Exact Solution of Some Non-Autonomous Nonlinear ODEs of Second Order

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Abstract: This paper shows the exact integrability analysis of two classes of non-autonomous and nonlinear differential equations. It has been possible to recover some equations of general relativity from the first class of equations and consequently to compute their solution in fashion way. The second class is shown to include the Emden-Fowler equation and its integrability analysis, performed with the first integral theory developed by Monsia et al. [16] allowed to compute the exact solution of some subclasses of Emden-Fowler equations.

keywords: Non-autonomous nonlinear differential equation; Emden-Fowler equation; first integral theory; nonlocal transformation; exact solution.

1 Introduction

Non-autonomous nonlinear ODEs are the nonlinear ODEs whose independent variable appears explicitly. These equations are often encountered in mathematical physics as the governing equations of considerable number of problems in mechanics, physics, and chemical engineering. The general solution of those equations can not usually be given explicitly. To fix this issue, some researchers employ a linearizing approach consisting to map a given nonlinear differential equation to some linear differential equation with known solution, so that the desired exact solution can be computed using the known solution of the linear equation. But the finding of such an appropriate transformation which ensures this mapping is not always evident. Other reseachers try to reduce the order of the considered equation by means of Lie point symmetries analysis to find its first integrals... In this context, consider the two classes of non-autonomous nonlinear ODEs

$$\ddot{x}(t) + (2\gamma - 1)\frac{\dot{x}^2(t)}{x(t)} - l\frac{g'(t)}{g(t)}\dot{x(t)} = 0 \tag{1}$$

and

$$u''(t) + A_1 u'(t) + B_1 u(t) + C_1 u^n(t) = 0 \tag{2}$$

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with

$$\begin{cases} A_1 = \dfrac{1}{t}\left[\rho - (\rho-1)\dfrac{\gamma-2\alpha}{\gamma+1}\right] \\ \\ B_1 = (\alpha+\alpha^2)\dfrac{(\rho-1)^2}{(\gamma+1)^2}\dfrac{1}{t^2} \\ \\ C_1 = b(-\dfrac{\gamma+1}{\rho-1})^{\dfrac{\sigma-2\gamma-\alpha(n-1)}{\gamma+1}} t^{-2\rho-\dfrac{(\rho-1)}{\gamma+1}[\sigma-2\gamma-\alpha(n-1)]} \end{cases}$$

where α, ρ, σ, γ, and n are arbitrary parameters $g(t) \neq 0$ is an arbitrary function of t .
The investigation of (1) can be useful, since it can be proved under certain condition to include some equations of general relativity as [1].

$$\ddot{x} - 3\frac{\dot{x}^2}{x} - \frac{\dot{x}}{t} = 0 \tag{3}$$

and other famous equations used to illustrate different methods of integration of differential equations, as for instance

$$\ddot{x} + \frac{\dot{x}^2}{x} + 3\frac{\dot{x}}{t} = 0 \tag{4}$$

Now the problem to investigate is to ask whether one can find some nonlocal transformation which maps the free particle equation into (1). This work predicts so. That transformation can appear so interesting since it will be used to compute the exact solution of equations (3) and (4) using the solution of the free particle equation.
On the other hand, the investigation of (2) is also useful since it may include the non-autonomous nonlinear ODE

$$\frac{d}{dt}(t^\rho\frac{du}{dt}) + bt^\sigma u^n = 0 \tag{5}$$

knowing in the literature as Emden-Fowler equation and which can be reduced under some appropriate transformation into the form [2,3]

$$y''(x) + bx^\sigma y^n(x) = 0 \tag{6}$$

(6) is known as a standard form of Emden-Fowler equation [4], where ρ, σ, n are arbitrary parameters. For $\sigma = 2$, $b = 1, \rho = 2$, equation (5) becomes the Lane-Emden equation

