Sinusoidal and Isochronous Oscillations of Dissipative Lienard type Equations

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Abstract

We present in this work a remarkable dissipative Lienard type equation. We show that its periodic solution can be expressed as a sinusoidal function. As a result this equation can be used to describe harmonic and isochronous oscillations of dynamical systems.

Keywords : Dissipative Lienard type equation, exact periodic solution, sinusoidal and isochronous oscillations.

Introduction

The differential equations play a fundamental role in the modern science. They gained this importance due to mainly their applications in physics and applied mathematics. Indeed, the modeling process in these branches of science leads often to solve differential equations to better understanding the behavior of involved dynamical systems. A special but very important differential equation solution, for example, in physics, is periodic solution since many dynamical processes are periodic. In this way the problem of finding periodic solutions mainly for nonlinear differential equations generated an attractive research field in pure and applied mathematics. Thus a vast literature exists on the theory of existence of periodic solutions for nonlinear differential equations. However, it is not difficult to see that there is a limited number of nonlinear differential equations that have exact and explicit general solutions. Therefore, the nonlinear Lienard type differential equation

$$\ddot{x} + h(x) = 0 \tag{1}$$

has been the object of important studies in the literature, where the overdot stands for a derivative with respect to time, and $h(x)$ is a nonlinear function of x. When $h(x) = \omega_0^2 x,$ the equation (1) is said to be the linear harmonic oscillator in physics. In this context, the equations of type (1) are widely used in physics to model conservative nonlinear systems or oscillators. A famous equation of the form (1) is the so-called cubic Duffing equation which has been used to describe many

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phenomena like the nonlinear resonance and chaos behavior in dynamical systems. The exact solution of the cubic Duffing equation without a forcing term is well-known to be a Jacobian elliptic function [1, 2]. However, it has been shown recently in several papers that such an equation can exhibit non-oscillatory solutions [3,4]. More recently it is also shown that the widely studied pendulum equation can have non-oscillatory behavior [5]. But these equations do not include a dissipative term of energy so that they could not model adequatly real world systems subject to damping forces. In this situation, the nonlinear dissipative Lienard type differential equation [6-9]

$$\ddot{x} + af(x)\dot{x} + h(x) = 0 \tag{2}$$

where a is an arbitrary parameter and $f(x)$ an arbitrary function of $x,$ is the object of intensive studies in the literaure. A celebrated equation of type (2) is the well-known Van der Pol equation which can exhibit limit cycles. Another famous equation of type (2) is the generalized and modified Emden type equation [7-9]

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 + \lambda x = 0 \tag{3}$$

where $f(x) = x, \alpha = a, h(x) = \beta x^3 + \lambda x,$ β and λ are arbitrary constants. The exact solution of the equation (3) where $\lambda = 0,$ has been calculated by several authors from different techniques [10,11]. In this case the obtained solutions are non-oscillatory as the dissipative term imposes its law. However, the authors in [7] succeeded to calculate an exact periodic solution to the equation (3) with the presence of the term $\lambda x.$ This has been a remarkable result that has motivated the publication of several papers on this equation. In [8] for example, the authors have shown that the equation (3) belongs to a more generalized equation exhibiting harmonic and isochronous periodic behavior. But, in a recent paper [9] Doutètien and coworkers investigated this equation. They found [9] that the so-called Lienard type nonlinear oscillator (3) with unusual properties as said by Chrandrasekar et al. [7] is not one, but a pseudo-oscillator as this equation (3) can exhibit unbounded periodic solution. As such, the identification of equations of type (2) able to exhibit non-isolated periodic solutions has become a vital research problem for the physics and mathematics. This motivates the work presented in [12] by Monsia and coworkers. In [12] the authors presented successfully a nonlinear oscillator of Lienard type (2) which can exhibit isochronous periodic oscillations. More recently, the same group, Monsia and coworkers [13] carried out a work [13] in which they show the existence of a Lienard type nonlinear oscillator with harmonic and isochronous oscillations. More interesting this Lienard type nonlinear oscillator and the linear harmonic oscillator have identical solutions expressed as a sine function with amplitude independent frequency [13].

