Popular Discussion of Classical and Finite Mathematics

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Abstract

Following our recently published book (F. Lev, *Finite mathematics as the foundation of classical mathematics and quantum theory. With application to gravity and particle theory.* Springer (2020)), we discuss different aspects of classical and finite mathematics and explain why classical mathematics is a special degenerate case of finite one.

Keywords: finite mathematics, classical mathematics, finite quantum theory

1 Problems with describing nature by classical mathematics

Mathematical education at physics departments develops a belief that classical mathematics (involving infinitesimals, limits, continuity etc.) is the most fundamental mathematics, while finite mathematics is something inferior what is used only in special applications. And many mathematicians have a similar belief.

Historically it happened so because more than 300 years ago Newton and Leibniz proposed the calculus of infinitesimals, and since that time a titanic work has been done on foundation of classical mathematics. This problem has not been solved till the present time (see below) but for the majority of physicists and many mathematicians the most important thing is not whether a rigorous foundation exists but that in many cases standard mathematical technique works with a very high accuracy.

The idea of infinitesimals was in the spirit of existed experience that any macroscopic object can be divided into arbitrarily large number of arbitrarily small parts, and even in the 19th century people did not know about atoms and elementary particles. But now we know that when we reach the level of atoms and elementary particles then standard division loses its usual meaning and in nature there are no arbitrarily small parts and no continuity.

For example, typical energies of electrons in modern accelerators are millions of times greater than the electron rest energy, and such electrons experience many collisions with different particles. If it were possible to break the electron into parts, then it would have been noticed long ago.

Another example is that if we draw a line on a sheet of paper and look at this line by a microscope then we will see that the line is strongly discontinuous because it consists of atoms. That is why standard geometry (the concepts of continuous lines and surfaces) can work well only in the approximation when sizes of atoms are neglected, standard macroscopic theory can work well only in this approximation etc.

Of course, when we consider water in the ocean and describe it by differential equations of hydrodynamics, this works well but this is only an approximation since water consists of atoms. However, it seems unnatural that even quantum theory is based on continuous mathematics. Even the name "quantum theory" reflects a belief that nature is quantized, i.e., discrete, and this name has arisen because in quantum theory some quantities have discrete spectrum (i.e., the spectrum of the angular momentum operator, the energy spectrum of the hydrogen atom etc.). But this discrete spectrum has appeared in the framework of classical mathematics.

I asked physicists and mathematicians whether in their opinion the indivisibility of the electron shows that in nature there are no infinitesimals, and standard division does not work always. Some mathematicians say that sooner or later the electron will be divided. On the other hand, as a rule, physicists agree that the electron is indivisible and in nature there are no infinitesimals. They say that, for example, dx/dt should be understood as $\Delta x/\Delta t$ where Δx and Δt are small but not infinitesimal. I ask them: but you work with dx/dt, not $\Delta x/\Delta t$. They reply that since mathematics with derivatives works well then there is no need to philosophize and develop something else (and they are not familiar with finite mathematics).

One of the key problems of modern quantum theory is the problem of infinities: the theory gives divergent expressions for the S-matrix in perturbation theory. In the so-called renormalized theories, the divergencies can be eliminated by subtracting one infinity from another. Although this is not a well founded mathematical operation, in some cases it results in excellent agreement with experiment. Probably the most famous case is that the results for the electron and muon magnetic moments obtained in quantum electrodynamics at the end of the 40th agree with experiment at least with the accuracy of eight decimal digits (see, however, a discussion in Ref. [1]). In view of this and other successes of quantum theory, the majority of physicists believe that agreement with the data is much more important than the rigorous mathematical substantiation.

At the same time, in the so called nonrenormalized theories, infinities cannot be eliminated by the renormalization procedure, and this a great obstacle for constructing quantum gravity. As the famous physicist and the Nobel Prize laureate Steven Weinberg writes in his book [2]: "*Disappointingly this problem appeared with even greater severity in the early days of quantum theory, and although greatly ameliorated by subsequent improvements in the theory, it remains with us to the present day*". The title of Weinberg's paper [3] is "Living with infinities".

In view of efforts to describe discrete nature by continuous mathematics, my friend told me the following joke: "A group of monkeys is ordered to reach the Moon. For solving this problem each monkey climbs a tree. The monkey who has reached the highest point believes that he has made the greatest progress and is closer to the goal than the other monkeys". Is it reasonable to treat this joke as a hint on some aspects of the modern science? Indeed, people invented continuity and infinitesimals which do not exist in nature, created problems for themselves and now apply titanic efforts for solving those problems. Below it will be explained on popular level (and the rigorous proof is given in Ref. [4]) that classical mathematics is a special degenerate case of finite mathematics.