$$\frac{d}{dt}(t^2\frac{du}{dt}) + t^2u^n = 0 \tag{7}$$

that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules [5]. Emden- Fowler equation arises in the study of gas dynamics and fluid mechanics [6,7]. More recently, the Emden Fowler equations also appear in the study of relativistic mechanics and in nuclear physics [6,8]. Emden-Fowler equation has been widely investigated in the literature from various point of view. Here, we propose to show that equation (2) can be considered as Emden-Fowler equation and may include (5), so that it can be considered as a generalized form of equation (5). We also investigate the exact integrability of the standard form of Emden-Fowler equation (6). A significant part of the relevant search in integrability analysis of (6) deals with the use of approximate methods [9–14] and phase plane analysis [15], Lie symmetry and Painlevé analysis of the Emden- Fowler equations has been performed in [4]. Recently, Monsia and coworkers introduced in the literature some theory of first integral analysis of differential equations [16], and whose importance have been highlighted in some recent publication [17], to calculate the exact general solution of the well known equation which is assumed to be a truly nonlinear oscillator. In this respect, it becomes reasonnable to ask whether that powerful theory can also be helpful in the exact integrability of equation (6). Therefore we purpose to investigate (6) from the point of view of first integral analysis via the first integral theory more recently introduced by Monsia and coworkers [16]. Hence, in order to investigate the exact integrability of the concerning two classes of nonlinear non-autonomous equations under consideration, we organize the paper in two sections. Section 2 presents the investigation of the class of equations (1) and Section 3, those of (2). We first in Section 2 establish the general theory which leads to equation (1) , and the usefulness of that theory is shown through the solving of target equations. As in Section 2, Section 3 starts by establishing the general theory from which derive the class of equations (2), whereafter the integrability analysis of (2) is done via the exact integrability of (6) under certain conditions. Finally a conclusion is addressed.

2 Exact integrability analysis of (1)

2.1 General theory

This section is devoted to the establishment of nonlinear and generalized equation (1) by nonlocal transformation of the free particle equation. Consider a free particle equation

$$y''(\tau) = 0 \tag{8}$$

with its well-known solution

$$y(\tau) = c_1\tau + c_2 \tag{9}$$

where c_1 and c_2 are constants. Let us consider the following nonlocal transformation

$$y(\tau) = x(t)^{2\gamma}, \;\; d\tau = g(t)^l dt \tag{10}$$

where, l and γ are real parameters, and $g(t)$ is an arbitrary function of t. Then, the following theorem may be proved.

Theorem 2.1 *Consider equation* (8)*. Then by application of* (10)*, equation*(8) *becomes equation*(1)*.*

Proof

Under the application of (10), the first derivative of $y(\tau)$ may be immediately written as

$$\frac{dy}{d\tau} = \frac{2\gamma}{g(t)^l}\dot{x}x^{2\gamma-1} \tag{11}$$

so that one may find

$$\frac{d^2y}{d\tau^2} = \frac{2\gamma x^{2\gamma-1}}{g(t)^{2l}}\left[\ddot{x} + (2\gamma-1)\frac{\dot{x}^2}{x} - l\frac{g'(t)}{g(t)}\dot{x}\right] \tag{12}$$

Inserting (12) into equation (8) and taking into account $\dfrac{2\gamma x^{2\gamma-1}}{g(t)^{2l}} \neq 0$ yields (8). Now some equations are considered in the following to illustrate the theory

2.2 A nonlinear differential equation of family of (1)

Let us consider equation (4). It has been considered in [18] to illustrate a method based on variational derivatives to find two integrating factors and therefore the general solution of the equation. The method used requires to solve a system of two coupled second-order partial differential equations. Here, we will recover equation (4) from (1), and consequently, compute its solution in a fashion way. Therefore, one may consider the following theorem

Theorem 2.2 *Let $\gamma = 1$, $l = -3$ and $g(t) = t$.Then equation* (1) *reduces to equation* (4).

Proof

The proof is evident. Indeed by substituting $\gamma = 1$, $l = -3$ and $g(t) = t$ into (1) leads immediately to (4).
Let us compute now the exact solution of equation (4). By application of theorem **2.2**, transformation(10) can be rewritten in this form

$$y(\tau) = x^2(t), \frac{d\tau}{dt} = t^{-3} \tag{13}$$

which yields

$$\begin{cases} x(t) = [y(\tau)]^{\frac{1}{2}} \\ \tau(t) = -\dfrac{1}{2t^2} + c_3. \end{cases} \tag{14}$$

c_3 is a constant which we set in the following to be zero. Taking into account (9) one may have

$$x(t) = [-\frac{c_1}{2t^2} + c_2]^{\frac{1}{2}} \tag{15}$$

2.3 An equation of general relativity

Consider the following theorem

Theorem 2.3 *Let $\gamma = -1$, $l = 1$ and $g(t) = t$.Then equation* (1) *reduces to equation* (3).

