now, the time-dependent thing is the simplest of complex exponentials and allow us to also nicely separate everything out into a real and an imaginary bit:

$$Re[\psi(\boldsymbol{r},t)] = \cos(\omega t + \varphi)\cdot\psi(\boldsymbol{r})$$

$$Im[\psi(\boldsymbol{r},t)] = \sin(\omega t + \varphi)\cdot\psi(\boldsymbol{r})$$

These are two orthogonal vectors in the complex plane[51] and we can, therefore, apply Pythagoras's Theorem:

$$[\cos(\omega t + \varphi)\cdot\psi(\boldsymbol{r})]^{2} + [\sin\cdot\psi(\boldsymbol{r})]^{2} = [\cos^{2}(\omega t + \varphi) + \sin^{2}(\omega t + \varphi)]\psi^{2}(\boldsymbol{r}) = \psi^{2}(\boldsymbol{r}) = |\psi(\boldsymbol{r})|^{2}$$

However, this just reminds us of the fact that the square of the modulus of a real number (the *absolute* square) is just the squared number itself. And taking the square root back allows for positive or negative (but always real-valued) amplitudes (spatial bit only), of course. But this does not add anything to our interpretation of the wavefunction. Can we add anything to the interpretation by trying to find some latus rectum formula? It might be possible, but we do not think so.[52] So that is it, then. We have:

1. Kinetic and potential energy sloshing back and forth and, obviously, adding up to the total energy; and

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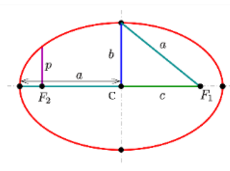
[50] We hope it helps the understanding of the *nature* of the wavefunction for the reader too, but that is for him or her to judge, of course.

[51] A complex *space* is usually associated with complex-valued coordinates or may have some other meaning. The complex *plane* is just two-dimensional Cartesian space, with the *x*-axis representing the real part (the axis with the cosine values) and the *y*-axis representing the imaginary part (the axis with the sine values).

[52] The latus rectum formula is $a\cdot p = b^2$, with *a*, *p* and *b* the lengths as depicted below.



The latus rectum formula popped up quite naturally in [our geometric interpretation of the *de Broglie* wavelength](), which was quite surprising and very interesting. However, our earlier [attempt to interpret Schrödinger's orbitals in terms of elliptical orbitals]() failed. We, therefore, regret this early paper remains popular, even if it gave us early ideas on the nature of Schrödinger's wave *equation* (not wavefunction) as an energy diffusion equation.

2. The sum of squares of the real and imaginary part adding up to give us the *energy density* (non-normalized wavefunction) at each point in space or, after normalization, a *probability* $P(\boldsymbol{r}) = |\psi(\boldsymbol{r})|^2$ to find the electron as a function of the position vector $\boldsymbol{r}$.

In short, the wavefunction is the pendant to the Planck-Einstein relation. To be precise, the example we explored showed how Schrödinger's orbitals incorporate a Planck-Einstein cycle or, we should say, Planck's quantum of action *tout court*: the energy, the frequency, the (linear and circular) momentum,… All comes out of the $E = h \cdot f = p \cdot \lambda$ equation (or its reduced form[53]) combined with Maxwell's equations written in terms of the scalar and vector potential.

We should note that the indeterminacy in regard to the position is statistical only: it arises because of the high velocity of the pointlike charge, which makes it impossible to accurately determine its position at any point in time. It would disappear if we would be able to do so. In that case, we would be able to define a precise (x_0, y_0, z_0, t_0) point – or, in polar coordinates – a (ρ_0, θ_0, Φ_0, t_0) point – with the pointlike charge actually crossing position (x_0, y_0, z_0) at time $t = t_0$. In other words, we would be able to determine the initial condition of the system which, in turn, would allow us to go from indefinite integrals to definite integrals, and so we would have a completely defined system.

Feynman once wrote this: "*These philosophers are always with us, struggling in the periphery to try to tell us something, but they never really understand the subtleties and depths of the problem.*" (Lectures on Physics, Vol. I, Ch. 19). I always found this remark rather disparaging, but he was right.

Conclusions

We think we found an awful lot of *meaning* but, yes, the question remains: do we really *get* this? Maybe. Maybe not. Can we do better – any more explaining that what we have done already? We do not think so but, of course, we invite the reader to think this through for him- or herself, and to check whether or not the bottom line is really this: the real and imaginary part of the wavefunction(s) – i.e. the solution(s) to the wave equation that applies to the situation at hand – combines not only the energy conservation law (potential and kinetic adding up to the (constant) total) but all of physics, plus Pythagoras's (complex number theory, that is), operator theory and all of the math in-between.

