The Lorentz Transformations: a Brief Review

Anamitra Palit

Physicist, Freelancer

P154 Motijheel Avenue, Flat C4, Kolkata 700074, India

Email: palit.anamitra@gmail.com

Cell:+919163892336

Abstract

With this article we arrive at a strange conclusion that the equations expressing the energy momentum transformations are not independent. This boils down to the fact that the equations portraying the Lorentz transformations are also not independent. Strange it may seem, this is true.

Introduction

We derive systematically at a strange fact that the equations expressing the relativistic energy momentum transformations are not independent. This boils down to the fact that that the equations portraying the Lorentz transformations are also not independent.

Calculations

We start with the transformation of the energy momentum four vector.[1][2] In the following calculations the reference frames translating in the x-x' direction. The particle being investigated is moving in arbitrary direction

$$p'_x = \gamma\left(p_x - \frac{v}{c^2}E\right) \quad (1.1)$$

$$E' = \gamma(E - vp_x) \ (1.2)$$

$$p'_y = p_y \quad (1.3)$$

$$p'_z = p_z \ (1.4)$$

[Details of formulas (1.1)..(1.4) available with Appendix]

A Short Summary of Relativistic Momentum[3][4]

$$p_x' = mv_x' = m_0\gamma_p{}'v_x'; p_y' = mv_y' = m_0\gamma_p{}'v_y'; p_z' = mv_z' = m_0\gamma_p'v_z'; E' = m_0\,\gamma_p{}'c^2 (2.1)$$

$$p_x = mv_x = m_0\gamma_p v_x; p_y = mv_y = m_0\gamma_p v_y; p_z = mv_z = m_0\gamma_p v_z; E = m_0\,\gamma_p c^2\ (2.2)$$

$$(v_x, v_y, v_z) \equiv \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right); (v_x', v_y', v_z') \equiv \left(\frac{dx'}{dt}, \frac{dy'}{dt}, \frac{dz'}{dt}\right)$$

$$\gamma_p = \frac{1}{\sqrt{1-\frac{{v_p}^2}{c^2}}}; \gamma_{p'} = \frac{1}{\sqrt{1-\frac{{v_p'}^2}{c^2}}}; v_p = \sqrt{{v_x}^2+{v_y}^2+{v_z}^2}, v_p' = \sqrt{{v_x'}^2+{v_y'}^2+{v_z'}^2}$$

$$m_0\gamma_p{}'v_x' = \gamma\left(m_0\gamma_p v_x - \frac{v}{c^2}m_0\,\gamma_p c^2\right)$$

$$m_0\,\gamma_p{}'c^2 = \gamma\left(m_0\,\gamma_p c^2 - v m_0\gamma_p v_x\right)$$

$$m_0\gamma_p{}'v_y' = m_0\gamma_p v_y$$

$$m_0\gamma_p'v_z' = m_0\gamma_p v_z$$

$$\gamma_p{}'v_x' = \gamma\gamma_p(v_x - v)\ (3.1)$$

$$\gamma_p' = \gamma\gamma_p\left(1-\frac{vv_x}{c^2}\right)(3.2)$$

$$\gamma_p{}'v_y' = \gamma_p v_y\ (3.3)$$

$$\gamma_p{}'v_z' = \gamma_p v_z\ (3.4)$$

Dividing (3.1) by (3.2) we obtain the standard velocity transformation formula[5] for the x-x' direction

$$v_x' = \frac{v_x - v}{1-\frac{vv_x}{c^2}}\ (4)$$

These equation hang together

$$\frac{\gamma_p{}'}{\gamma_p} = \frac{\gamma(v_x - v)}{v_x{}'} = \frac{v_y}{v_y{}'} = \frac{v_z}{v_z{}'}\ (5.1)$$

$$\frac{{\gamma_p}'^2}{{\gamma_p}^2} = \frac{\gamma^2(v_x - v)^2}{{v_x}'^2} = \frac{{v_y}^2}{{v_y}'^2} = \frac{{v_z}^2}{{v_z}'^2}\ (5.2)$$

