Popular Discussion of Classical and Finite Mathematics

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Abstract

Following the book [1] we discuss on a popular level different aspects of classical and finite mathematics and explain why classical mathematics is a special degenerate case of finite one.

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1 Introduction

Mathematical education at physics departments develops a mentality that classical mathematics (involving infinitely small, limits, continuity etc.) is the most fundamental mathematics while finite mathematics is something inferior what is used only in special applications. And many mathematicians have a similar mentality.

Historically it happened so because more than 300 years ago Newton and Leibniz proposed the calculus of infinitely small, and since that time a titanic work has been done on foundation of classical mathematics. This problem has not been solved till the present time (see below) but for the majority of physicists and many mathematicians the most important thing is not whether a rigorous foundation exists but that in many cases standard mathematical technique works with a very high accuracy.

The idea of infinitely small was in the spirit of existed experience that any macroscopic object can be divided into arbitrarily large number of arbitrarily small parts, and even in the 19th century people did not know about atoms and elementary particles. But now we know that when we reach the level of atoms and elementary particles then standard division loses its usual meaning and in nature there are no arbitrarily small parts and no continuity. For example, it is not possible to divide by two the electron or neutrino. Another example is that if we draw a line on a sheet of paper and look at this line by a microscope then we will see that the line is strongly discontinuous because it consists of atoms. That is why standard geometry (the notions of continuous lines and surfaces) can work well only in the approximation when sizes of atoms are neglected, standard macroscopic theory can work well only in this approximation etc.

Of course when we consider water in the ocean and describe it by differential equations of hydrodynamics, this works well but this is only an approximation since water consists of atoms. However, it seems unnatural that even quantum theory is based on continuous mathematics. Even the name "quantum theory" reflects a belief that nature is quantized, i.e. discrete, and this name has arises because in quantum theory some quantities has discrete spectrum (i.e. the spectrum of the angular momentum operator, the energy spectrum of the hydrogen atom etc.). But this discrete spectrum has appeared in the framework of standard mathematics.

I asked physicists and mathematicians whether in their opinion the indivisibility of the electron shows that in nature there are no infinitely small quantities and standard division does not work always. Some mathematicians say that sooner or later the electron will be divided. On the other hand, as a rule, physicists agree that the electron is indivisible and in nature there are no infinitely small parts. They say that, for example, dx/dt should be understood as $\Delta x/\Delta t$ where Δx and Δt are small but not infinitely small. I ask them: but you work with dx/dt, not $\Delta x/\Delta t$. They reply that since math with derivatives works well then there is no need to philosophize and develop something else (and they are not familiar with finite mathematics).

Of course, the founders of quantum theory and physicists who essentially contributed to this theory were highly educated scientists. But they used only classical mathematics, and even now finite mathematics is not a part of standard education for physicists. The development of quantum theory has shown that the theory contains anomalies and divergences. Physicists persistently try to solve those problems in the framework of continuous mathematics and refuse to acknowledge that those difficulties arise just because this mathematics is applied.

Some facts of classical mathematics seem to be unnatural from the point of view of common sense. For example, $tg(x)$ is one-to-one reflection of $(-\pi/2, \pi/2)$ onto $(-\infty, \infty)$, i.e. the impression might arise that the both intervals have the same numbers of elements although the first interval is a nontrivial part of the second one. Another example is the Hilbert paradox with an infinite hotel. But mathematicians even treat those facts as pretty ones. For example, Hilbert said: "No one shall expel us from the paradise that Cantor has created for us". And this is in spite of the fact that the problem of foundation of classical mathematics has not been solved yet.

In view of efforts to describe discrete nature by continuous mathematics, my friend told me the following joke: "A group of monkeys is ordered to reach the Moon. For solving this problem each monkey climbs a tree. The monkey who has reached the highest point believes that he has made the greatest progress and is closer to the goal than the other monkeys". He says that he knew this joke even when we studied at the Moscow Institute of Physics and Technology at the of the 60th but I did not know this joke.

