Affirmative resolve of the Riemann Hypothesis

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Abstract

In this paper, we prove the proposition about the Mobius function equivalent to the Riemann Hypothesis.

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Handles propositions equivalent to the Riemann Hypothesis. I express the Riemann Hypothesis as R.H, and the Mobius function as $\mu(n)$.

Next theorem is well-known
Theorem

$$\sum_{n=1}^{m}\mu(n) = O(m^{\frac{1}{2}+\epsilon}) \Leftrightarrow R.H$$

I will prove Left hand formula.

Lemma1.1

$$\sum_{n|m}\mu(n) = 1(m=1), \sum_{n|m}\mu(n) = 0(m \neq 1)$$

Proof. First, if $m = 1$, it is $\sum_{n|m}\mu(n) = \mu(1) = 1$. Second case. There is a little explanation for this. Let m's prime factorization be $m = p_1^{n_1}p_2^{n_2}p_3^{n_3}\cdots p_k^{n_k}$. Then it becomes $\sum_{n|m}\mu(n) =_k C_0 -_k C_1 +_k C_2 -_k C_3 + \cdots_k C_k = (1-1)^k = 0$. □

Theorem1

$$\sum_{n\leq m}\mu(n)[\frac{m}{n}] = 1$$

Proof. $\sum_{m'=1}^{m}\sum_{n|m'}\mu(n) = 1$ is from Lemma1.1

$$1 = \sum_{m'=1}^{m}\sum_{n|m'}\mu(n) = (\mu(1)) + (\mu(1)+\mu(2)) + (\mu(1)+\mu(3))$$

$$+(\mu(1)+\mu(2)+\mu(4))+\cdots$$

See $\mu(n)$ in this expression as a character. $\mu(1)$ appears m times in the expression. $\mu(2)$ appears $[\frac{m}{2}]$ times that is a multiple of 2 less than m. In general, the number of occurrences of $\mu(n)(n<m)$ in this expression is the number $\left[\frac{m}{n}\right]$ that is a multiple of n below m. I get $\sum_{n\le m}\mu(n)[\frac{m}{n}]=1$. □

example
$m=10$ case, $10-5-3-2+1-1+1=1$.$m=13$ case, $13-6-4-2+2-1+1-1-1=1$ etc..

Theorem2

$$\sum_{n=1}^{x}\frac{m}{n}\mu(n) \textit{ changes sign at } n_0\in[m^{\frac{1}{2}(1-\epsilon')},m^{\frac{1}{2}}](m>\exists m_{\epsilon'})$$

Proof. $\sum_{n=1}^{x}\frac{m}{n}\mu(n)$ changes sign in the interval $[m^{\frac{1}{2}(1-\epsilon')},m^{\frac{1}{2}}],m>\exists m_{\epsilon'}$ ([1]). □

lemma3.1

$$-1<f(n)<1,(n=1,\cdots m)\Rightarrow|\sum_{n=1}^{m}f(n)|<m$$

Proof. negative terms sum is (I call it F_1) satisfy $|F_1|<m$. positive terms sum is (I call it F_2) satisfy $|F_2|<m$. $|F_1+F_2|<m$. In other words, summation of m elements that is absolute value 1 or less is less than m. □

Theorem3

$$|\sum_{n=1}^{m}\mu(n)|<Km^{\frac{1}{2}+\epsilon}$$

and R.H. is true.

Proof. From theorem1

$$\sum_{n\le n_0}\mu(n)[\frac{m}{n}]+\sum_{n_0<n\le m}\mu(n)[\frac{m}{n}]=1$$

($n_0<\sqrt{m}$ is the sign change point of $\sum_{n\le x}\frac{m}{n}\mu(n)$.) By lemma3.1, Using $\sum_{n\le n_0}\mu(n)[\frac{m}{n}]$ and $\sum_{n\le n_0}\mu(n)\frac{m}{n}$. These are $n_0(<\sqrt{m})$ terms, so the difference of size is less than $\sqrt{m}$.

The following is obtained by calculation for $\sum_{n_0<n\le m}\mu(n)[\frac{m}{n}]$. $[\sqrt{m}]$ term

is sum of all terms satisfy $[\frac{m}{n}] = [\sqrt{m}] - 1$, $m/\sqrt{m} = \sqrt{m} \geq [\sqrt{m}]$ and $m/(m/(\sqrt{m}-1)) = \sqrt{m}-1 \geq [\sqrt{m}-1]$, $(m/(m/(\sqrt{m}-1)+1) = m(\sqrt{m}-1)/(m+\sqrt{m}-1) < \sqrt{m}-1,)$ so the range is $\sqrt{m}$ to $m/(\sqrt{m}-1)$. Next term is sum of all terms satisfy $[\frac{m}{n}] = [\sqrt{m}] - 2$, $m/(m/(\sqrt{m}-2)) \geq [\sqrt{m}-2]$. The range is $m/(\sqrt{m}-1)$ to $m/(\sqrt{m}-2)$. The last term satisfy $[\frac{m}{n}] = 1$, that is $\frac{m}{2}$ to m.

