On the paradoxical Summations of the infinite Series

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There are four paradoxical summations of the infinite series, $1-2+3-4+\cdots=\frac{1}{4}$ as even Leonhard Euler admitted this infinite series, the Ramanujan summation $1+2+3+4+\cdots=-\frac{1}{12}$ which is widely cited, especially in Riemann zeta function as if it were correct, Grandi's series $1-1+1-1+\cdots=\frac{1}{2}$, and $1+1+1+1+\cdots=-\frac{1}{2}$. These infinite series are inconsistent. (Subject Class: 40A30, 65B10)

A. The Paradoxical Infinite Series $1-2+3-4+\cdots=\frac{1}{4}$

The sum of the infinite series of $1-2+3-4+\cdots=\frac{1}{4}$ [1] is not a kind of paradox as known but nonsense. Because the sum of $1-2+3-4+\cdots$ fluctuates between $-\infty$ and $+\infty$.

First of all, the following is a review of the series. For the summation of an infinite series, we can first, find the partial sum of an infinite series from the first term up to the n^{th} term, and then see how the sum changes if n tends towards infinity.

Let the sum of the infinite series be S, it shows

$$\begin{aligned} S_1 &= 1-2+3-4+5+\cdots+(-1)^{n-1}n, \\ 2S_2 &= 0+2-4+6-8+\cdots+2(-1)^{n-2}(n-1), \\ S_3 &= 0+0+1-2+3+\cdots+(-1)^{n-3}(n-2), \\ S_1+2S_2+S_3 &= 1, \end{aligned} \tag{1}$$

where n is greater than 2 so that these equations are valid, and the equation (1) stands for the sum of the above three equations.

It can be seen that the righthand side of the sum is always 1 regardless of n terms. But if we look into the lefthand side, there are S_1, S_2 and S_3. If $S_1=S_2=S_3$, then S_1 can be simply calculated to be $\frac{1}{4}$, i.e., $1-2+3-4+\cdots=\frac{1}{4}$. However, S_1 is not equal to S_2 or S_3. By observing n^{th} term on the righthand side shows that S_2 is the partial sum of the series from 0 to $(n-1)$ less than the last term of S_1, and S_3 is the partial sum from 0 to $(n-2)$ less than two terms of S_1 or less than one term of S_2. So, to recapitulate,

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$$\begin{aligned} S_2 &= S_1 - (-1)^{n-1}n, \\ S_3 &= S_1 - (-1)^{n-2}(n-1) - (-1)^{n-1}n. \\ &= S_2 - (-1)^{n-2}(n-1). \end{aligned} \tag{2}$$

If n goes to infinity, each of S_2 and S_3 diverges to $-\infty$ or $+\infty$. Therefore it is not true that $S_1 = S_2 = S_3$.

In addition, if $S_1 = S_2$, then we get the following result by subtracting S_2 from S_1,

$$\begin{aligned} S_1 - S_2 &= 1 - 3 + 5 - 7 + 9 - \cdots \\ &= \lim_{n\to\infty} (-1)^{n-1}n \\ &= \pm\infty \\ &\neq 0. \end{aligned} \tag{3}$$

This is against the condition mentioned above, because subtraction of infinity series results *constant* or $\pm\infty$.

In fact, $1 - 2 + 3 - 4 + \cdots$ is an infinite series derived from the subtraction of the sum of even number series from the sum of odd number series (See below for more detailed explanation). By the use of sigma summation notation, the infinite series can be written as follows

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} (2k-1) - \sum_{k=1}^{n} 2k \right). \tag{4}$$

Since this series with $2n$ elements consists of the sum of n pieces odd numbers and the sum of n pieces even numbers, it can be expressed as follows.

$$\begin{aligned} & 1 - 2 + 3 - 4 + \cdots + (-1)^{2n-2}(2n-1) + (-1)^{2n-1}(2n) \\ = & \sum_{k=1}^{n} (2k-1) - \sum_{k=1}^{n} 2k \\ = & -n. \end{aligned} \tag{5}$$

As the sum is $-n$, this diverges to $-\infty$.

The next $n+1$ term becomes odd number term, and we can get

$$\sum_{k=1}^{n+1}(2k-1)-\sum_{k=1}^{n}2k=n+1. \tag{6}$$

This diverges to $+\infty$. Accordingly, $1-2+3-4+\cdots\neq\frac{1}{4}$, but tends to $\pm\infty$ (Refer to below Grandi's series for another disproof).