Is it reasonable to treat this joke as a hint on some aspects of the modern science? Indeed, people invented continuity and infinity which do not exist in nature, created problems for themselves and now apply titanic efforts for solving those problems. Below it will be explained on popular level (and the rigorous proof is given in Ref. [1]) that classical mathematics is a special degenerate case of finite mathematics.

The problem of foundation of classical mathematics has been investigated by many great mathematicians like Cantor, Fraenkel, Gödel, Hilbert, Kronecker, Russell, Zermelo and others. The philosophy of those mathematicians was based on macroscopic experience in which the notions of infinitely small/large, continuity and standard division are natural. However, as noted above, those notions contradict the existence of elementary particles and are not natural in quantum theory. The illusion of continuity arises when one neglects the discrete structure of matter.

The fact that foundational problems of classical mathematics cannot be resolved follows, in particular, from Gödel's incompleteness theorems which state that no system of axioms can ensure that all facts about natural numbers can be proved. Moreover, the system of axioms in classical mathematics cannot demonstrate its own consistency. The theorems are written in highly technical terms of mathematical logics. However, simple arguments in Ref. [1] show that foundational problems of classical mathematics follow from simple considerations, and below we give those arguments.

In the 20s of the 20th century the Viennese circle of philosophers under the leadership of Schlick developed an approach called logical positivism which contains verification principle: *A proposition is only cognitively meaningful if it can be definitively and conclusively determined to be either true or false* (see e.g. Refs. [2]). However, this principle does not work in standard classical mathematics. For example, it cannot be determined whether the statement that $a+b=b+a$ for all natural numbers a and b is true or false.

As noted by Grayling [3], *"The general laws of science are not, even in principle, verifiable, if verifying means furnishing conclusive proof of their truth. They can be strongly supported by repeated experiments and accumulated evidence but they cannot be verified completely"*. So, from the point of view of classical mathematics and classical physics, verification principle is too strong.

Popper proposed the concept of falsificationism [4]: *If no cases where a claim is false can be found, then the hypothesis is accepted as provisionally true.* In particular, the statement that $a+b=b+a$ for all natural numbers a and b can be treated as provisionally true until one has found some numbers a and b for which $a+b \neq b+a$.

According to the philosophy of quantum theory, there should be no statements accepted without proof and based on belief in their correctness (i.e. axioms). The theory should contain only those statements that can be verified, where by "verified" physicists mean an experiment involving only a finite number of steps. So the philosophy of quantum theory is similar to verificationism, not falsificationism. Note that Popper was a strong opponent of the philosophy of quantum theory and supported Einstein in his dispute with Bohr.

From the point of view of verificationism and the philosophy of quantum theory, classical mathematics is not well defined not only because it contains an infinite number of numbers. For example, let us pose a problem whether 10+20 equals 30. Then we should describe an experiment which should solve this problem. Any computing device can operate only with a finite amount of resources and can perform calculations only modulo some number p. Say $p=40$, then the experiment will confirm that 10+20=30 while if $p=25$ then we will get that 10+20=5.

So the statements that 10+20=30 and even that $2 \cdot 2 = 4$ are ambiguous because they do not contain information on how they should be verified. On the other hand, the statements

$$10+20=30\,(mod\,40),\;\;10+20=5\,(mod\,25),$$

$$2\cdot 2=4\,(mod\,5),\;\;2\cdot 2=2\,(mod\,2)$$

are well defined because they do contain such an information. So only operations modulo some number are well defined.

We believe the following observation is very important: although classical mathematics (including its constructive version) is a part of our everyday life, people typically do not realize that *classical mathematics is implicitly based on the assumption that one can have any desired amount of resources*. So classical mathematics is based on the implicit assumption that we can consider an idealized case when a computing device can operate with an infinite amount of resources. In other words, standard operations with natural numbers are implicitly treated as limits of operations modulo p when $p\to\infty$. As a rule, every limit in mathematics is thoroughly investigated but in the case of standard operations with natural numbers it is not even mentioned that those operations are limits of operations modulo p. In real life such limits even might not exist if, for example, the Universe contains a finite number of elementary particles.

