On Minkowski Diagrams and Newtonian Gravitation

Aryamoy Mitra

Abstract

This paper is bifurcated into two elements: a) synthesizing a differential understanding of classical mechanics, and in particular *Gravity*, and b) a collection of thoughts on the underpinnings of Minkowski diagrams, light cones and space-time intervals. Neither bears remarkably significant consequences, but is nonetheless not entirely trivial.

Gravitational observations (that are constrained to classical domains) can oftentimes be demonstrated to engender an acceleration, in the form of differential equations and second positional derivatives. This comment reiterates that idea with (approximate) prototypical examples, in celestial contexts.

Secondly, Minkowski diagrams can facilitate geometric interpretations of space-time - a characteristic brought to the fore by their mathematically amenable nature.
For instance, by revolving a linear *worldline* around its *ct* axis, one can construct its corresponding light cone.

Space-time intervals, traditionally characterized by 3 dimensions, can also be reconstituted in the form of generalized, *n-dimensional* spatial coordinates.

1. Newtonian Gravitation

Envisage, for argument, the classical approximations:

$$F = MA \text{ and } F = \frac{GM_1M_2}{r^2}$$

These arguments sustain themselves (for all practical intents) amidst non-accelerating and low-velocity (in comparison to *c*) reference frames. In truth, the second formulation is equivalent to the first – and is solely repurposed to be consistent with the gravitational parameter *GM.*

Additionally, if one were to disintegrate an observed gravitational acceleration into its positional consequences, they'd invoke the differential form:

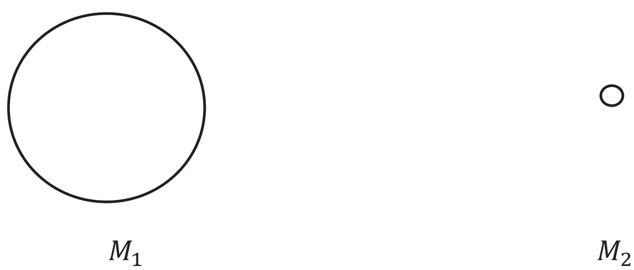
$$F = M\frac{d^2s}{dt^2}$$

wherein *s*, in a given context, characterizes a *positional* function of a massive body.

In creating the equivalence:

$$F = M\frac{d^2s}{dt^2} = \frac{GM_1M_2}{r^2}$$

Imagine (maintaining the above notion) an asteroid, of negligible mass M_2, approaching (directly) a larger celestial body (a planet). For calculating objectives, presuppose that no other noticeable forces operate on either of them.



$$M = M_2$$

$$\frac{F}{M} = \frac{d^2s}{dt^2} = \frac{GM_1}{r^2}$$

Furthermore, identify that prior to immersing in M_1's gravitational field, M_2 was entirely stationary.

Consequently, if *s* is the positional function (with respect to time) that describes the distance that M_2 has travelled (towards M_1);

$$s(0) = 1$$

$$s'(0) = 1$$

While M_1 experiences an interchangeable gravitational force, its consequent acceleration, and therefore the distance it travels towards M_2, may be discounted.

With these assumptions, one observes that:

$$\frac{d^2s}{dt^2} = \frac{GM_1}{r^2}$$

r^2, albeit a seemingly independent parameter, can be reformulated in terms of $s(t)$.

Instantaneously, r plainly refers to the distance that separates two masses at any point in time.

If one assigns an initial separation of $I\ at\ t = 0$, then:

$$r = I - \ s(t)$$

since $s(t)$ does not alter direction.