B. Inconsistent Ramanujan Summation ($1+2+3+4+\cdots=-\frac{1}{12}$)

The Ramanujan summation[2] [3] [5] is about subtracting two times of the sum of even numbers from the sum of natural numbers. The product is to be the subtraction of the sum of even numbers from the sum of odd numbers. In other words, subtracting the sum of even numbers from the sum of natural numbers results the sum of odd numbers. And one more time, if we subtract the sum of even numbers from the sum of odd numbers, we should have the result of $\pm\infty$, because both of the sum of odd number and the sum of even number are incontestably divergent. Ramanujan had erroneously made a guess that -3 times of the sum of odd numbers less the sum of even numbers made the sum of natural numbers. The sum of natural number, the sum of even number, and the sum of odd number are different from each other, and Ramanujan miscalculated by quoting inconsistent of infinite series $1-2x+3x^2-4x^3+\cdots=\frac{1}{(1+x)^2}$ at $x=1$.
Subtracting two times of the sum of even numbers from the sum of natural numbers results subtracting the sum of even numbers from the sum of odd numbers. Ramanujan summation was described a shape of subtraction of infinite series as follows

$$\begin{aligned}
C &= 1+2+3+4+5+6+\cdots\,, \\
4C &= \quad\ 4 \quad + \quad 8 \quad + \quad 12 \quad +\cdots\,, \\
-3C &= 1-2+3-4+5-6+\cdots\,.
\end{aligned} \tag{7}$$

Intuitively, it is difficult to approach the above conclusion. It rather approaches negative infinite series as follows,

$$\begin{aligned}
C &= 1+2+3+4+\cdots\,, \\
4C &= 4+8+12+16+\cdots\,, \\
-3C &= -3(1+2+3+4+5+\cdots).
\end{aligned} \tag{8}$$

Ramanujan should have subtracted infinite series by using a different infinite series C_2 as follows,

$$\begin{aligned} C_1 &= 1+2+3+4+5+\cdots+2n, \\ 4C_2 &= 0+4+0+8+0+\cdots+4n, \\ C_1 - 4C_2 &= 1-2+3-4+5+\cdots-2n. \end{aligned} \tag{9}$$

Ramanujan might have inferred heuristically the value of $\frac{1}{4}$ by assuming the above series were the same value of the formal power series $\frac{1}{(1+x)^2}$ with x defined as one(see Grandi's series below). Furthermore, on the lefthand side, it is assumed that C_2 were the same value as C_1. Accordingly the product of C_1 led to a misunderstanding of $1+2+3+4+\cdots = -\frac{1}{12}$.
If we look closely, we find C_2 is the sum of the even numbers that C_1 contains. And of course, if we observe $(2n)^{th}$ term that C_1 runs from 1 to $2n$, while C_2 may only run from 1 to n, and the sum from $(n+1)^{th}$ to $(2n)^{th}$ is excluded, so C_2 cannot be equal to C_1. By using the sigma summation notation, we can calculate the sum as follows,

$$\sum_{k=1}^{2n} k - 2\sum_{k=1}^{n} 2k \tag{10}$$
$$= \sum_{k=1}^{n}(2k-1) - \sum_{k=1}^{n} 2k = -n.$$

If n goes to infinity, the result goes to negative infinity as well.

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}(2k-1) - \sum_{k=1}^{n} 2k\right) = -\infty. \tag{11}$$

Now, if we add the next term $(2n+1)$ to the equation (10), which is an odd term, the equation (10) provides with

$$\sum_{k=1}^{2n+1} k - 2\sum_{k=1}^{n} 2k = \sum_{k=1}^{n+1}(2k-1) - \sum_{k=1}^{n} 2k = n+1. \tag{12}$$

Accordingly, this is divergent to $+\infty$.

Therefore, $1-2+3-4+\cdots \neq \frac{1}{4}$ as shown as the above, as well as $1+2+3+4+\cdots \neq -\frac{1}{12}$. And as we know, the sum of $1+2+3+4+\cdots+n$ is only,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \tag{13}$$

And when n goes to infinity, the sum diverges to infinity, too.

For this reason, Ramanujan summation is inconsistent and to be abandoned.

C. Incorrect Grandi's Series $1-1+1-1+\cdots = \frac{1}{2}$

Grandi's series $1-1+1-1+\cdots$ is also an infinite series[4], as written as follows,

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} (-1)^{n-1}. \tag{14}$$

We may develop this series as seen as the above style of the infinite series $1-2+3-4+\cdots$ by substituting with G,

$$\begin{aligned} G_1 &= 1-1+1-1+1+\cdots+(-1)^{n-1}, \\ G_2 &= 0+1-1+1-1+\cdots+(-1)^{n-2}, \\ G_1+G_2 &= 1. \end{aligned} \tag{15}$$

Here, G_2 has 0 added as the first term of the infinite series G_1, and one less integer than G_1 , which is $(-1)^{n-1}$ of the infinite series. The righthand side is always 1 regardless of n. Therefore if $G_1 = G_2$, then one may easily find that $G_1 = \frac{1}{2}$. This is what we see Grandi's series. But the result shows that they added constant items, because both G_1 and G_2 are divergent[4] to 1 or 0, even though they are infinite series. Taking another close look, we can find if the sum of G_1 is 1, then G_2 is enforcedly to equal to zero and vice versa, and we see that the sum of G_1+G_2 is always 1, but incontestably $G_1 \neq G_2$. It is clear that the sum of Grandi's series has two points at one and zero as similar as the infinite series, $1-2+3-4+\cdots$, has two points at $\pm\infty$.