In the *technique* of classical mathematics there is no number ∞, infinity is understood only as a limit (i.e. as a potential infinity) and, as a rule, legitimacy of every limit is thoroughly investigated. However, the *basis* of classical mathematics involves actual infinity from the very beginning. For example, the ring of integers Z is involved from the very beginning and, even in standard texbooks, it is not even posed a problem whether Z should be treated as a limit of finite rings. Moreover, Z is the starting point for constructing the sets of rational, real and complex numbers and the sets with greater and greater cardinalities.

For solving the problem of infinities, different kinds of arithmetic have been proposed, e.g. Peano arithmetic, Robinson arithmetic, finitistic arithmetic and others. The latter deals with all natural numbers but only finite sets are allowed. However, finite mathematics rejects infinities from the beginning. This mathematics starts from the ring $R_p=(0,1,2,...p-1)$ where addition, subtraction and multiplication are performed as usual but modulo p, and p is called the characteristic of the ring. In the literature the ring R_p is usually denoted as Z/p. In our opinion this notation is not adequate because it may give a wrong impression that finite mathematics starts from the infinite set Z and that Z is more general than R_p. However, although Z has more elements than R_p, Z cannot be more general than R_p because Z does not contain operations modulo a number. We will see below that, although R_p has a lesser number of elements than Z, but R_p is the more general set than Z, and Z is a special degenerate case of R_p in the formal limit $p\to\infty$.

For understanding this fact, it is useful to note that in physics a typical situation is as follows:

Definition: Let theory A contain a finite parameter and theory B be obtained from theory A in the formal limit when the parameter goes to zero or infinity.

Suppose that, with any desired accuracy, A can reproduce any result of B by choosing a value of the parameter. On the contrary, when the limit is already taken then one cannot return back to A and B cannot reproduce all results of A. Then A is more general than B and B is a special degenerate case of A.

Known examples are that nonrelativistic theory (NT) is a special degenerate case of relativistic one (RT) in the special case $c \to \infty$, classical theory is is a special degenerate case of quantum one in the special case $\hbar \to 0$, and RT is a special degenerate case of de Sitter invariant theory in the special case $R \to \infty$ where R is the parameter of contraction from the Poincare Lie algebra to the de Sitter Lie algebra. In the literature those facts are explained from physical considerations but, as shown in Ref. [1], they follow from mathematics of Lie algebras.

Let us consider in more details that NT is a special degenerate case of RT in the special case $c \to \infty$. According to **Definition**, this implies that RT can reproduce any result of NT with any accuracy if c is chosen to be sufficiently large. However, NT cannot reproduce all results of RT because RT also describes phenomena where it is important that c is finite. From the naive point of view, one might think that NT is more general than RT because NT corresponds to the case $c = \infty$, i.e. one might think that NT describes more cases than RT where c is finite. However, NT gives the same results as RT only when speeds are much less than c, but when they are comparable to c then it is known that NT does not work.

Let us also note the following analogy between RT and finite mathematics. If tachyons are not taken into account (at present it is not known whether they exist) then in RT no speed can be greater than c. Let us consider, for example, the following problem. Let some reference frame move relative to us with the speed $V = 0.6c$, and in this frame a body moves in the same direction with the same speed. Then the speed of the body relative to us is not $v = 1.2c$, as one might think from naive considerations, but $v \approx 0.882c$, and if, for example, $V = 0.99c$ then $v \approx 0.9999495c$, i.e. there is no possibility to get $v > c$. Here there is an analogy with finite mathematics where the results are the same as in classical mathematics if the numbers in question are much less than p but, since in finite mathematics all operations are modulo p then it is not possible to get the result greater than p.

Since mathematical mentality of the absolute majority of physicists is based on classical mathematics, many physicists might decide that calculations modulo a number are nonphysical. However, as noted above, just calculations modulo a number are more physical than calculations in classical mathematics.