Consequently,

$$\frac{d^2s}{dt^2} = \frac{GM_1}{I - \ s(t)^2}$$

Abbreviating variable notation from $s(t)$ to s:

$$\frac{d^2s}{dt^2} = \frac{GM_1}{I^2 + s^2 - 2Is}$$

$$GM = \frac{d^2s}{dt^2}(I^2 + s^2 - 2Is)$$

$$GM = \frac{d^2s}{dt^2}I^2 + \frac{d^2s}{dt^2}s^2 - \frac{d^2s}{dt^2}2Is$$

Abbreviating variable notation from $\frac{d^2s}{dt^2}$ to s'':

$$GM = s''I^2 + s''s^2 - 2Is''s$$

which constitutes a non-linear, second order ordinary differential equation, whose solution (if existent) is predicated on the *gravitational parameter (GM)* of a massive body.

In any event, its veracity is contingent to whether a given circumstance is favorable to its described motion. In almost every instance, this necessitates the existence of a negligible (or point) mass, in conjunction with few (if any) exogenous forces – both of which are commensurate with celestial and astrophysical environments. Nevertheless, it ceases to trace the *orbit* of a given mass, if one emerges, with regards to another.

One can elicit this equation in innumerable contexts - one of which may comprise an asteroid steadily approaching Earth, from 100 kilometers afar:

$$GM_{Earth} = s''I^2 + s''s^2 - 2Is''s$$

$$3.986\ x\ 10^{14} = s''{10^5}^2 + s''s^2 - 2Is''s$$

$$10^{10}\ s'' + s''s^2 - (2x10^5)s''s - 3.986\ x\ 10^{14} = 0$$

wherein s delineates the distance traversed by the asteroid, with regards to a progression in time.

By referring to the constancy associated with:

$$GM = s''I^2 + s''s^2 - 2Is''s$$

One may also derive a generic, universal differential equivalency with regards to all gravitationally inspired positional functions;

$$\frac{d}{dt}GM = \frac{d}{dt}\ s''I^2 + s''s^2 - 2Is''s = 0$$

$$\frac{d}{dt}s''I^2 + \frac{d}{dt}s''s^2 - \frac{d}{dt}2Is''s = 0$$

$$s'''I^2 + (s'''s^2 + 2ss's'') - 2I(s'''s + s's'') = 0$$

$$I^2s''' + s^2s''' + 2ss's'' - 2I(ss''' + s's'') = 0$$

wherein I expresses the initial separation between any two celestial entities, and s (and its derivatives) describe iterations of the distance either one has travelled (with respect to time).

This construction isn't necessarily exclusive to celestial, for that matter gravitational interactions.

Any physical framework that describes a mode of acceleration, can be redefined to the domain of a second positional derivative, and, by the transitive association, positional functions. Naturally, these approximations capitulate at high velocities, consequent to time dilation and length contraction.

Position-functions, which evolve with regards to time, are inherently dependent on objective conceptions of length.

Furthermore, the Newtonian conceptualization of a gravitational *force* is inherently flawed; as movements along a gravitational field are instead rationalized by geodesic transformations and Langrangian variants of the Stationary-Action Principle (ex: The Einstein-Hilbert Action).

As far as special relativity is concerned, masses (irrespective of the gravitational potential they are embroiled in) are not immune to relativistic phenomena, either.

Gravitational forces can be redefined (in relativistic paradigms) by virtue of:

$$F = \frac{d}{dt}\vec{p}$$

insofar as $\vec{p}$ entails a relativistic momentum vector, whose derivation instantiates a Lorentz factor.

2. <u>Minkowski Diagrams</u>

Space-time diagrams (that are constrained to two dimensions) are ubiquitous schematics invoked in the description of *worldlines* (migrations across a space-time fabric) and *events.* While they are canonically two-dimensional (one axis assembling space, and another characterizing time), their expressions are routinely thought of as being analogous to observations across three spatial dimensions.

A) Firstly, contemplate any two events, entitled $\emptyset$ and β, that occur in a time-like circumstance.

