On the existence of odd perfect numbers

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January 4, 2021

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"Entia non sunt multiplicanda praeter necessitatem" (Ockam, W.)

"Te doy gracias, Padre, porque has ocultado estas cosas a los sabios y entendidos y se las has revelado a la gente sencilla" (Mt 11,25)

Abstract

In this brief paper it is proved the inexistence of odd perfect numbers using elementary methods. From the definition of a perfect number P, and operating with the set of proper divisors less than $\sqrt{P}$, the existence of some odd perfect number is linked to the existence of solution of a particular egyptian fraction with an special restriction. Proving that such an egyptian fraction with that restriction can not exist, it is concluded that no odd perfect number does exist.

2010MSC: 11A99

1 Introduction

In number theory, a perfect number is a positive integer equal to the sum of its positive proper divisors, excluding itself. In about 300 BC Euclid showed that if $2^p - 1$ is prime then $2^{p-1}(2^p - 1)$ is perfect. Two millennia later, Euler proved that all even perfect numbers are of this form. This is known as the Euclid–Euler theorem.

However, it is not known whether there are any odd perfect numbers, although there are a good number of well-known results regarding the conditions that it should satisfy. In this sense, in 1888, Sylvester stated that *"... a prolonged meditation on the subject has satisfied me that the existence of any one such [odd perfect number]—its escape, so to say, from the complex web of conditions which hem it in on all sides—would be little short of a miracle".*

In this paper it is proved the inexistence of odd perfect numbers using only elementary methods, and none of the previous "complex web of conditions" that previous papers have found. As a result, this paper has no references.

The final conclusion of this paper can be expressed with the following:

Theorem. *No odd perfect number can exist.*

2 Proof of the inexistence of odd perfect numbers

Firstly, we need some basic definitions and well-known lemmas; we skip the proof for the shake of briefness:

1. A perfect number must be composite, as the sum of all proper divisors of any prime number excluding itself is 1.

2. A perfect number can not be a square; therefore, a perfect number can not be expressed as the product of two equal factors.

3. Every composite number C expressed as the product of two distinct factors a and b, such that $a < b$, has the property that $a < \sqrt{C}$ and $b > \sqrt{C}$, $b = \frac{C}{a}$.

4. All the proper divisors of any odd composite number C are odd.

Let P be some perfect number.

Let $R = \{d_1, d_2, ..., d_n\}$ be the set of proper divisors of P less than $\sqrt{P}$ excluding 1, and $S = \left\{\frac{P}{d_1}, \frac{P}{d_2}, ...\frac{P}{d_n}\right\}$ be the set of proper divisors of P greater than $\sqrt{P}$ excluding P. As P is a perfect number,

$$1 + d_1 + d_2 + ... + d_n + \frac{P}{d_n} + ... + \frac{P}{d_2} + \frac{P}{d_1} = P \tag{1}$$

Operating,

$$1 + d_1 + d_2 + ... + d_n = P - \frac{P}{d_1} - \frac{P}{d_2} - ... - \frac{P}{d_n}$$

$$1 + d_1 + d_2 + ... + d_n = P\left(1 - \frac{1}{d_1} - \frac{1}{d_2} - ... - \frac{1}{d_n}\right)$$

$$\frac{1 + d_1 + d_2 + ... + d_n}{1 - \frac{1}{d_1} - \frac{1}{d_2} - ... - \frac{1}{d_n}} = P$$

As $1 + d_1 + d_2 + ... + d_n$ is an integer, it follows that $\frac{1}{1-\frac{1}{d_1}-\frac{1}{d_2}-...-\frac{1}{d_n}}$ must be integer. It is trivial to show $\sqrt{P} < 1 + d_1 + d_2 + ... + d_n < P$. Therefore, P is a perfect number only if both $1 + d_1 + d_2 + ... + d_n$ and $\frac{1}{1-\frac{1}{d_1}-\frac{1}{d_2}-...-\frac{1}{d_n}}$ are proper divisors of P.

By the lemma 3, one of the two expression is less than $\sqrt{P}$, and therefore belongs to R, and the other is greater than $\sqrt{P}$ and belongs to S. As $1+d_1+d_2+...+d_n$ is greater than the greatest element of R, subsequently we can state that

$$\left(\frac{1}{1 - \frac{1}{d_1} - \frac{1}{d_2} - ... - \frac{1}{d_n}}\right) = d_k \in R \tag{2}$$

$$1 + d_1 + d_2 + ... + d_n = \frac{P}{d_k} \in S \tag{3}$$

Now we are in position to prove the following useful

Lemma 5. *If P is some odd perfect number, then* $\frac{d_n}{d_k} < 3$

Proof.