Proof: Equation (3) is immediately recovered by substituting $\gamma = -1$, $l = 1$ and $g(t) = t$ into (1).
Equation(3) is found in [19] providing an illustration of searching the first integral from symmetries. Originally that equation was found by Buchdal in its study of a relativistic fluid sphere in the theory of general relativity [1], wherein he indicates its solution whithout any proof. Recently Yessoufou et al. [20] have computed the same solution by applying a more general nonlocal transformation to the harmonic oscillator equation. Let us show the solution of equation(3) in the form highlighted by Buchdal [1]. By application of theorem **2.3** , transformation (10) can be rewritten in this form

$$y(\tau) = x^{-2}(t), \frac{d\tau}{dt} = t \tag{16}$$

which yields

$$\begin{cases} x(t) = [y(\tau)]^{\frac{-1}{2}} \\ \tau(t) = -\dfrac{1}{2}t^2 + c_3. \end{cases} \tag{17}$$

c_3 is a constant which we set to be zero in the following. Using (9) one may have

$$x(t) = [\frac{c_1}{2}t^2 + c_2]^{\frac{-1}{2}} \tag{18}$$

which can be arranged in the form

$$x(t) = c_2^{-\frac{1}{2}}(1 + \frac{c_1}{2c_2}t^2)^{\frac{-1}{2}} \tag{19}$$

Let a, c, and k be the new constants such that $c_2 = 4\frac{c^4}{a^2}$ $a \neq 0$ and $k = \frac{c_1}{2c_2}$. Then the solution is rewritten in the form

$$x(t) = \frac{1}{2}c^{-2}a(1 + kt^2)^{-\frac{1}{2}} \tag{20}$$

which is the solution of equation(3) highlighted by Buchdal [1].

3 Exact integrability analysis of equation (2)

3.1 General theory

This section is devoted to the establishment of nonlinear and generalized equation (2) by nonlocal transformation of the standard form of Emdem- Fowler equation. Consider the standard form of Emdem- Fowler equation (6) which is rewritten as

$$y'' + f(x)y^n = 0 \tag{21}$$

with $f(x) = bx^\sigma$, $x \in (o, \infty)$, b, and σ the arbitrary parameters.
We propose here to apply some nonlocal transformation to (21) to obtain some equation which includes (5) as particular case. Thus, the equation obtained will be considered as a more generalized one than (5). In this way, let

$$y(x) = u(t)e^{-\alpha\varphi(x)}; \;\; t^{-\rho}dt = e^{\gamma\varphi(x)}dx \tag{22}$$

be a nonlocal transformation. Then the following theorem may be proved

Theorem 3.1 *Consider equation* (21). *Then by application of* (22), *equation* (21) *may become*

$$u''(t) + Au'(t) + Bu(t) + Cu^n(t) = 0 \tag{23}$$

with

$$\begin{cases} A = \frac{\rho}{t} + (\gamma - 2\alpha)t^{-\rho}\varphi'(x)e^{-\gamma\varphi(x)} \\ \\ B = \left[-\alpha\varphi''(x) + \alpha^2\varphi'^2(x)\right] t^{-2\rho}e^{-2\gamma\varphi(x)} \\ \\ C = \left[f(x)t^{-2\rho}e^{-[2\gamma+(n-1)\alpha]\varphi(x)}\right] \end{cases}$$

Proof
Under the application of (22), let us evaluate $y^n(x)$, $y'(x)$ and $y''(x)$ as

$$y^n(x) = u^n(t)e^{-n\alpha\varphi(x)} \tag{24}$$

and

$$y' = \frac{dy}{dx}$$

that is

$$\begin{aligned} y' &= \frac{d}{dx}\left[u(t)e^{-\alpha\varphi(x)}\right] \\ &= -\alpha\varphi'(x)e^{-\alpha\varphi(x)}u(t) + \frac{du}{dt}\frac{dt}{dx}e^{-\alpha\varphi(x)} \end{aligned}$$