Again applying (3.2) and (3.4) on $\frac{\gamma_p{}^2}{\gamma_p{}'^2} = \frac{1-\frac{v_x'^2+v_y'^2+v_z{}^{2\prime}}{c^2}}{1-\frac{v_x{}^2+v_y{}^2+v_z{}^2}{c^2}}$

we obtain,

$$\frac{\gamma_p{}^2}{\gamma_p{}'^2} = \frac{1-\frac{v_x'^2+v_y'^2+v_z{}^{2\prime}}{c^2}}{1-\frac{v_x{}^2+v_y{}^2+v_z{}^2}{c^2}} = \frac{v_y{}'^2}{v_y{}^2} = \frac{v_z{}'^2}{v_z{}^2}$$

$$\frac{\gamma_p{}^2}{\gamma_p{}'^2} = \frac{c^2-v_x'^2-v_y'^2-v_z{}^{2\prime}}{c^2-v_x{}^2-v_y{}^2-v_z{}^2} = \frac{v_y{}'^2}{v_y{}^2} = \frac{v_z{}'^2}{v_z{}^2} = \frac{c^2-v_x'^2}{c^2-v_x{}^2}$$

$$\frac{\gamma_p{}^2}{\gamma_p{}'^2} = \frac{c^2-v_x'^2}{c^2-v_x{}^2} \ (5.3)$$

Equation (4) can be deduced independently from each (3.1) and (3.2). We can move in the reverse direction too[just by retracing the steps]. Therefore (3.1) and (3.2) are equivalent.This fact has been proved in the following:

We have from (3.1)

$$\gamma_p{}'v_x' = \gamma\gamma_p(v_x - v)$$

$$v_x' = \gamma\frac{\gamma_p}{\gamma_p'}(v_x - v)$$

We have proved

$$\frac{\gamma_p}{\gamma_p'} = \sqrt{\frac{c^2-v_x{}^2}{c^2-v_x'^2}}$$

irrespective of velocity components in the y and the z directions so long as the reference frames translate in the x-x’ direction.

Therefore

$$v_x' = \gamma\sqrt{\frac{c^2-v_x'^2}{c^2-v_x{}^2}}(v_x - v)$$

$$v_x'^2 = \gamma^2\left(\frac{c^2-v_x'^2}{c^2-v_x{}^2}\right)(v_x - v)^2$$

$$v_x'^2(c^2 - v_x^2)\left(1 - \frac{v^2}{c^2}\right) = (c^2 - v_x'^2)(v_x - v)^2$$

$$v_x'^2\left(c^2 - v^2 - v_x^2 + \frac{v_x^2 v^2}{c^2}\right) = (c^2 - v_x'^2)(v_x^2 + v^2 - 2vv_x)$$

$$v_x'^2\left(c^2 - v^2 - v_x^2 + \frac{v_x^2 v^2}{c^2}\right) = c^2(v_x - v)^2 - v_x'^2(v_x - v)^2$$

$$v_x'^2\left(c^2 - v^2 - v_x^2 + \frac{v_x^2 v^2}{c^2} + (v_x - v)^2\right) = c^2(v_x - v)^2$$

$$v_x'^2\left(c^2 - v^2 - v_x^2 + \frac{v_x^2 v^2}{c^2} + v_x^2 + v^2 + 2vv_x\right) = c^2(v_x - v)^2$$

$$v_x'^2\left(c^2 + \frac{v_x^2 v^2}{c^2} - 2vv_x\right) = c^2(v_x - v)^2$$

$$v_x'^2\left(1 + \frac{v_x^2 v^2}{c^4} - 2\frac{vv_x}{c^2}\right) = (v_x - v)^2$$

$$v_x'^2\left(1 - \frac{vv_x}{c^2}\right)^2 = (v_x - v)^2$$

$$v_x'^2 = \left(\frac{v_x - v}{1 - \frac{vv_x}{c^2}}\right)^2 \quad (6.1)$$

$$v_x' = \pm\frac{v_x - v}{1 - \frac{vv_x}{c^2}} \quad (6.2)$$

We may reverse our steps to the initial formula

$$[\gamma_p' v_x' = \gamma\gamma_p(v_x - v)] \Leftrightarrow \left[v_x'^2 = \left(\frac{v_x - v}{1 - \frac{vv_x}{c^2}}\right)^2\right] \quad (7)$$