From (3), and operating,

$$1 + \sum_{j=1}^{n} d_j = \frac{P}{d_k}$$

$$P = d_k\left(1 + \sum_{j=1}^{n} d_j\right)$$

As by definition we have that $d_n < \sqrt{P}$, just substituting we can set that

$$d_n < \sqrt{d_k \left(1 + \sum_{j=1}^{n} d_j\right)}$$

$$d_n^2 < d_k \left(1 + \sum_{j=1}^{n} d_j\right)$$

$$\frac{d_n}{d_k} < \frac{1 + \sum_{j=1}^{n} d_j}{d_n} \tag{4}$$

Other hand, taking (1) and dividing by $\sqrt{P}$, we get that

$$\frac{1}{\sqrt{P}} + \frac{d_1}{\sqrt{P}} + \frac{d_2}{\sqrt{P}} + ... + \frac{d_n}{\sqrt{P}} + \frac{P}{d_n\sqrt{P}} + ... + \frac{P}{d_2\sqrt{P}} + \frac{P}{d_1\sqrt{P}} = \sqrt{P}$$

$$\frac{1}{\sqrt{P}} + \frac{d_1}{\sqrt{P}} + \frac{d_2}{\sqrt{P}} + ... + \frac{d_n}{\sqrt{P}} + \frac{\sqrt{P}}{d_n} + ... + \frac{\sqrt{P}}{d_2} + \frac{\sqrt{P}}{d_1} = \sqrt{P}$$

As $\forall d_j \; \frac{\sqrt{P}}{d_j} > 1$, and $1 + \sum_{j=1}^{n} d_j > \sqrt{P}$, then we get inmediately that $n < \sqrt{P} - 1$; otherwise, the sum of all the terms would be greater than $\sqrt{P}$ and P would not be a perfect number. Thus, we can affirm that

$$n \leq \sqrt{P} - 2 \tag{5}$$

Also, we can state that $d_n \geq 2n + 1$, as the minimum gap between consecutive elements of S is 2, and the minimum possible value of d_1 is 3.

Additionally, the maximum sum of elements of S with the minimum gap between them is

$$d_n + (d_n - 2) + (d_n - 4) + ... + (d_n - 2\,(n-1))$$

Therefore, we can establish that

$$1 + \sum_{j=1}^{n} d_j \leq d_n + (d_n - 2) + (d_n - 4) + ... + (d_n - 2\,(n-1))$$

$$1 + \sum_{j=1}^{n} d_j \leq n d_n - \left((n-1)^2 + (n-1)\right)$$

Subsequently, we get that

$$\frac{1+\sum_{j=1}^{n} d_j}{d_n} \leq \frac{nd_n - \left((n-1)^2 + (n-1)\right)}{d_n}$$

$$\frac{1+\sum_{j=1}^{n} d_j}{d_n} \leq n - \frac{\left((n-1)^2 + (n-1)\right)}{d_n}$$

Substituting n by the inequality obtained in (5), we get that

$$\frac{1+\sum_{j=1}^{n} d_j}{d_n} \leq \sqrt{P} - 2 - \frac{\left(\left(\sqrt{P}-2-1\right)^2 + \left(\sqrt{P}-2-1\right)\right)}{d_n}$$

$$\frac{1+\sum_{j=1}^{n} d_j}{d_n} \leq \sqrt{P} - 2 - \frac{\left(\left(\sqrt{P}-3\right)^2 + \left(\sqrt{P}-3\right)\right)}{d_n} \tag{6}$$

Operating with the numerator of the third term of the right handside of (6), we get that

$$\left(\sqrt{P}-3\right)^2 + \left(\sqrt{P}-3\right) = P + 9 - 6\sqrt{P} + \sqrt{P} - 3 =$$

$$= P - 5\sqrt{P} + 6$$

As by definition $d_n < \sqrt{P}$, we can affirm that

$$\frac{P - 5\sqrt{P} + 6}{d_n} \geq \sqrt{P} - 5 + \frac{6}{\sqrt{P}}$$

Therefore, substituting at (6), we get that

$$\frac{1+\sum_{j=1}^{n} d_j}{d_n} \leq \sqrt{P} - 2 - \left(\sqrt{P} - 5 + \frac{6}{\sqrt{P}}\right)$$

$$\frac{1+\sum_{j=1}^{n} d_j}{d_n} \leq 3 - \frac{6}{\sqrt{P}}$$

Subsequently,

$$\frac{1+\sum_{j=1}^{n} d_j}{d_n} < 3$$

$$1 + \sum_{j=1}^{n} d_j < 3d_n$$

Finally, substituting in (4), we get the desired result

$$\frac{d_n}{d_k} < 3$$

Operating with (2), we get that, if P is some odd perfect number, then

$$1 - \frac{1}{d_1} - \frac{1}{d_2} - ... - \frac{1}{d_n} = \frac{1}{d_k}$$

$$\frac{2}{d_k} + \frac{1}{d_1} + \frac{1}{d_2} + ... = 1 \tag{7}$$

Now, we are going to prove that the Egyptian fraction of (7) with the constraint of $\frac{d_n}{d_k} < 3$ set by Lemma 5 can not exist, and subsequently that the existence of odd perfect numbers is not possible.