Furthermore, if $G_1 = G_2$, then we get by subtracting G_2 from G_1 as follows

$$\begin{aligned} G_1 - G_2 &= 1-2+2-2+2-2+\cdots \\ &= 1 \quad or -1 \\ &\neq 0. \end{aligned} \tag{16}$$

This is a violation of precondition. Therefore, $G_1 \neq G_2$.

Another disproof of the infinite series $1-2+3-4+5+\cdots$, the Ramanujan summation that quoted the value $\frac{1}{4}$, and Grandi's series that cited the value $\frac{1}{2}$, is as follows
A power series $1-x+x^2-x^3+x^4+\cdots$ is divergent if $1<x$, and convergent if $0<x<1$. In case $x=1$, we get Grandi's series of the above .
In general, we may have a partial sum of the power series from 1 to $(n-1)$ term as follows,

$$\begin{aligned} S_n &= 1-x+x^2-x^3+x^4+\cdots+(-1)^{n-1}x^{n-1} \\ &= \frac{1+(-1)^{n-1}x^n}{1+x}. \end{aligned} \tag{17}$$

In case $0<x<1$, the power series converges to $\frac{1}{1+x}$ if n tends towards infinity. i.e.,

$$1-x+x^2-x^3+x^4+\cdots=\frac{1}{1+x}, \quad 0<x<1. \tag{18}$$

If one substitutes with $x=1$ enforcedly to this case, they may get $1-1+1-1+\cdots=\frac{1}{2}$, it is inappropriate because range of x is less than one.
Furthermore, by differentiating this power series, it shows

$$1-2x+3x^2-4x^3+\cdots=\frac{1}{(1+x)^2}, \quad 0<x<1. \tag{19}$$

If one puts $x=1$ by force regardless of range lies on $0<x<1$, they may get $1-2+3-4+\cdots=\frac{1}{(1+1)^2}$. This is what Ramanujan incorrectly quoted to derive out $1-2+3-4+\cdots=\frac{1}{4}$, and is another disapproval of the aforementioned three paradoxes.

D. $1+1+1+1+\cdots=-\frac{1}{2}$

The sum of this infinite series can be written as follows,

$$\begin{aligned} T_1 &= 1+1+1+1+1+\cdots \\ 2T_2 &= 0+2+0+2+0+\cdots \\ T_1-2T_2 &= 1-1+1-1+1-\cdots. \end{aligned} \tag{20}$$

If $T_1=T_2$, and $1-1+1-1+1-\cdots=\frac{1}{2}$, one gets $T_1=-\frac{1}{2}$.
However, both are false, because $T_1\neq T_2$, and Grandi's series is false, too. If n^{th} term of the series T_1 and T_2 is even number term, then $T_1=T_2$ becomes 0, but the next odd term becomes 1. Therefore the subtraction of the two is divergent between 1 and 0.

This equation comes from the following sum of infinite series by manipulating the Riemann zeta function[7] $\zeta(s)$ and Dirichlet eta function $\eta(s)$[2][9]

$$\begin{aligned} \zeta(s) &= 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \cdots \\ 2 \times 2^{-s}\zeta(s) &= \quad 2 \times 2^{-s} \quad + 2 \times 4^{-s} \quad + 2 \times 6^{-s} + \cdots \\ (1 - 2^{1-s})\zeta(s) &= 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots = \eta(s). \end{aligned} \tag{21}$$

If $s = 0$, this summation equation becomes similar to the above equation (20). To add or subtract algebraic infinite series, domain of a variant is needed to calculate the infinite series. For the above algebraic sum of infinite series, domain of s is greater than 1, i.e., $s > 1$. In this case, both $\zeta(s)$ and $\eta(s)$ are convergent. If $0 < s \leq 1$, $\zeta(s)$ is divergent while $\eta(s)$ is convergent, and when $s \leq 0$, both are divergent. Subtraction of infinite series which is divergent results $\infty - \infty = constant \quad or \quad \pm\infty$.

[1] https://en.wikipedia.org/wiki/1 − 2 + 3 − 4 + ···
[2] https://en.wikipedia.org/wiki/1 + 2 + 3 + 4 + ···
[3] https://en.wikipedia.org/wiki/Ramanujan summation
[4] https://en.wikipedia.org/wiki/Grandi's series
[5] https://en.wikipedia.org/wiki/Harmonic series (mathematics)
[6] https://en.wikipedia.org/wiki/1 + 1 + 1 + 1 + ···
[7] https://en.wikipedia.org/wiki/Riemann_zeta_function
[8] https://en.wikipedia.org/wiki/Functional_equation
[9] https://en.wikipedia.org/wiki/Dirichlet_eta_function