In light of these events, the space-time interval that they exhibit (in terms of 1, and 3 spatial dimensions respectively) is defined by:

$$ds^2 = dx^2 - (c\Delta t)^2$$

$$ds^2 = -(c\Delta t)^2 + (dx^2 + dy^2 + dz^2)\ with\ the\ metric\ signature\ (-,+,+,+)$$

$$ds^2 = (dx^2 + dy^2 + dz^2) - (c\Delta t)^2$$

If one were to rewrite these spatial differentials in the form of their respective coordinates, with the ordered triples $(\emptyset_x, \emptyset_y, \emptyset_z)$ and $(\beta_x,\ \beta_y,\ \beta_z)$, they'd derive:

$$ds^2 = (\emptyset_x - \beta_x)^2 + (\emptyset_y - \beta_y)^2 + (\emptyset_z - \beta_z)^2 - (c\Delta t)^2$$

$$ds^2 = {\emptyset_x}^2 + {\beta_x}^2 - 2\emptyset_x\beta_x + {\emptyset_y}^2 + {\beta_y}^2 - 2\emptyset_y\beta_y + {\emptyset_z}^2 + {\beta_z}^2 - 2\emptyset_z\beta_z - (c\Delta t)^2$$

$$ds^2 = {\emptyset_x}^2 + {\beta_x}^2 + {\emptyset_y}^2 + {\beta_y}^2 + {\emptyset_z}^2 + {\beta_z}^2 - (2\emptyset_x\beta_x + 2\emptyset_y\beta_y + 2\emptyset_z\beta_z) - (c\Delta t)^2$$

Therefore, constructing summations, replacing *(x, y and z)* with the familiar variants *(i, j and k)*, and encapsulating every spatial term engenders:

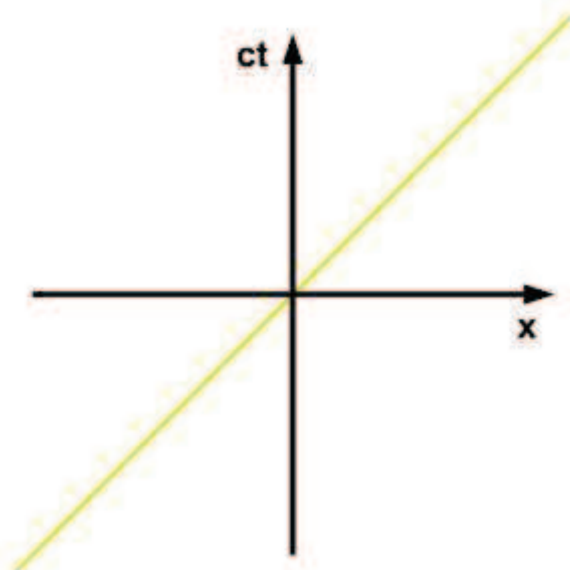
$$ds^2 = \sum_{N=i}^{k} {\emptyset_n}^2 + {\beta_n}^2 - \sum_{N=i}^{k} 2\emptyset_n\beta_n - (c\Delta t)^2$$

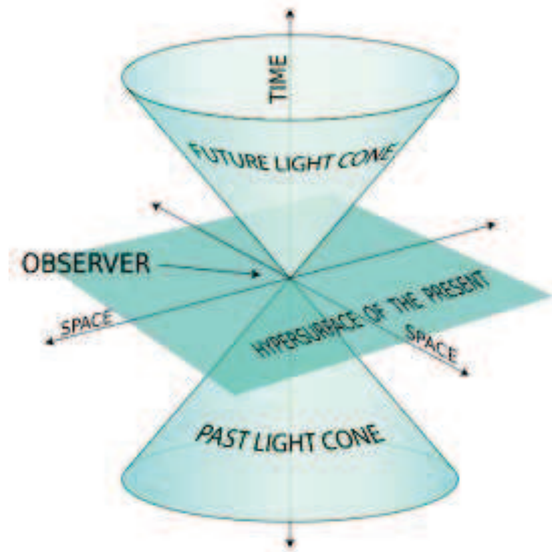
If the invariance of a stated space-time interval remains, by definition, undeterred, and its expansion indifferent to its spatial degree, one can reasonably postulate a generalized, coordinate summation that is n-dimensional in character.