The founders of quantum theory and scientists who essentially contributed to it were highly educated. But they used only classical mathematics, and even now finite mathematics is not a part of standard education for physicists. The development of quantum theory has shown that the theory contains anomalies and divergences. Physicists persistently try to solve those problems in the framework of classical mathematics and refuse to acknowledge that they arise just because this mathematics is applied. Nevertheless, several famous scientists, e.g., the Nobel Prize laureates Gross, Nambu and Schwinger discussed the idea that the ultimate quantum theory will be based on finite mathematics (see e.g., Ref. [5]).

2 Why finite mathematics is more natural than classical one

We will now discuss **whether it is justified to use mathematics with infinitesimals although in nature there are no infinitesimals**. First we note that a typical situation in physics is that there are two theories, A and B, and the problem arises when B can be treated as a special degenerate case of A. In Ref. [4] we have proposed the following

Definition: *Let theory A contain a finite parameter and theory B be obtained from theory A in the formal limit when the parameter goes to zero or infinity. Suppose that, with any desired accuracy, A can reproduce any result of B by choosing a value of the parameter. On the contrary, when the limit is already taken then one cannot return back to A and B cannot reproduce all results of A. Then A is more general than B and B is a special degenerate case of A.* Known examples are that:

1) nonrelativistic theory (NT) is a special degenerate case of relativistic one (RT) in the special case $c \to \infty$ (where c is the speed of light);

2) classical theory is a special degenerate case of quantum one (QT) in the special case $\hbar \to 0$ (where $\hbar$ is the Planck constant);

3) RT is a special degenerate case of de Sitter invariant theories in the special case $R \to \infty$ where R is the parameter of contraction from the de Sitter groups or Lie algebras to the Poincare group or Lie algebra, respectively.

In the literature those facts are explained from physical considerations but, as shown e.g., in the famous Dyson's paper "Missed Opportunities" [6], 1) follows from the pure mathematical fact that the Galilei group can be obtained from the Poincare one by contraction $c \to \infty$, and 3) follows from the pure mathematical fact that the Poincare group can be obtained from the de Sitter groups by contraction $R \to \infty$. However, as argued in Ref. [4], on quantum level symmetry should be defined not by groups but by the corresponding Lie algebras, and as shown in Ref. [4], the statements 1)-3) follow from the facts that the Galilei Lie algebra can be obtained from the Poincare one by contraction $c \to \infty$, classical Lie algebra can be obtained from the quantum one by contraction $\hbar \to 0$ and the Poincare Lie algebra can be obtained from the de Sitter Lie algebras by contraction $R \to \infty$. So, in general, theory B is a special degenerate case of theory A if the symmetry algebra for theory B can be obtained from the symmetry algebra for theory A by contraction.

Let us consider in more details that NT is a special degenerate case of RT in the special case $c \to \infty$. According to **Definition**, this implies that RT can reproduce any result of NT with any accuracy if c is chosen to be sufficiently large. However, NT cannot reproduce all results of RT because RT also describes phenomena where it is important that c is finite. From the naive point of view, one might think that NT is more general than RT because NT corresponds to the case $c = \infty$, i.e., one might think that NT describes more cases than RT where c is finite. However, NT gives the same results as RT only when speeds are much less than c, but when they are comparable to c then NT does not work.

Since in many cases speeds are much less than c then, for describing those cases, NT works with a very high accuracy and there is no need to apply RT: although in principle RT describes those cases, typically describing them by RT involves unnecessary complications. In particular, there is no need to apply RT for describing everyday life. At the same time, when speeds are comparable to c, it is important that c is not infinitely large but finite, and only RT can be applied.

Let us consider, for example, the following problem. Suppose that some reference frame moves relative to us with the speed $V = 0.6c$, and in this frame a body moves in the same direction with the same speed. Then the speed of the body relative to us is not $v = 1.2c$, as one might think from naive considerations, but $v \approx 0.882c$, and if, for example, $V = 0.99c$ then $v \approx 0.9999495c$, i.e., there is no way to get $v > c$.

Analogously, for describing almost all phenomena on macroscopic level, there is no need to apply QT. In particular, there is no need to describe the motion of the Moon by the Schrödinger equation. In principle this is possible but results in unnecessary complications. At the same time, microscopic phenomena can be correctly described only in the framework of QT.

In view of those examples, the following problem arises: is it justified to always use mathematics with infinitesimals for describing nature in which infinitesimals do not exist? There is no doubt that the technique of classical mathematics is very powerful, and in many cases describes physical phenomena with a very high accuracy. However, a problem arises whether there are phenomena which cannot be correctly described by mathematics involving infinitesimals.