It may look like some miracle that, somehow, all laws of physics – and all of geometry, of course! – combine into Euler's function, but so that is it then: there is no further explanation, and we should just marvel at the fact that we sort of intuitively *get* this. And that is all of the mystery of quantum mechanics, then. No weird metaphysical uncertainty. And there is also no need for 'hidden variables' – because all is determined and any indeterminism that is there is of a statistical nature only: think of the airplane propeller again! And, yes, of course we should thank and remember Leonhard Euler for inventing the $e^{i \cdot \theta} = \cos\theta + \sin\theta$ formula[54], without which physicists would have had an awful lot of

[53] The Compton wavelength is a linear concept, and the Compton *radius* of an electron is just its reduced form: $r_C = \lambda_C/2\pi$. The fine-structure constant relates the various radii of the electron: radius of the pointlike charge ($r_e = \alpha \cdot r_C$), radius of the free electron ($r_C = \alpha \cdot r_B$), and the Bohr radius (r_B). It, therefore, makes sense that the fine-structure constant is *not* one of Nature's (independent) constants but a combination of them: the electron charge, lightspeed, and Planck's quantum of action.

[54] We may usefully quote one of the other great polymaths of history here, Pierre-Simon Laplace, who is said to have said: "Read Euler, read Euler, he is the master of us all!" While he stood on the shoulders of other giants (the Wikipedia article on complex numbers offers a useful short historical introduction), such as Descartes and de

trouble to concisely model and represent Nature's fundamental *cycle* which – in turn – is represented by the Planck-Einstein relation.

So what equations should we show a visiting alien as part of the earliest discussions when trying to communicate?[55] The equations of modern physics, of course: Maxwell's equations (preferably in four-vector notation), Schrödinger's wave (probably the more general one for an electron in an electromagnetic field[56]), and the Planck-Einstein relation. But I think we should scribble a few math formulas in the margin too, perhaps. Which ones? Pythagoras's Theorem or – closely related – Euler's formula. 😊

Jean Louis Van Belle, 4 November 2020

Moivre, Euler's formula remains Euler's formula: "the most remarkable formula in mathematics", according to Feynman, that is.

[55] We refer to Feynman's story about the Martian, in the context of his very insightful discussion of (a)symmetries and matter-antimatter in the last chapter of his first volume of lectures. Needless to say, before talking, we should make sure he or she or it does not feel threatened, so it is not tempted to blow us away – literally !

[56] As mentioned in other papers, we think Schrödinger's wave equation might be relativistically correct, because the ½ factor does *not* refer to a (non-relativistic) concept of kinetic energy. The factor is there because we are basically modelling the motion of *two* electrons with opposite spin. And what Schrödinger wave equation or Hamiltonian should we show our Martian? The complete one: $\mathrm{H} = i\hbar\frac{\partial}{\partial \mathrm{t}} = -\frac{1}{2\mathrm{m}}(i\hbar\boldsymbol{\nabla} + \mathrm{q}\boldsymbol{A})^2 + \mathrm{q}\boldsymbol{A}$.

Annex: Spin and the sign of the imaginary unit in the wavefunction

When thinking of spin as *physical* angular momentum, one can easily integrate the concept of spin in the elementary wavefunction by thinking about the *direction* of motion, as illustrated below (**Figure 5**): we can go from the +1 to the –1 position on the unit circle taking opposite directions.

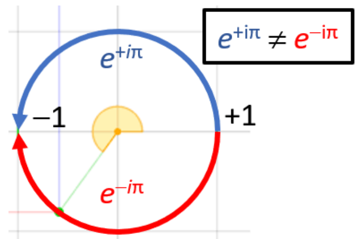


Figure 5: $e^{+i\pi} \neq e^{-i\pi}$

Hence, combining the + and – sign for the imaginary unit with the direction of travel, we get four mutually exclusive structures for our electron wavefunction (see Table 1).

