On the linear harmonic oscillator solution for a quadratic Lienard type equation

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Abstract

We present in this paper a quadratic Lienard type equation which is explicitly integrable. We show that it has the isochronous periodic solution of the linear harmonic oscillator.

Keywords: Linear harmonic oscillator, quadratic Lienard type equation, isochronous periodic solution, exact general solution.

Introduction

The quadratic Lienard type equation

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^{2}+h(x)=0 \tag{1}$$

where overdot means a derivative with respect to time, and prime denotes derivative with respect to the argument and $g(x)$ and $h(x)$ are arbitrary functions of x, has been for a long time investigated in the literature [1-4]. In [1] Lagrangian and phase plane methods are used to secure in 1974 a celebrated equation of the form (1) having a harmonic periodic solution but with amplitude dependent frequency. Recently Akande et al. [3] have shown the existence of a class of equations of the form (1) containing equations which can exhibit harmonic periodic solutions but with amplitude dependent frequency. In [4] the authors investigated the Painlevé-Gambier XVII equation

$$\ddot{x}-\frac{(m-1)}{m}\frac{\dot{x}^{2}}{x}=0 \tag{2}$$

and recovered by non-point transformation the non-periodic solution given by Ince in his book [5]. The equation (2) is of the reduced form

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$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2=0 \tag{3}$$

In this context it is suitable to ask whether there exists a function $g(x)$ which ensures the existence of periodic solution for (3). In this work we predict the existence of such a function so that the general solution of (3) is harmonic periodic with isochronous properties. In this perspective we establish the equation and calculate its sinusoidal periodic solution in section (2). Finally a conclusion is done for the work.

2- Model of equation (3)

In this case we consider the general class of mixed Lienard type equations

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+a\,\ell\frac{f(x)}{g(x)}\,x^{\ell-1}\dot{x}+ab\,x^{\ell}\,\frac{f'(x)}{g^2(x)}-a^2\,x^{2\ell}\,\frac{f'(x)f(x)}{g^2(x)}=0 \tag{4}$$

introduced recently in [6]. The corresponding first-order differential equation reads [6]

$$g(x)\dot{x}+a\,f(x)x^{\ell}=b \tag{5}$$

Substituting $a=0$, into (4) and (5) yields respectively

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2=0 \tag{6}$$

and

$$g(x)\dot{x}=b \tag{7}$$

Applying $g(x)=\left(\mu^2-x^2\right)^{-1/2}$, where μ is an arbitrary parameter, turns (6) and (7) respectively into

$$\ddot{x}+\frac{x}{\mu^2-x^2}\dot{x}^2=0 \tag{8}$$

and

$$\left(\mu^2-x^2\right)^{-1/2}\dot{x}=b \tag{9}$$

The equation (9) can be written

$$\frac{dx}{\sqrt{\mu^2 - x^2}} = bdt \tag{10}$$

By integration, one can find

$$\sin^{-1}\left(\frac{x}{\mu}\right) = b(t + K) \tag{11}$$

where K is a constant of integration. From (11) one can get the desired harmonic and isochronous periodic solution as

$$x(t) = \mu \sin[b(t + K)] \tag{12}$$

where $\mu > 0$.

Conclusion

In this paper we have presented a quadratic Lienard type nonlinear equation. We have shown that this equation has the solution of the linear harmonic oscillator equation.

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