In the present work the question is to ask whether there are algebraic functions $f(x)$ such that the solution of the dissipative Lienard type equation (8) can be expressed as a sinusoidal function of time. In this perspective the objective in this paper is to show the existence of such equations. To do so, we review briefly the first integral method [14-16] introduced recently in the literature by Monsia and his group (section 2) and present the Lienard type nonlinear oscillator and its solution (section 3). Finally we give a conclusion for the work.

2. The review of the theory

According to [14-16], using the first integral

$$b = g(x)\,\dot{x} + a\,f(x)x^{\ell} \tag{4}$$

one may secure by differentiation the corresponding second-order dissipative Lienard type equation

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a\,\ell x^{\ell-1}\frac{f(x)}{g(x)}\,\dot{x} + ab\,x^{\ell}\,\frac{f'(x)}{g^2(x)} - a^2\,x^{2\ell}\,\frac{f'(x)f(x)}{g^2(x)} = 0 \tag{5}$$

where b and ℓ are arbitrary constants and $g(x)$ an arbitrary function of x. The prime means differentiation with respect to the argument. When $g(x)=1$, the equation (5) becomes

$$\ddot{x} + a\,\ell\,f(x)x^{\ell-1}\dot{x} - a^2x^{2\ell}f'(x)f(x) + ab\,x^{\ell}f'(x) = 0 \tag{6}$$

The general solution of the equation (6) is given by the quadrature

$$t + K = \int \frac{dx}{b - a\,f(x)x^{\ell}} \tag{7}$$

Applying $\ell = 1$, and $b = 0$, the equations (6) and (7) become respectively

$$\ddot{x} + a\,f(x)\,\dot{x} - a^2x^2f'(x)f(x) = 0 \tag{8}$$

and

$$-a(t+K) = \int \frac{dx}{f(x)x} \tag{9}$$

As can be seen, the equation (8) is the nonlinear Lienard type equation (2) where $h(x) = -a^2 f'(x)f(x)x^2$. Now we are able to show in the following section the existence of the Lienard type nonlinear oscillator equation of interest by an appropriate choice of $f(x)$.

3- The Lienard type oscillator and its solution

Let us consider $f(x) = \sqrt{c_1 + \frac{c_2}{x}}$. Then the equation (8) transforms into the desired Lienard type equation

$$\ddot{x} + a\sqrt{c_1 + \frac{c_2}{x}}\,\dot{x} + \frac{a^2 c_2}{2} = 0 \tag{10}$$

where c_1 and c_2 are arbitrary constants. In this condition the equation (9) takes the form

$$-a(t+K) = \int \frac{dx}{\sqrt{c_1 x^2 + c_2 x}} \tag{11}$$

which becomes after integration of the right side member [17]

$$\sin^{-1}\left(\frac{2c_1 x + c_2}{c_2}\right) = a\sqrt{-c_1}\,(t+K) \tag{12}$$

from which one can ensure the periodic solution

$$x(t) = \frac{c_2}{2c_1}\left[-1 + \sin\left(a\sqrt{-c_1}\,(t+K)\right)\right] \tag{13}$$

where $c_1 \prec 0$, and K is a constant of integration. The solution (13) is periodic but not isochronous. To make this solution isochronous, it is needed to put $c_1 = -1$, such that the expression (13) becomes isochronous periodic solution and takes the definitive form

$$x(t) = \frac{c_2}{2}\left[1 - \sin\left[a(t+K)\right]\right] \tag{14}$$

Thus the formula (14) is sinusoidal solution but with a shifting factor $\frac{c_2}{2}$.

Conclusion

We study in this paper the dissipative Lienard type nonlinear differential equation. We have successfully shown the existence of a Lienard type nonlinear oscillator which can exhibit sinusoidal and isochronous behavior but with a shifting factor.

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