One can rigorously prove [1] that any result in Z can be reproduced in R_p if p is chosen to be sufficiently large, and that is why Z can be treated as a limit of R_p when $p \to \infty$. This result looks natural from the following considerations. Since all operations in R_p are modulo p, then R_p can be treated as a set $(-(p-1)/2, ... -1, 0, 1, ...(p-1)/2)$ if p is odd and as a set $(-p/2+1, ... -1, 0, 1, ...p/2)$ if p is even. In this representation, for numbers with the absolute values much less than p, the results of addition, subtraction and multiplication are the same in R_p and in Z, i.e. for such numbers it is not manifested that in R_p operations are modulo p.

On the other hand, in Z it is not possible to reproduce all results in

R_p since in Z there are no operations modulo a number. Therefore, as follows from **Definition**, the theory with R_p is more general than the theory with Z, and the latter is a special degenerate case of the former in the formal limit $p \to \infty$. Therefore, the theory with R_p is more general than the theory with Z in spite of the fact that R_p contains less elements than Z. This situation is analogous to that discussed above that RT is more general than NT and to other cases discussed above when theory A is more general than theory B.

The fact that the theory with R_p is more general than the theory with Z means that, even from purely mathematical point of view, the notion of infinity cannot be fundamental since when we introduce infinity, we get the degenerate theory where all operations modulo a number disappear.

The proof that $R_p \to Z$ when $p \to \infty$ is analogous to standard proof that a sequence (a_n) of natural numbers goes to infinity when $n \to \infty$ if $\forall M > 0\ \exists n_0$ such that $a_n > M\ \forall n > n_0$. Therefore the proof in Ref. [1] that $R_p \to Z$ when $p \to \infty$ does not involve actual infinity.

At the same time, the fact that $R_p \to Z$ when $p \to \infty$ can be proved in the framework of the theory of ultraproducts described in a vast literature. As noted by Zelmanov [5], infinite fields of zero characteristic (and Z) can be embedded in ultraproducts of finite fields. This approach is in the spirit of mentality of many mathematicians that sets of characteristic 0 are more general that finite sets, and for investigating infinite sets it might be convenient to use properties of simpler sets of positive characteristic.

The theory of ultraproducts is essentially based on classical results on infinite sets involving actual infinity. In particular, the theory is based on Łoŝ' theorem involving the axiom of choice. Therefore the theory of ultraproducts cannot be used in proving that finite mathematics is more general than classical one.

Let us also note that standard terminology that Z and the fields constructed from Z (e.g. the fields of rational, real and complex numbers) are sets of characteristic 0 reflects the usual spirit that classical mathematics is more fundamental than finite one. I think it is natural to say that Z is the ring of characteristic ∞ because it is a limit of rings of characteristic p when $p \to \infty$. The characteristic of the ring p is understood such that all operations in the ring are modulo p but operations modulo 0 are meaningless. Usually the characteristic n of the ring is defined as the smallest positive number n such that the sum of n units $1+1+1...$ in the ring equals zero if such a number exists and 0 otherwise. However, this sum can be written as $1 \cdot n$ and the equality $1 \cdot 0 = 0$ takes place in any ring.

Consider now the following question. Does the fact that R_p is more general than Z mean that in applications finite mathematics is more general than classical one? Indeed, in applications not only rings (involving sums, subtractions and multiplications) are used but also fields which also contain division. For example, if p is prime then R_p becomes the Galois field F_p in which division is defined as usual but modulo p.

As note above, for numbers with the absolute values much less than p, the results of summation, subtraction and multiplication are the same in R_p and Z.

That is why if an experiment deals only with such numbers, and the theory describing this experiment involves only sums, subtractions and multiplications then the results of the experiment cannot answer the question what mathematics is more adequate for the description of this experiment: classical or finite. However, in the case of division the difference is essential. For example, $1/2$ in the field F_p equals $(p+1)/2$, i.e. a very large number if p is large. That is why an impression may arise that finite mathematics is not adequate for decription of experimental data. Let us consider this problem in more details.