Spin and direction of travel	**Spin up (J = +ħ/2)**	**Spin down (J = –ħ/2)**
Positive *x*-direction	$\psi = \exp[i(kx-\omega t)]$	$\psi^* = \exp[-i(kx-\omega t)] = \exp[i(\omega t-kx)]$
Negative *x*-direction	$\chi = \exp[-i(kx+\omega t)] = \exp[i(\omega t-kx)]$	$\chi^* = \exp[i(kx+\omega t)]$

Table 1: Occam's Razor: mathematical possibilities versus physical realities

Unfortunately, the mainstream interpretation of quantum mechanics does not integrate the concept of particle spin from the outset because the + or – sign in front of the imaginary unit (i) in the elementary wavefunction ($a \cdot e^{-i \cdot \theta}$ or $a \cdot e^{+i \cdot \theta}$) is thought as *a mathematical convention* only. This non-used *degree of freedom* in the mathematical description then leads to the *false* argument that the wavefunction of spin-½ particles has a 720-degree symmetry. Indeed, physicists treat –1 as a *common phase factor* in the argument of the wavefunction.[57] However, we should think of –1 as a complex number itself: the phase factor may be +π or, alternatively, –π: when going from +1 to –1 (or vice versa), it matters how you get there—as illustrated above.[58]

What are the implications? Physicists should go about their calculations more carefully, drag a ± sign along, and inverse it when appropriate. And they should carefully think about the physics when getting rid of a $n \cdot \pi$ factor: the concept of parity is important, and should be integrated in the analysis from the outset.

[57] Mainstream physicists therefore think one can just multiply a set of amplitudes – let us say *two* amplitudes, to focus our mind (think of a beam splitter or alternative paths here) – with –1 and get the same *physical states*.
[58] The quantum-mechanical argument is technical, and so I am not going to reproduce it here. I do encourage the reader to glance through it, though. See: [Euler's Wavefunction: The Double Life of – 1](). Note that the $e^{+i\pi} \neq e^{-i\pi}$ expression may look like *horror* to a mathematician! However, if he or she has a bit of a sense for geometry and the difference between identity and equivalence relations, there should be no surprise. If you are an amateur physicist, you should be excited: it is, effectively, the secret key to unlocking the so-called mystery of quantum mechanics. Remember Aquinas' warning: *quia parvus error in principio magnus est in fine*. A small error in the beginning can lead to great errors in the conclusions, and we think of this as a rather serious error in the beginning!

Spin versus orbital angular momentum

Spin is angular momentum. When analyzing a free electron or, to be precise, *a pointlike charge as such*[59], one should think of its spin angular momentum: the charge of the electron spins around its own axis, thereby generating a magnetic moment. However, Schrödinger's wave equation – in the context of electron orbitals – does not take this into account: we think the 1/2 factor may be explained because it actually models an electron pair – *two* electrons with opposite spin, effectively lowering their joint energy as a pair. There is no good reason to assume such electron pair should be essentially different from, say, a Cooper pair in perpetual currents in superconductors, when there is no thermal motion or radiation to prevent such pairs from forming. Schrödinger's wave equation does, however, take *orbital* angular momentum into account. In fact, that is why the spin number *l* comes out when solving it for definite energy states.

These *subshell* solutions are, in essence, Schrödinger's great new addition to the Rutherford-Bohr model. In fact, Sommerfeld had already made major steps in this direction.[60] However, Schrödinger's wave equation is not *complete* as an explanation either: while it explains the gross and fine structure of a hydrogen atom, it does not explain the hyperfine splitting of spectral lines, which results from the coupling of the spin angular momentum of the electron and the spin angular momentum of the nucleus, which is just one proton in the case of the hydrogen atom.[61] It, therefore, continues to model the electron as a pointlike charge only, with no substructure of its own. That is fair enough because it would be difficult to integrate such substructure in these 'equations of motion' (we are just using Dirac's characterization of wavefunctions here): when everything is said and done, one should not expect *models* to incorporate each and every detail, right? However, the fact remains such incomplete models trigger their own set of questions. One is this: Schrödinger's orbitals imply the electron spends most of its time right on top of the proton, so how should we think of that?

We could, perhaps, imagine some short-range repulsive force here but such solution would inject entirely new dynamics and, therefore, looks pretty unacceptable: assuming the electron, somehow, does go straight through the center or, else, bounces back – fully elastically, because momentum and energy should be conserved – is the only solution but, of course, raises the obvious question: *how* does that work, *exactly*? We have no answer to that, but the assumption that the electron and the proton should have a substructure that allows them to go right through each other makes more sense than one might expect at first. Personally, we have no difficulty whatsoever to accept the radius of the electron in its orbital motion is nothing but the Compton radius, and that the classical electron radius is just the radius of the oscillating pointlike charge inside of Compton's electron. To explain the point, we copy – once again – the illustration of the spiral-like motion of the *Zitterbewegung* electron (Figure 6).