Some facts of classical mathematics seem to be unnatural from the point of view of common sense. For example, $tg(x)$ is one-to-one reflection of $(-\pi/2, \pi/2)$ onto $(-\infty, \infty)$, i.e. the impression might arise that the both intervals have the same numbers of elements although the first interval is a nontrivial part of the second one. Another example is the Hilbert paradox with an infinite hotel. But mathematicians even treat those facts as pretty ones. For example, Hilbert said: "No one shall expel us from the paradise that Cantor has created for us". And this is in spite of the fact that the problem of foundation of classical mathematics has not been solved yet.

This problem has been investigated by many great mathematicians: Cantor, Fraenkel, Gödel, Hilbert, Kronecker, Russell, Zermelo and others. Their philosophy was based on macroscopic experience in which the concepts of infinitesimals, continuity and standard division are natural. However, as noted above, those concepts contradict the existence of elementary particles and are not natural in quantum theory. The illusion of continuity arises when one neglects the discrete structure of matter.

The fact that foundational problems of classical mathematics cannot be resolved, follows, in particular, from Gödel's incompleteness theorems which state that no system of axioms can ensure that all facts about natural numbers can be proved. Moreover, the system of axioms in classical mathematics cannot demonstrate its own consistency. The theorems are written in highly technical terms of mathematical logics. However, simple arguments in Ref. [4] show that foundational problems of classical mathematics follow from simple considerations, and below we give those arguments.

In the 20s of the 20th century, the Viennese circle of philosophers under the leadership of Schlick developed an approach called logical positivism which contains verification principle: *A proposition is only cognitively meaningful if it can be definitively and conclusively determined to be either true or false* (see e.g., Refs. [7]). However, this principle does not work in standard classical mathematics. For example, it cannot be determined whether the statement that $a + b = b + a$ for all natural numbers a and b is true or false.

As noted by Grayling [8], *"The general laws of science are not, even in principle, verifiable, if verifying means furnishing conclusive proof of their truth. They can be strongly supported by repeated experiments and accumulated evidence but they cannot be verified completely".* So, from the point of view of classical mathematics and classical physics, verification principle is too strong.

Popper proposed the concept of falsificationism [9]: *If no cases where a claim is false can be found, then the hypothesis is accepted as provisionally true.* In particular, the statement that $a + b = b + a$ for all natural numbers a and b can be treated as provisionally true until one has found some numbers a and b for which $a + b \neq b + a$.

Before discussing the foundation of mathematics and physics in greater details, I would like to make several remarks about problems in accepting new theories.

I think that the main problem is probably the following. Our experience is based on generally acknowledged theories and everything not in the spirit of this experience is treated as contradicting common sense. A known example is that, from the point of view of classical mechanics, it seems unreasonable that the speed $0.999c$ is possible while the speed $1.001c$ is not. The reason of this judgement is that the experience based on everyday life works only for speeds which are much less than c and extrapolating this experience to cases where speeds are comparable to c is not correct.

One of the examples is the paradox of twins in the theory of relativity: one of the brothers flew to a distant star, and when he returned being 10 years older, he found that 1000 years had passed on Earth. From the point of view of "common sense" this seems meaningless, but this seems so because our experience based on everyday life is extrapolated to the case of speeds comparable to c and this experience does not work there.

Another example is the following. If we accept that physics in our world is described by finite mathematics with characteristics p then this can be treated as the statement that p is the greatest possible number in nature. The argument attributed to Euclid is that there can be no greatest number because if p is such a number than $p+1$ is greater than p. This is again an example where our experience based on rather small numbers is extrapolated to numbers where it does not work.

According to the philosophy of quantum theory, there should be no statements accepted without proof and based on belief in their correctness (i.e. axioms). The theory should contain only those statements that can be verified, where by "verified" physicists mean an experiment involving only a finite number of steps. This philosophy is the result of the fact that quantum theory describes phenomena which, from the point of view of "common sense", seem meaningless but they have been experimentally verified.

So the philosophy of quantum theory is similar to verificationism, not falsificationism. Note that Popper was a strong opponent of quantum theory and supported Einstein in his dispute with Bohr.

From the point of view of verificationism and the philosophy of quantum theory, classical mathematics is not well defined not only because it contains an infinite number of numbers. Consider, for example, whether the rules of standard arithmetic can be justified.

We can verify that 10+10=20 and 100+100=200, but can we verify that, say $10^{1000000}+10^{1000000}=2\cdot 10^{1000000}$? One might think that this is obvious, but this is only a belief because based on extrapolating our everyday experience to numbers where it is not clear whether the experience still works.