$$\sum_{n_0<n\leq m} \mu(n)[\frac{m}{n}] = ([m/(n_0)] - 1) \times \sum_{m/([m/(n_0)])<n\leq m/([m/(n_0)]-1)} \mu(n) + \cdots +$$

$$([\sqrt{m}])\times \sum_{m/(\sqrt{m}+1)<n\leq m/\sqrt{m}} \mu(n)+([\sqrt{m}]-1)\times \sum_{\sqrt{m}<n\leq m/(\sqrt{m}-1)} \mu(n)+([\sqrt{m}]-2)\times$$

$$\sum_{m/(\sqrt{m}-1)<n\leq m/(\sqrt{m}-2)} \mu(n) + \cdots + 1 \times \sum_{m/2<n\leq m} \mu(n)$$

By induction, there are "almost correct" formulas. $|N\times\sum_{m/(N+1)<n\leq m/N} \mu(n)| < \frac{N}{N+1}K(\frac{m}{N})^{\frac{1}{2}(1+\epsilon')}|1 \times \sum_{m/2<n\leq m} \mu(n)| < \frac{1}{2}(K(m-1)^{\frac{1}{2}(1+\epsilon')} + 1)$

example: $m = 10000$ case,

$$-95 = 107+106+105+0-103+0+0+0-99-98-97+0-95+94-93+0-91$$

$$+0 + 0 - 88 - 87 + 86 + 0 + 0 + 84 \times 2 + \cdots$$

$(84 \times 2$ means $\mu(118) = 1, [10000/118] = 84, \mu(119) = 1, [10000/119] = 84)$will transform

$$-95+85-84 = 107+106+105+0-103+0+0+0-99-98-97+0-95+94-93+0$$

$$-91 + 0 + 0 - 88 - 87 + 86 + 0 + 85 + 84 \times 2 - 84 + \cdots$$

$$([\sqrt{m}] - 1) \times \sum_{\sqrt{m}<n\leq m/(\sqrt{m}-1)} \mu(n)$$

From here. (It might be 0.) $|\sum_{\sqrt{m}<n\leq m/(\sqrt{m}-1)} \mu(n)|$ is less than $\frac{1}{[\sqrt{m}]-1}K[\frac{m}{n_0}]$
Later, I calculate real example.
example:$m = 100$ case.

$$-6 = 10 - 9 + 0 + 0 + 6 - 5 \times 2 - 4 - 3 + 2 + 1 \times 4$$

$1\times 4-4+2-3$, $4-1+1-1 = 3$ give the almost value of $\sum_{[100/n_0]<n\leq 100} \mu(n)$. Actualy, $\sum_{9<n\leq 100} \mu(n) = 2$. This gives $|\sum_{[100/n_0]<n\leq 100} \mu(n)| < K[100/n_0] =$

$K \times 10$,
example:$m = 10000$ case.

$$-95 = 107+106+105+0-103+0+0+0-99-98-97+0-95+94-93+0-91$$

$$+0+0-88-87+86+0+0+84\times2+0+0+81\times2+0+0-78+77-76\times2+75$$
$$+74+0-72\times2-71+70\times2+69+68\times2-67-66+0+0+0+62\times2-61+0-59$$
$$-58-57\times2+56\times2-55\times2+54+0-52\times2-51-50\times2+49\times3+48\times2+47+46$$
$$\times4+45\times2-44-43\times3+0-41\times2+40+0-38+0-36\times2-35\times3-34+33\times6$$
$$-32-31+30\times4+29\times3-28\times4-27+0+25\times3+0-23\times6+22-21\times2+0+19$$
$$\times4+18\times7+17-16\times9-15\times9+14\times9+0+0+11\times2+10\times3-9\times15+8\times10+7$$
$$\times12-6\times20+5\times16-4\times6+3\times18-2\times15-1\times25$$

$-1\times25+3\times8-2\times15+3\times10$, $-25+8-15+10=-22$ gives the almost value of $\sum_{94<n\leq 10000}\mu(n)=-22$. This gives $|\sum_{[100/n_0]<n\leq 100}\mu(n)| < K[100/n_0] = K\times 107$.

$$\frac{1}{4}K(\frac{m}{3})^{\frac{1}{2}(1+\epsilon')}\frac{A\times 2+B\times 1}{A+B} < Km^{\frac{1}{2}(1+\epsilon')} - Km^{\frac{1}{4}+\epsilon}$$

$$\frac{1}{2}K((m-1)^{\frac{1}{2}(1+\epsilon')}+1)\frac{A(1-\frac{1}{2})+B(1-\frac{1}{3})}{A+B} < Km^{\frac{1}{2}(1+\epsilon')} - Km^{\frac{1}{4}+\epsilon}$$

These formulas hold. If there is needed 4 or more terms, "almost correct" formula can be taken as flexible. It becomes to 3 terms case.

$$|\sum_{n\leq m}\mu(n)| < Km^{\frac{1}{4}+\epsilon} + K[m^{\frac{1}{2}(1+\epsilon')}] - Km^{\frac{1}{4}+\epsilon}$$

$$|\sum_{n\leq m}\mu(n)| < Km^{\frac{1}{2}+\epsilon}$$

So R.H. is got. □

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References

[1] Oscillatory properties of arithmetical functions. I Kaczorowski and Pintz (Acta Math. Hungar. 48 (1986))173-185