Now

$$\frac{dt}{dx} = t^{\rho}e^{\gamma\varphi(x)}$$

then it yields

$$y'(x) = \big[-\alpha\varphi'(x)u(t) + t^{\rho}u'(t)e^{\gamma\varphi(x)}\big]e^{-\alpha\varphi(x)} \tag{25}$$

where

$$y'' = \frac{d}{dx}y'$$

that is to say

$$\begin{aligned} y''(x) &= \frac{d}{dx}\big[-\alpha\varphi'(x)u(t) + t^{\rho}u'(t)e^{\gamma\varphi(x)}\big]e^{-\alpha\varphi(x)} \\ &= -\alpha\varphi'(x)e^{-\alpha\varphi(x)}\left[-\alpha\varphi'(x)u(t) + t^{\rho}u'(t)e^{\gamma\varphi(x)}\right] + e^{-\alpha\varphi(x)}\left[-\alpha\varphi''(x)u(t) - \alpha\varphi'(x)u'(t)t^{\rho}e^{\gamma\varphi(x)}\right] \\ &+ e^{-\alpha\varphi(x)}\left[\rho t^{2\rho-1}u'(t)e^{2\gamma\varphi(x)} + t^{2\rho}e^{2\gamma\varphi(x)}u''(t) + \gamma t^{\rho}\varphi'(x)u'(t)e^{\gamma\varphi(x)}\right] \end{aligned}$$

which is

$$\begin{aligned} y''(x) = u''(t)\left[t^{2\rho}e^{(2\gamma-\alpha)\varphi(x)}\right] + u'(t)\left[(\gamma-2\alpha)t^{\rho}\varphi'(x)e^{(\gamma-\alpha)\varphi(x)} + \rho t^{2\rho-1}e^{(2\gamma-\alpha)\varphi(x)}\right] \\ + u(t)\left[-\alpha\varphi''(x)e^{-\alpha\varphi(x)} + \alpha^2\varphi'^2(x)e^{-\alpha\varphi(x)}\right] \end{aligned} \tag{26}$$

Therefore, Equation (21) is rewritten in this form

$$\begin{aligned} u''(t)\left[t^{2\rho}e^{(2\gamma-\alpha)\varphi(x)}\right] + u'(t)\left[(\gamma-2\alpha)t^{\rho}\varphi'(x)e^{(\gamma-\alpha)\varphi(x)} + \rho t^{2\rho-1}e^{(2\gamma-\alpha)\varphi(x)}\right] \\ + u(t)\left[-\alpha\varphi''(x)e^{-\alpha\varphi(x)} + \alpha^2\varphi'^2(x)e^{-\alpha\varphi(x)}\right] + u^n(t)\left[f(x)e^{-n\alpha\varphi(x)}\right] = 0 \end{aligned} \tag{27}$$

which may be arranged in the form (23). To obtain our desired generalized equation, consider the following theorem

Theorem 3.2 *If* $\varphi(x) = \ln x$*, equation* (23) *becomes* (2)

Proof

From (22), one may have by integration

$$\frac{-1}{\rho-1}t^{-(\rho-1)} = \int e^{\gamma\varphi(x)}dx + I$$

Setting $I = 0$, and for $\varphi(x) = \ln x$, it yields

$$x = \left[-\frac{\gamma+1}{\rho-1}\right]^{\frac{1}{\gamma+1}} t^{-\frac{\rho-1}{\gamma+1}} \tag{28}$$

and

$$x^{\sigma} = \left[-\frac{\gamma+1}{\rho-1}\right]^{\frac{\sigma}{\gamma+1}} t^{-\frac{\sigma(\rho-1)}{\gamma+1}} \tag{29}$$

Since $\varphi(x) = \ln x$, then

$$\varphi'(x) = \frac{1}{x}$$

and

$$\varphi''(x) = -\frac{1}{x^2}$$

Therefore A, B, and C are respectively rewritten in the form A_1, B_1, and C_1. Thus equation (2) is found. (2) is our desired generalized equation. To observe that one may recover (5) from (2), consider the following theorem

Theorem 3.3 *If $\alpha = -1$, $\gamma = -2$, then equation (2) becomes (5).*

Proof
For $\alpha = -1$, $\gamma = -2$, $A_1 = \frac{\rho}{t}$, $B_1 = 0$, and

$$c_1 = b(\rho-1)^{\sigma+n+3} t^{[-2\rho+(\rho-1)(\sigma+n+3)]}$$

then, equation (2) becomes

$$u''(t) + \frac{\rho}{t}u' + u^n(t)\left[b(\rho-1)^{\sigma+n+3} t^{[-2\rho+(\rho-1)(\sigma+n+3)]}\right] = 0$$