Recalling (3.2)

$$\gamma_p' = \gamma\gamma_p\left(1 - \frac{vv_x}{c^2}\right)$$

$$\frac{\gamma_p'}{\gamma_p} = \gamma\left(1 - \frac{vv_x}{c^2}\right)$$

$$\left(\frac{\gamma_p'}{\gamma_p}\right)^2 = \gamma^2\left(1 - \frac{vv_x}{c^2}\right)^2$$

$$\frac{c^2 - {v_x}^2}{c^2 - {v_x'}^2} = \frac{1}{1 - \frac{v^2}{c^2}}\left(1 - \frac{vv_x}{c^2}\right)^2$$

$$c^2 - {v_x'}^2 = \left(1 - \frac{v^2}{c^2}\right)(c^2 - {v_x}^2)\frac{1}{\left(1 - \frac{vv_x}{c^2}\right)^2}$$

$${v_x'}^2 = c^2 - \left(1 - \frac{v^2}{c^2}\right)(c^2 - {v_x}^2)\frac{1}{\left(1 - \frac{vv_x}{c^2}\right)^2}$$

$$= \frac{c^2\left(1 - \frac{vv_x}{c^2}\right)^2 - \left(1 - \frac{v^2}{c^2}\right)(c^2 - {v_x}^2)}{\left(1 - \frac{vv_x}{c^2}\right)^2}$$

$$= \frac{c^2 + \frac{v^2{v_x}^2}{c^2} - 2vv_x - \left(c^2 - {v_x}^2 + \frac{v^2{v_x}^2}{c^2} - v^2\right)}{\left(1 - \frac{vv_x}{c^2}\right)^2}$$

$$= \frac{{v_x}^2 + v^2 - 2vv_x}{\left(1 - \frac{vv_x}{c^2}\right)^2} = \frac{(v_x - v)^2}{\left(1 - \frac{vv_x}{c^2}\right)^2}$$

$${v_x'}^2 = \frac{(v_x - v)^2}{\left(1 - \frac{vv_x}{c^2}\right)^2} \quad (8)$$

We can move in the reverse direction also to reach the starting formula.

Therefore

$$\left[\gamma_p' = \gamma\gamma_p\left(1 - \frac{vv_x}{c^2}\right)\right] \Leftrightarrow \left[{v_x'}^2 = \frac{(v_x - v)^2}{\left(1 - \frac{vv_x}{c^2}\right)^2}\right] \quad (9)$$

Thus from (7) and (9) we conclude

$$\left[{\gamma_p}' v_x' = \gamma\gamma_p(v_x - v)\right] \Leftrightarrow \left[\gamma_p' = \gamma\gamma_p\left(1 - \frac{vv_x}{c^2}\right)\right] \quad (10.1)$$

Therefore

$$\left[p_x' = \gamma\left(p_x - \frac{v}{c^2}E\right)\right] \Leftrightarrow [E' = \gamma(E - vp_x)] \;\; (10.2)$$

Therefore the Lorentz transformations are also not independent. Starting from the Lorentz transformations we may arrive at the energy momentum transformation laws;we can also move in the reverse direction[see Appendix].

Alternatively without assuming /considering equation(4) we may consider equations (3.1) and (3.2): they have to be identical. Else we have specific values for each v_x andv_x' .

[If equations (3.1) and (3.2) were independent then it would not be possibly for $v_x$and v_x' *to take on an infinitely many number of values* as understood from the physical context. Therefore they should not be independent. If they are identical then it would be possible for each v_x andv_x'to take on infinitely possible values as demanded by physical requirements.]

The following two criteria cannot go together

1. The velocity component v_x(and v_x' for that matter)has an infinitude of values.
2. Equations (3.1) and (3.2) are independent.

Point (1) is valid from the physical[observation] point of view. That negates point two. We had seen earlier by brute force calculations that equations (3.1) and (3.2) are identical

Conclusions

As asserted at the outset the equations expressing the relativistic energy momentum transformations and consequently the Lorentz transformations are not independent

Appendix

1. From the Lorentz Transformations[taking differentials on each side for a given relative velocity of translation I the x-x' diction],we obtain

$$dx' = \gamma(dx - vdt) \; (11.1)$$

$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right) \; (11.2)$$

$$dy' = dy \; (11.3)$$

$$dz' = dz(11.4)$$

or,

$$\frac{dx'}{d\tau} = \gamma\left(\frac{dx}{d\tau} - v\frac{dt}{d\tau}\right)$$

$$\frac{dt'}{d\tau} = \gamma\left(\frac{dt}{d\tau} - \frac{v}{c^2}\frac{dx}{d\tau}\right)$$

$$\frac{dy'}{d\tau} = \frac{dy}{d\tau}$$

$$\frac{dz'}{d\tau} = \frac{dy}{d\tau}$$

$d\tau$:time interval measured in the particle's rest frame. It is independent of the choice of reference frame.