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[59] We should qualify this statement somewhat. To explain the spin angular moment of a free electron - and the anomaly in the magnetic moment – one must assume the free electron has a substructure. We elaborated the model in several places, but a concise presentation of the argument can be found in our classical explanation of the Lamb shift. The point is this: if the free electron has a substructure (a point charge within the point charge), one might – once again – distinguish between spin and orbital angular momentum. Oliver Consa suggests such fractal structure in his helical solenoid model of an electron.

[60] See the notes on the Bohr-Sommerfeld theory in the Wikipedia article on 'old quantum theory', which we think is not old (in the sense of irrelevant) at all!

[61] We refer once again to the above-mentioned paper on the hyperfine structure and the Lamb shift.

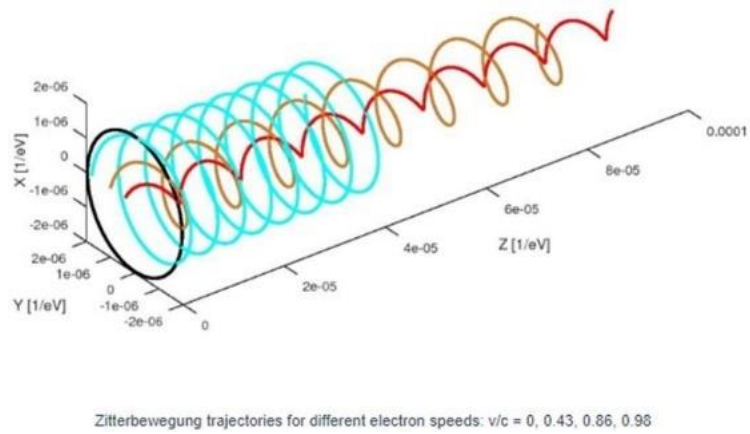

Figure 6: The Compton radius must decrease with increasing velocity

The idea is simple enough: a free electron combines a pointlike charge with zero rest mass orbiting around some center at lightspeed[62]. This motion generate the (rest) mass of the electron[63] *as well as its magnetic moment*. However, the radius of this orbital motion must decrease when adding linear motion because the velocity of the pointlike charge cannot exceed the speed of light, which is shown in the illustration above.[64]

We can then assume the magnetic moment of the electron and the magnetic moment of the proton will then line up as these two particles approach each other at each and every passage through the center of the atom and this must, somehow, avoid the collision between the charges themselves. Indeed, it is hard to avoid the conclusion that electrons and protons do not engage in mutual annihilation – as opposed to electrons and positrons – because the electron and the proton have very different *geometries*. We must refer our reader to yet another paper with speculative thoughts on the nature of anti-matter here[65] because we have not done much thinking on this, and so this is all what we can reasonably say about these questions right now.

[62] The velocity must be lightspeed because its rest mass is zero. This pointlike *Zitterbewegung* charge is, therefore, photon-like. However, unlike a photon, it does carry charge: the elementary charge, to be precise – which is why electrons are electrons and photons are photons.

[63] It is a mass without mass model: the mass is just the equivalent energy of motion – so it is a measure of inertia only. As mentioned above, we know this may all sound rather fantastical to our reader and so we do not want to dwell on this: we refer to our other papers for detail.

[64] The illustration is used – with permission – from Vassallo, G., Di Tommaso, A. O., and Celani, F, *The Zitterbewegung interpretation of quantum mechanics as theoretical framework for ultra-dense deuterium and low energy nuclear reactions*, in: Journal of Condensed Matter Nuclear Science, 2017, Vol 24, pp. 32-41. The reader should not worry about the rather weird distance scale (1×10^{-6} eV^{-1}). Time and distance can be expressed in *inverse* energy units when using so-called *natural units* ($c = \hbar = 1$). We are not very fond of this because we think it does *not* necessarily clarify or simplify relations. Just note that 1×10^{-9} eV^{-1} = 1 GeV^{-1} ≈ 0.1975×10^{-15} m. As you can see, the *zbw* radius is of the order of 2×10^{-6} eV^{-1} in the diagram, so that is about 0.4×10^{-12} m, which is what we calculated: $a \approx 0.386\times10^{-12}$ m.

[65] See: The Ring Current Model for Antimatter and Other Questions.