According to principles of quantum theory, the statement that $10^{1000000}+10^{1000000}=2\cdot 10^{1000000}$ is true or false depends on whether this statement can be verified. Is there a computer which can verify this statement? Any computing device can operate only with a finite number of resources and can perform calculations only modulo some number p. If our universe is finite and contains only N elementary particles, then there is no way to verify that $N+N=2N$. So, if, for example, our universe is finite, then in principle it is not possible to verify that the rules of arithmetic are valid for any numbers.

That's why the statement a+b=c is ambiguous because it does not contain information on the computing device which will verify this statement. For example, let us pose a problem whether 10+20 equals 30. Suppose that our computing devise is such that $p = 40$. Then the experiment will confirm that 10+20=30 while if $p = 25$ then we will get that 10+20=5.

So the statements that 10+20=30 and even that $2 \cdot 2 = 4$ *are ambiguous because they do not contain information on how they should be verified.* On the other hand, the statements

$$10 + 20 = 30\,(mod\,40),\;\; 10 + 20 = 5\,(mod\,25),$$

$$2 \cdot 2 = 4\,(mod\,5),\;\; 2 \cdot 2 = 2\,(mod\,2)$$

are well defined because they do contain such an information. So only operations modulo a number are well defined.

I believe the following observation is very important: although classical mathematics (including its constructive version) is a part of our everyday life, people typically do not realize that *classical mathematics is implicitly based on the assumption that one can have any desired amount of resources.* So, classical mathematics is based on the implicit assumption that we can consider an idealized case when a computing device can operate with an infinite number of resources. In other words, standard operations with natural numbers are implicitly treated as limits of operations modulo p when $p \to \infty$. As a rule, every limit in mathematics is thoroughly investigated but, in the case of standard operations with natural numbers, it is not even mentioned that those operations are limits of operations modulo p. In real life such limits even might not exist if, for example, the universe contains a finite number of elementary particles.

3 A sketch of the proof that finite mathematics is more general than classical one

In the *technique* of classical mathematics there is no number ∞, infinity is understood only as a limit (i.e. as a potential infinity) and, as a rule, legitimacy of every limit is thoroughly investigated. However, the *basis* of classical mathematics involves actual infinity from the very beginning. For example, the ring of integers Z is involved from the very beginning and, even in standard textbooks, it is not even posed a problem whether Z should be treated as a limit of finite rings. Moreover, Z is the starting point for constructing the sets of rational, real and complex numbers and the sets with greater and greater cardinalities.

For solving the problem of infinities, different kinds of arithmetic have been proposed, e.g., Peano arithmetic, Robinson arithmetic, finitistic arithmetic and others. The latter deals with all natural numbers but only finite sets are allowed. However, finite mathematics rejects infinities from the beginning. This mathematics starts from the ring $R_p = (0, 1, 2, ...p - 1)$ where addition, subtraction and multiplication are performed as usual but modulo p, and p is called the characteristic of the ring. In the literature the ring R_p is usually denoted as $Z/(pZ)$. In my opinion, this notation is not adequate even because finite mathematics should not involve infinite sets. The notation may give a wrong impression that finite mathematics starts from the infinite set Z and that Z is more general than R_p. However, although Z has more elements than R_p, Z cannot be more general than R_p because Z does not contain operations modulo a number. We will see below that, although R_p has a lesser number of elements than Z, but the concept of R_p is more general than the concept of Z, and Z is a special degenerate case of R_p in the formal limit $p \to \infty$.

In the above discussion of the relation between NT and RT, we noted that those theories give close results when speeds are much less than c, but the results are considerably different when speeds are comparable to c, and in RT it is not possible to get $v > c$. Analogously, the results in finite and classical mathematics are the same if the numbers in question are much less than p but, since in finite mathematics all operations are modulo p, it is not possible to get the result greater than p. Physicists might think that calculations modulo a number are nonphysical but, as noted above, just such calculations are more physical than calculations in classical mathematics.

One can rigorously prove [4] that any result in Z can be reproduced in R_p if p is chosen to be sufficiently large, and that is why Z can be treated as a limit of R_p when $p \to \infty$. This result looks natural from the following considerations. Since all operations in R_p are modulo p, then R_p can be treated as a set $(-(p-1)/2, ... -1, 0, 1, ...(p-1)/2)$ if p is odd and as a set $(-p/2+1, ... -1, 0, 1, ...p/2)$ if p is even. In this representation, for numbers with the absolute values much less than p, the results of addition, subtraction and multiplication are the same in R_p and in Z, i.e. for such numbers it is not manifested that in R_p operations are modulo p.