$$\frac{dx'}{dt'}\frac{dt'}{d\tau} = \gamma\left(\frac{dx}{dt}\frac{dt}{d\tau} - v\frac{dt}{d\tau}\right)$$

$$\frac{dt'}{d\tau} = \gamma\left(\frac{dt}{d\tau} - \frac{v}{c^2}\frac{dx}{dt}\frac{dt}{d\tau}\right)$$

$$\frac{dy'}{dt'}\frac{dt'}{d\tau} = \frac{dy}{dt}\frac{dt}{d\tau}$$

$$\frac{dz'}{dt'}\frac{dt'}{d\tau} = \frac{dy}{dt}\frac{dt}{d\tau}$$

[dt and $dt'$are the time intervals in the unprimed and the primed frames respectively.One should also take not of the fact that for any given world line there is a one to one correspondence between the points of the world line and the points on the time axis that lets us consider relationships like $\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau}, \ldots.; \frac{dx'}{d\tau} = \frac{dx'}{dt\prime}\frac{dt\prime}{d\tau}\frac{dy'}{d\tau} = \frac{dy'}{dt\prime}\frac{dt\prime}{d\tau}$ … ..]With our calculations we do have a single worldline n consideration.

$$\frac{dx'}{dt'}\gamma_p{}' = \gamma\left(\frac{dx}{dt}\gamma_p - v\frac{dt}{d\tau}\right)$$

$$\gamma_p{}' = \gamma\left(\gamma_p - \frac{v}{c^2}\frac{dx}{dt}\gamma_p\right)$$

$$\frac{dy'}{dt}\gamma_p{}' = \frac{dy}{dt}\gamma_p$$

$$\frac{dz'}{dt}\gamma_p{}' = \frac{dy}{dt}\gamma_p$$

or,

$$v_x'\gamma_p' = \gamma\left(v_x\gamma_p - \frac{v}{c^2}\gamma_p c^2\right) (12.1)$$

$$\gamma_p' = \gamma\left(\gamma_p - \frac{vv_x}{c^2}\gamma_p\right)(12.2)$$

$$v_y'^{\gamma_p'} = v_y\gamma_p \ (12.3)$$

$$v_z'\gamma_p' = \gamma_p v_z(12.4)$$

$$m_0 v_x'\gamma_p' = \gamma\left(m_0 v_x\gamma_p - \frac{vm_0}{c^2}\gamma_p c^2\right)(13.1)$$

$$m_0\gamma_p' = \gamma\left(m_0\gamma_p - \frac{vv_x}{c^2}m_0\gamma_p\right) \equiv c^2 m_0\gamma_p' = \gamma\left(m_0\gamma_p c^2 - vv_x m_0\gamma_p\right)(13.2)$$

$$m_0 v_y'^{\gamma_p'} = v_y m_0\gamma_p(13.3) m_0 v_z'\gamma_p' = m_0 v_z\gamma_p(13.3)$$

Noting

$$p_x' = mv_x' = m_0\gamma_p' v_x'; p_y' = mv_y' = m_0\gamma_p' v_y'; p_z' = mv_z' = m_0\gamma_p' v_z'; E' = m_0\ \gamma_p' c^2$$

$$p_x = mv_x = m_0\gamma_p v_x; p_y = mv_y = m_0\gamma_p v_y; p_z = mv_z = m_0\gamma_p v_z; E = m_0\ \gamma_p c^2$$

we arrive at (1.1) thorough (1.4)

References

1. Hyper Physics: http://hyperphysics.phy-astr.gsu.edu/hbase/Relativ/vec4.html#:~:text=In%20the%20literature%20of%20relativity,invariant%20under%20a%20coordinate%20transformation.&text=The%20length%20of%20a%204,same%20in%20every%20inertial%20frame. Accessed on 8th January 2021
2. https://hepweb.ucsd.edu/ph110b/110b_notes/node54.html, Accessed on 8th January 2021
3. Griffiths D. J., Introduction to Electrodynamics,Pearson, Pearson Education India Ltd India8th impression,Section 2.2Relativistic Energy and Momentum,p 512-513
4. Resnick R.,Introduction to Special Relativity,Wiley India,Chapter 3: Relativistic Dynamics,p 118
5. Resnick R.,Introduction to Special Relativity,Wiley India,Chapter2: Relativiastic Kinematics,p 83