Now it is accepted that the most general physical theory is quantum one, i.e. any classical theory is a special case of quantum one. This fact has been already mentioned above. Therefore the problem arises whether quantum theory based on real and complex numbers containing division (and also quantum theory based on their generalizations, e.g. p-adic numbers or quaternions) can be a special case of quantum theory based on finite mathematics.

In quantum theory a state of a system is described by the wave function $\Psi = c_1e_1 + c_2e_2 + ...$ where the e_j $(j = 1, 2, ...\infty)$ are the elements of the bazis of the Hilbert space, and the c_j are complex coefficients. Usually basis elements are normalized on unity: $||e_j|| = 1$ and then the probability for a system to be in the state e_j is $|c_j|^2$. However, normalization on unity is only the question of convention but not the question of principle. The matter is that not the probability itself but only relative probabilities of different events have a physical meaning. That is why spaces in quantum theory are projective: $const \cdot \Psi$ and Ψ describe the same state if $const \neq 0$.

That is why one can choose the basis where all the $||e_j||$ are positive integers. Then we use the theorem which is proved in standard textbooks on Hilbert spaces: any element of the Hilbert space can be approximated with any desired accuracy by a finite linear combination $\Psi = c_1e_1 + c_2e_2 + ...c_ne_n$ where the coefficients are rational numbers. Finally, using the fact that spaces in quantum theory are projective, one can multiply Ψ by the common denominator of all the coefficients and get the case when all the complex coefficients $c_j = a_j + ib_j$ are such that all the numbers a_j and b_j are integers.

Therefore, although formally Hilbert spaces in quantum theory are complex, but with any required accuracy any state can be described by a set of coefficients which are elements of Z. Hence the description of states by means of Hilbert spaces is not optimal since such a description contains a big redundancy of elements which are not needed for a full description.

Now we use that theory with R_p is more general than theory with Z and describe quantum states not by elements of Hilbert spaces but by elements of spaces over a finite ring $R_p + iR_p$, i.e. now all the a_j, b_j and $||e_j||$ are elements of R_p. Let us call this theory FQT - finite quantum theory. As follows from the abovementioned, FQT is more general than standard quantum theory: when the absolute values of all the a_j, b_j and $||e_j||$ are muh less than p then the both theories give the same results but if the absolute values of some of those quantities are comparable to p then the descriptions are different because in standard theory there are no operations modulo p. In Ref. [1] we considered some phenomena where it is important that p is finite. Such phenomena cannot be described by standard quantum theory. As noted above, there is an analogy with the fact that nonrelativistic theory cannot describe phenomena in which it is important that c is finite.

Let us note that in FQT there are no infinities in principle and that is why divergences are absent in principle. In addition, probabilistic interpretation of FQT is only approximate: it applies only to states described by the numbers a_j, b_j and $||e_j||$ which are all much less than p.

This situation is a good illustration of the famous Kronecker's expression: "God made the natural numbers, all else is the work of man". In view of the above discussion, we propose to reformulate this expression as: "God made only finite sets of natural numbers, all else is the work of man". For illustration, consider a case when some experiment is conducted N times, the first event happens n_1 times, the second one — n_2 times etc. such that $n_1 + n_2 + ... = N$. Then the experiment is fully described by a finite set of natural numbers. But people introduce rational numbers $w_i = n_i/N$, introduce the notion of limit and define probabilities as limits of the quantities $w_i(N)$ when $N \to \infty$.

Of course, when classical and finite mathematics are considered only as abstract sciences then the question what mathematics is more general (fundamental) does not have a great meaning. However, the above discussion shows that, from the point of view of applications, finite mathematics is more general (fundamental) than classical one. The conclusion from the above consideration can be formulated as:

Mathematics describing nature at the most fundamental level involves only a finite number of numbers, while the notions of limit, infinitely small/large and continuity are needed only in calculations describing nature approximately.

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