These results are natural from the following graphical representation of the sets Z and R_p. If elements of Z are depicted as integer points on the x axis of the xy plane then, if p is odd, the elements of R_p can be depicted as points of the circle in Figure 1 and analogously if p is even.

This figure is natural from the following historical analogy. For many years people believed that the Earth was flat and infinite, and only after a long period of time they realized that it was finite and curved. It is difficult to notice the curvature when we deal only with distances much less than the radius of the curvature. Analogously, when we deal with numbers the modulus of which is much less than p then the results are the same in Z and R_p, i.e. we do not notice the "curvature" of R_p. This "curvature" is manifested only when we deal with numbers the modulus of which is comparable to p.

As noted above, Dyson's idea [6] is that theory A is more general than theory B if the symmetry in B can be obtained from the symmetry in A by contraction. It is clear from Figure 1 that R_p has a higher symmetry than Z. Mathematically this follows from the following facts. When we take an element $a \in R_p$ and successively add 1 to it then sooner or later we will get all elements of R_p by analogy with the fact that when we move along a circle in the same direction then sooner or later we will return to the starting point. However, all elements of Z can be obtained from an element $a \in Z$ only in two infinite stages when the fist stage is successively adding 1

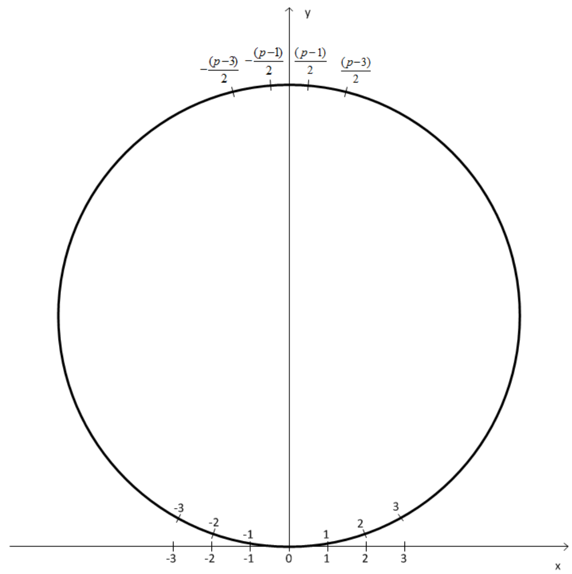

Figure 1: Relation between R_p and Z

to a and the second stage is successively adding -1 to a.

It is seen from Figure 1 that in R_p the elements $(p-1)/2$ and $-(p-1)/2$ are close to each other, and mathematically this follows from the fact that $(p-1)/2+1 = (p+1)/2 = -(p-1)/2 \ (mod\ p)$. The set Z can be treated as obtained from R_p as follows. First we break the circle in Figure 1 at the top and move the points $(p-1)/2$ and $-(p-1)/2$ a great distance from each other. Then in the formal limit $p \to \infty$ the part $(1, 2, ...(p-1)/2)$ of the circle becomes the part $(1, 2, ...\infty)$ of the straight line, and the part $(-1, -2, ... -(p-1)/2)$ of the circle becomes the part $(-1, -2, ... -\infty)$ of the straight line. Finally, by adding 0 we obtain the set Z.

This observation can be treated as an illustration of Dyson's idea because it becomes clear why R_p has a higher symmetry than Z. In Z it is not possible to reproduce all results in R_p since in Z there are no operations modulo a number. As proved in Ref. [4], it follows from **Definition** that the theory with R_p is more general than the theory with Z, and the latter is a special degenerate case of the former in the formal limit $p \to \infty$. This is in spite of the fact that R_p contains less elements than Z. This situation is analogous to that discussed above that RT is more general than NT and to other cases discussed above when theory A is more general than theory B.

The fact that the theory with R_p is more general than the theory with Z implies that, even from purely mathematical point of view, the concept of infinity cannot be fundamental since when we introduce infinity, we get the degenerate theory where all operations modulo a number disappear.

The proof that $R_p \to Z$ when $p \to \infty$ is analogous to standard proof that a sequence (a_n) of natural numbers goes to infinity when $n \to \infty$ if $\forall M > 0\ \exists n_0$ such that $a_n > M\ \forall n > n_0$. Therefore the proof in Ref. [4] that $R_p \to Z$ when $p \to \infty$ does not involve actual infinity.