Multiplying by t^{ρ}, it yields

$$t^{\rho}u''(t) + \rho t^{\rho-1}u'(t) + \left[b(\rho-1)^{\sigma+n+3} t^{[-\rho+(\rho-1)(\sigma+n+3)]}\right] u^n(t) = 0$$

which is written in the self adjoint form as follows

$$\frac{d}{dt}(t^{\rho}\frac{du}{dt}) + \beta t^{\sigma'} u^n(t) \tag{30}$$

where $\sigma' = -\rho + (\rho-1)(\sigma+n+3)$ and $\beta = b(\rho-1)^{\sigma+n+3}$
Thus, the generalized equation (5) is recovered from (2). Therefore the exact integrability analysis of (2) can be provided by the exact integrability analysis of (21).

3.2 Exact solution of a subclass of equation (21)

The purpose of this section is to introduce first integral theory developed by Monsia and coworkers as an alternative to existing methods in solving nonlinear Emden-Fowler type equation. Here, we propose to use these techniques to obtain the exact solution of equation (21). Consider equation (21). Setting $\sigma = 0$, then one has

$$y''(x) + by^n(x). = 0 \tag{31}$$

We choose to investigate the exact integrability of (31) using the first integral theory [16], for $n^2 \neq 1$. In this way, let us consider the first integral recently introduced by Monsia et al. [16]

$$I = I(y, y') = y'^l y^\mu - \lambda y^r \tag{32}$$

where l, μ, r, and λ are arbitrary constants. By differentiation with respect to x, one may have [17].

$$y'' + \frac{\mu}{ly}\left[\lambda y^{r-\mu} + I y^{-\mu}\right]^{\frac{2}{l}} - \frac{\lambda r}{l} y^{r-\mu-1}\left[\lambda y^{r-\mu} + I y^{-\mu}\right]^{\frac{2-l}{l}} = 0 \tag{33}$$

Using (33), exact integrability theorem of (31) may be formulated.

3.2.1 Integrability theorem of the equation (31)

To formulate the theorem which assures the exact and general solution to (31), let us consider $\mu = 0$, $l = 1$, $r = \frac{n+1}{2}$. Thus (33) reduces to the nonlinear equation

$$y'' - I\frac{\lambda(n+1)}{2} y^{\frac{n-1}{2}} - \frac{\lambda^2(n+1)}{2} y^n = 0 \tag{34}$$

For $I = 0$, and $b = -\frac{\lambda^2(n+1)}{2}$, one may recover (31). From (34), one may state the following theorem

Theorem 3.4 *If $I = 0$, and $b = -\frac{\lambda^2(n+1)}{2}$, then equation (34) turns into (31), and is exactly integrable and the exact and general solution is*

$$y(x) = \left[\frac{-b(n-1)^2}{2(n+1)}(x + K_1)^2\right]^{\frac{1}{1-n}} \tag{35}$$

where K_1 is an arbitrary constant

Proof

In the context of (31), (32) becomes

$$y' - \lambda y^{\frac{n+1}{2}} = 0 \tag{36}$$

which leads to

$$\frac{2}{1-n} y^{\frac{1-n}{2}} = \lambda(x + K_1) \tag{37}$$

so that

$$y(x) = \left[\lambda^2 \frac{(1-n)^2}{4}(x + K_1)^2\right]^{\frac{1}{1-n}} \tag{38}$$

Substituting $\lambda^2 = \frac{-2b}{n+1}$ into (38), one may secure the exact general solution to (31) as (35). Thus the above theorem is proved.

4 Conclusion

In this paper a nonlocal transformation has been developed to map the free particle equation into a class of second order non-autonomous and nonlinear differential equation. It is found that this class of equation includes some famous non-autonomous differential equations of literature. As consequence, their solutions have been computed using the generalized equation. Doing so, the usefullness of the theory has been shown. In the other hand, following the first integral theory developed by Monsia and coworkers [16], the exact integrability analysis of Emden-Fowler equation is performed.

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