Let us define set $S = \{1, 2, ..., n\}$. Operating with (7), we get that

$$\frac{\sum_{\substack{j=1 \\ j \neq k}}^{n} \left(\prod_{\substack{s \in S \\ s \neq j}} d_s \right) + 2 \prod_{\substack{s \in S \\ s \neq k}} d_s}{\prod_{s \in S} d_s} = \frac{\prod_{s \in S} d_s}{\prod_{s \in S} d_s}$$

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \left(\prod_{\substack{s \in S \\ s \neq j}} d_s \right) + 2 \prod_{\substack{s \in S \\ s \neq k}} d_s = \prod_{s \in S} d_s \tag{8}$$

It is easy to see that this implies the following:

Lemma 6. For each $d_{j \in S}$, we have that

$$d_j \mid \prod_{\substack{s \in S \\ s \neq j}} d_s$$

This property, considered jointly with Lemma 5, has the following direct implication:

Lemma 7. *d_k is some composite number.*

Proof. If d_k were some prime number, as by Lemma 6 $d_k \mid \prod_{\substack{s \in S \\ s \neq k}} d_s$ and all the proper divisors of P are distinct, then some $d_{s \neq k}$ must be some odd composite number multiple of d_k. As the minimum possible multiple of d_k distinct of d_k is $3d_k$, then we have that some $d_{s \neq k} \geq 3d_k$. However, as we have from Lemma 5 that $d_n < 3d_k$, there can not exist any $d_{s \neq k} \geq 3d_k$. Subsequently, d_k must be composite.

Lemma 6 and Lemma 7 considered jointly imply that $d_k = d_i d_j$, where d_i and d_j can be either prime numbers or composite numbers, but in any case such that $3d_i \leq d_k$ and $3d_j \leq d_k$.

Now, we are in a position to prove the final

Lemma 8. *If $d_k \leq d_n < 3d_k$, it can not exist any solution to the egyptian fraction $\frac{2}{d_k} + \frac{1}{d_1} + \frac{1}{d_2} + ... = 1$.*

Proof. For each $d_{j \in S}$, dividing each of the terms of (8) by all $d_{s \neq k,j}$, we get that

$$d_j d_k = 2d_j + d_k + d_j d_k \left(\sum_{\substack{i=1 \\ i \neq k,j}}^{n} \frac{1}{d_i} \right) \tag{9}$$

Operating with (9), we get that

$$d_j \left(d_k - 2 \right) = d_k \left(1 + d_j \left(\sum_{\substack{i=1 \\ i \neq k,j}}^{n} \frac{1}{d_i} \right) \right)$$

$$d_j = \frac{d_k \left(1 + d_j \left(\sum_{\substack{i=1 \\ i \neq k,j}}^{n} \frac{1}{d_i} \right) \right)}{d_k - 2}$$

As d_k and $d_k - 2$ are odd integers, it follows that $\gcd(d_k, d_k - 2) = 1$. As each d_j is some odd positive integer, then necessarily for each d_j we have one of the following two options: either $d_k = 3$, or $d_k - 2$ divides $1 + d_j \left(\sum_{\substack{i=1 \\ i \neq k,j}}^{n} \frac{1}{d_i} \right)$. For each d_j we can discard the first option, as 3 is a prime number and by Lemma 7 d_k is some composite number. Looking at the second option left, it can be noticed that $\sum_{\substack{i=1 \\ i \neq k,j}}^{n} \frac{1}{d_i} < 1$, because precisely the original egyptian fraction set in (7) states that $\left(\sum_{\substack{i=1 \\ i \neq k,j}}^{n} \frac{1}{d_i} \right) + \frac{1}{d_j} + \frac{2}{d_k} = 1$. But this implies that, if $d_k - 2$ divides $1 + d_j \left(\sum_{\substack{i=1 \\ i \neq k,j}}^{n} \frac{1}{d_i} \right)$, then necessarily

$$d_k - 2 < 1 + d_j$$

$$d_k - 3 < d_j$$

As $d_k - 3$ is an even number, then

$$d_k - 2 \leq d_j \tag{10}$$

As this must be true for each d_j, we reach a contradiction between the bound set in (10) and Lemma 7; according to the bound, d_k cannot be some composite number of two other divisors d_i and d_j and must be some prime number. Subsequently, it follows that the egyptian fraction in (7) with the inequality of Lemma 5 can not exist.

As both the egyptian fraction in (7) and the inequality of Lemma 5 are necessary for an odd perfect number to exist, then if follows that the existence of odd perfect numbers is not possible.

Q.E.D. ¡D.G.!