At the same time, the fact that $R_p \to Z$ when $p \to \infty$ can be proved in the framework of the theory of ultraproducts described in a vast literature. As noted by Zelmanov [10], infinite fields of zero characteristic (and Z) can be embedded in ultraproducts of finite fields. This approach is in the spirit of belief of many mathematicians that sets of characteristic 0 are more general that finite sets, and for investigating infinite sets it might be convenient to use properties of simpler sets of positive characteristics.

The theory of ultraproducts is essentially based on classical results on infinite sets involving actual infinity. In particular, the theory is based on Łoŝ' theorem involving the axiom of choice. Therefore the theory of ultraproducts cannot be used in proving that finite mathematics is more general than classical one.

Let us also note that standard terminology that Z and the fields constructed from Z (e.g. the fields of rational, real and complex numbers) are sets of characteristic 0 reflects the usual spirit that classical mathematics is more fundamental than finite one. I think it is natural to say that Z is the ring of characteristic ∞ because it is a limit of rings of characteristic p when $p \to \infty$. The characteristic of the ring p is understood such that all operations in the ring are modulo p but operations modulo 0 are meaningless. Usually the characteristic n of the ring is defined as the smallest positive number n such that the sum of n units $1+1+1...$ in the ring equals zero if such a number exists and 0 otherwise. However, this sum can be written as $1 \cdot n$ and the equality $1 \cdot 0 = 0$ takes place in any ring.

Consider now the following question. Does the fact that R_p is more general than Z mean that in applications finite mathematics is more general (fundamental) than classical one? Indeed, in applications not only rings are used but also fields which contain division. For example, if p is prime then R_p becomes the Galois field F_p in which division is defined as usual but modulo p.

As note above, for numbers with the absolute values much less than p, the results of summation, subtraction and multiplication are the same in R_p and Z. That is why if an experiment deals only with such numbers, and the theory describing this experiment involves only sums, subtractions and multiplications then the results of the experiment cannot answer the question what mathematics is more adequate for describing this experiment: classical or finite. However, in the case of division the difference is essential. For example, $1/2$ in F_p equals $(p+1)/2$, i.e. a very large number if p is large. That is why an impression may arise that finite mathematics is not adequate for describing experimental data. Let us consider this problem in more details.

Now it is accepted that the most general physical theory is quantum one, i.e., any classical theory is a special case of quantum one. This fact has been already mentioned above. Therefore the problem arises whether quantum theory based on real and complex numbers containing division (and also quantum theories based on their generalizations, e.g., p-adic numbers or quaternions) can be a special case of a quantum theory based on finite mathematics.

In standard quantum theory (SQT) a state of a system is described by the wave function $\Psi = c_1e_1 + c_2e_2 + ...$ where the e_j $(j = 1, 2, ...\infty)$ are the elements of the basis of the Hilbert space, and the c_j are complex coefficients. Usually basis elements are normalized to one: $||e_j|| = 1$ and then the probability for a system to be in the state e_j is $|c_j|^2$. However, normalization to one is only the question of convention but not the question of principle. The matter is that not the probability itself but only relative probabilities of different events have a physical meaning. That is why spaces in quantum theory are projective: $const \cdot \Psi$ and Ψ describe the same state if $const \neq 0$.

Hence one can choose the basis where all the $||e_j||$ are positive integers. Then we use the theorem proved in standard textbooks on Hilbert spaces: any element of the Hilbert space can be approximated with any desired accuracy by a finite linear combination $\Psi = c_1e_1 + c_2e_2 + ...c_ne_n$ where the coefficients are rational numbers. Finally, using the fact that spaces in quantum theory are projective, one can multiply Ψ by the common denominator of all the coefficients and get the case when all the complex coefficients $c_j = a_j + ib_j$ are such that all the numbers a_j and b_j are integers.

Therefore, although formally Hilbert spaces in quantum theory are complex, but, with any required accuracy, any state can be described by a set of coefficients which are elements of Z. Hence the description of states by means of Hilbert spaces is not optimal since such a description contains a big redundancy of elements which are not needed for a full description.

Now we use that theory with R_p is more general than theory with Z and describe quantum states not by elements of Hilbert spaces but by elements of spaces over a finite ring $R_p + iR_p$, i.e. now all the a_j, b_j and $||e_j||$ are elements of R_p. I call this theory FQT - finite quantum theory. As follows from the abovementioned, FQT is more general than SQT: when the absolute values of all the a_j, b_j and $||e_j||$ are much less than p then both theories give the same results but if the absolute values of some of those quantities are comparable to p then the descriptions are different because in SQT there are no operations modulo p.