Consequently, with regards to any space-time continuum (that encompasses the entirety of $N = 1$ to $N = k$):

$$ds^2 = \sum_{N=1}^{K} {\emptyset_n}^2 + {\beta_n}^2 - \sum_{N=1}^{K} 2\emptyset_n\beta_n - (c\Delta t)^2$$

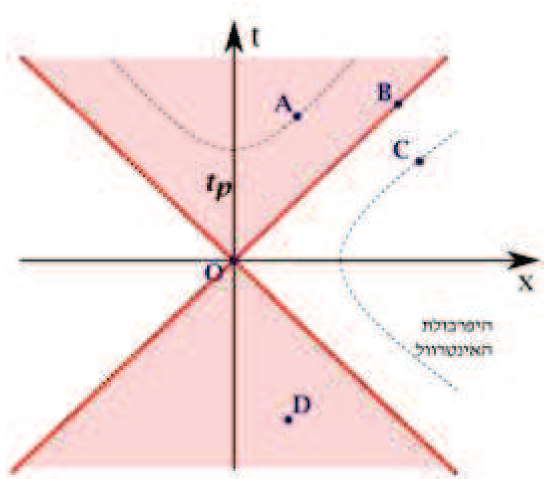
B) Secondly, envisage a space-time diagram, that illustrates the *worldline* of a photon immersed in a one-dimensional space, besides the light-cone of a three-dimensional particle:





Naturally, one can't create a meaningful equivalence between the two, since the two-dimensional plane of a light cone is a hypersurface. Nonetheless, there exists a geometric resonance between them; light-cones, and their volumes, serve as proportional analogies of the space-time fabric that is accessible to an observer, at any point in time. If one were to redefine the second figure's hypersurface as a regular, two-plane space, then integrating *across* a cross-sectional image of a photon's worldline (around its *ct* axis) would independently derive both segments of its light-cone. Mathematically, this constitutes a *solid of revolution.*

By integrating across a Minkowski diagram, one can derive the *light-cone* equivalent of a body encapsulated in a one-dimensional space, with a temporal axis.



To commence, one may define *ct* as a function of *x* (solely with the objective of integrating it).

While one can't easily derive a function that describes two light-cone segments, $ct = |x|$ suffices for the upper segment.

Having distilled this function, one can invoke for a solid of revolution, and geometrically confirm its legitimacy.

Nevertheless, one mustn't conflate the existence of a functional relationship, with the existence of a causal determination; the former is solely geometric in intent.

After replacing *ct* with *y*, and integrating across the entirety of *y;*

$Vol_{revolution} = \pi \int_o^y x^2 dy$

$$y = |x|$$

$$y^2 = x^2$$

$$x = \pm\sqrt{y^2}$$

$$Vol_{revolution} = \pi \int_o^y \pm\sqrt{y^2}^2 dy = \pi \int_o^y y^2 dy = \pi \frac{y^3}{3} \left| \begin{matrix} y \\ o \end{matrix} \right| = \pi \frac{y^3}{3} = \pi \frac{(ct)^3}{3} = \pi \frac{c^3 t^3}{3}$$

One can re-verify by bringing to the fore:

$$Vol_{revolution} = \pi \frac{r^2 h}{3} = \pi \frac{x^2 ct}{3}$$

$$x^2 ct = (ct)^2 ct = c^3 t^3;\ Vol_{revolution} = \pi \frac{c^3 t^3}{3}$$

These conclusions, albeit not path-breaking, do comprise meaningful abstractions in the context of special relativity. By rediscovering the space-time fabrics associated with fictitious, *n-dimensional* surfaces and investigating them, one may still remain adherent to the constraints of light-speed invariance, simultaneity and time dilation.