4 Examples when finite mathematics can solve problems which classical mathematics cannot

In Ref. [4] we considered phenomena where it is important that p is finite. They cannot be described by SQT, and this is analogous to the fact that nonrelativistic theory cannot describe phenomena in which it is important that c is finite. Below we describe several such phenomena.

One example is as follows. As noted above, standard quantum theory cannot describe gravity because the theory is nonrenormalizable. But in our approach, the universal law of gravitation can be derived as a consequence of FQT in semiclassical approximation [4]. In this case the gravitational constant G depends on p as $1/ln(p)$. By comparing the result with the experimental value, one gets that $ln(p)$ is of the order of 10^{80} or more, and therefore p is a huge number of the order of $exp(10^{80})$ or more. One might think that since p is so huge then in practice p can be treated as an infinite number. However, since G depends on p as $1/ln(p)$, and $ln(p)$ is "only" of the order of 10^{80}, gravity is observable. In the formal limit $p \to \infty$, G becomes zero and gravity disappears. Therefore, in our approach, gravity is a consequence of finiteness of nature.

Another case is the famous Dirac vacuum energy problem. The vacuum energy should be zero, but in standard theory the sum for this energy diverges. In Sec. 8.8 of Ref. [4], I take the standard expression for this sum; then I explicitly calculate this sum in finite mathematics without any assumptions or philosophy, and, since all the calculations are modulo p, I get zero as it should be.

I will now consider the following interesting example. In quantum theory, elementary particles are described by irreducible representations (IRs) of the symmetry algebra. The algebras are such that their IRs contain either only positive energies or only negative energies. In the first case the particles are called particles and in the second one – antiparticles. In the first case the spectrum of energies in standard theory contains the values $(m_1, m_1+1, m_1+2, \cdots \infty)$, and in the second case – the values $(-m_2, -m_2-1, -m_2-2, \cdots - \infty)$, where $m_1 > 0$, $m_2 > 0$, m_1 is called the mass of a particle and m_2 is called the mass of the corresponding antiparticle. Experimentally $m_1 = m_2$ but in standard theory, IRs with positive and negative energies are fully independent of each other. The usual statement is that $m_1 = m_2$ follows from the fact that local covariant equations are CPT invariant. However, as discussed in detail in Ref. [4], the argument x in local field functions does not have a physical meaning because it is not associated with any operator. So, in fact standard theory cannot explain why $m_1 = m_2$.

For understanding this problem, the following observation from particle theory may be helpful. In the formal case when electromagnetic and weak interactions are absent, isotopic invariance is exact, and the proton and the neutron have equal masses simply because they are different states in the same IR of the isotopic algebra. Therefore, the equality of the masses has nothing to do with locality.

Consider now what happens in quantum theory over finite mathematics. For definiteness, we consider the case when p is odd, and the case when p is even can be considered analogously. One starts constructing the IR as usual with the value m_1, and, by acting on the states by raising operators, one gets the values $m_1+1, m_1+2, \cdots$. However, now we are moving not along the straight line but along the circle in Figure 1. When we reach the value $(p-1)/2$, then, as explained above, the next value is $-(p-1)/2$, i.e. one can say that by adding 1 to a large positive number $(p-1)/2$ one gets a large negative number $-(p-1)/2$. By continuing this process, one gets the numbers $-(p-1)/2+1 = -(p-3)/2$, $-(p-3)/2+1 = -(p-5)/2$ etc. The explicit calculation shows that the procedure ends when the value $-m_1$ is reached.

Therefore, finite mathematics gives a clear proof of the experimental fact that $m_1 = m_2$, and this is analogous to the above observation that two states have equal masses if they belong to the same IR of the symmetry algebra. In addition, finite mathematics shows that, instead of two independent IRs in classical mathematics, one gets only one IR describing both, a particle, and its antiparticle. The case described by classical mathematics can be called degenerate because in the formal limit $p \to \infty$ one IR in finite mathematics splits into two IRs in classical mathematics. So, in the case $p \to \infty$ we get symmetry breaking. This example is a beautiful illustration of Dyson's idea [6] that theory A is more general than theory B if the symmetry in B can be obtained from the symmetry in A by contraction. The example is fully in the spirit of this idea because it shows that classical mathematical can be obtained from finite one by contraction of the symmetry in the formal limit $p \to \infty$. This example also shows that even the very concept of particle-antiparticle is only approximate and is approximately valid only when p is very large. Consequently, constructing complete quantum theory based on finite mathematics will be difficult because the construction should be based on new principles.

The above examples demonstrate that there are phenomena which can be explained only in finite mathematics because for those phenomena it is important that p is finite and not infinitely large. So, we have an analogy with the case that relativistic theory can explain phenomena where c is finite while nonrelativistic theory cannot explain such phenomena. In Ref. [4] I also consider other examples where the results given by finite mathematics considerably differ from ones given by classical mathematics.

5 Discussion

In the literature an idea is discussed that space and time should be quantized. However, as discussed in detail in Ref. [4], the concept of space-time has a physical meaning only on classical level, i.e., when first FQT is approximated by SQT in the formal limit $p \to \infty$, and then SQT is approximated by classical theory in the formal limit $\hbar \to 0$.

One can also pose the following problem. If the laws of physics are described in finite mathematics with some p then a question arises whether there are reasons for p to be as is or the value of p is a result of pure random circumstances. As noted above, every computing device can perform mathematical operations only modulo some number p which is defined by the number of bits this device can operate with. It is reasonable to believe that finite mathematics describing physics in our universe is characterized by a characteristic p which depends on the current state of the universe, i.e., the universe can be treated as a computer. Therefore, it is reasonable to believe that the number p is different at different stages of the universe.

The problem of time is one of the most fundamental problems of quantum theory, and this problem is discussed in a vast literature (see e.g. Ref. [4] and references therein). In quantum theory it is not correct to operate with the time t which is a continuous quantity belonging to the interval $(-\infty, +\infty)$. In Ref. [4] it has been discussed a conjecture that standard classical time t manifests itself because the value of p changes, i.e. t is a function of p. We do not say that p changes over time because classical time t cannot be present in quantum theory; we say that we feel t because p changes. As discussed in Ref. [4] and will be discussed in more details in a separate publication, with such an approach, the known problem of the baryon asymmetry of the universe has a natural solution.

Let us note that in FQT there are no infinities in principle and that is why divergences are absent in principle. In addition, probabilistic interpretation of FQT is only approximate: it applies only to states described by the numbers a_j, b_j and $||e_j||$ which are much less than p.

This situation is a good illustration of the famous Kronecker's expression: "God made the natural numbers, all else is the work of man". In view of the above discussion, I propose to reformulate this expression as: "God made only finite sets of natural numbers, all else is the work of man". For illustration, consider a case when some experiment is conducted N times, the first event happens n_1 times, the second one — n_2 times etc. such that $n_1 + n_2 + ... = N$. Then the experiment is fully described by a finite set of natural numbers. But people introduce rational numbers $w_i = w_i(N) = n_i/N$, introduce the concept of limit and define probabilities as limits of the quantities $w_i(N)$ when $N \to \infty$.

Of course, when classical and finite mathematics are considered only as abstract sciences then the question what mathematics is more general (fundamental) does not have a great meaning. However, the above discussion shows that, from the point of view of applications, finite mathematics is more general (fundamental) than classical one. The conclusion from the above consideration can be formulated as:

Mathematics describing nature at the most fundamental level involves only a finite number of numbers, while the concepts of limit, infinitesimals and continuity are needed only in calculations describing nature approximately.

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References

[1] O. Consa, *The Unpublished Feynman Diagram IIc.* arXiv:2010.10345 (2020).

[2] S. Weinberg. *The Quantum Theory of Fields*, Vol. I, Cambridge University Press: Cambridge, UK (1999).

[3] S. Weinberg, *Living with Infinities*, arXiv:0903.0568 (2009).

[4] F. Lev, *Finite mathematics as the foundation of classical mathematics and quantum theory. With application to gravity and particle theory.* ISBN 978-3-030-61101-9. Springer, https://www.springer.com/us/book/9783030611002 (2020).

[5] Y. Nambu, *Field Theory of Galois Fields*, pp. 625-636 in *Quantum Field Theory and Quantum Statistics.* I.A. Batalin et. al. eds. Adam Hilger: Bristol (1987).

[6] F. G. Dyson, *Missed Opportunities.* Bull. Amer. Math. Soc. **78**, 635–652 (1972).

[7] C. J. Misak, *Verificationism: Its History and Prospects*. Routledge: N.Y. (1995); A.J. Ayer, *Language, Truth and Logic*, in "Classics of Philosophy". Oxford University Press: New York - Oxford (1998) pp. 1219-1225; G. William, *Lycan's Philosophy of Language: A Contemporary Introduction*. Routledge: N.Y. (2000).

[8] A.C. Grayling, *Ideas That Matter*. Basic Books: New York (2012).

[9] Karl Popper, in Stanford Encyclopedia of Philosophy.

[10] E. Zelmanov, private communication (2020).

[11] M. Burgin and M. Czachor, *Non-Diophantine Arithmetics in Mathematics, Physics and Psychology*. ISBN: 978-981-121-432-5, World Scientific (2021).