Affirmative resolve of the Riemann Hypothesis

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Abstract

Riemann Hypothesis has been the unsolved conjecture for 170 years. This conjecture is the last one of conjectures without proof in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse"(B. Riemann). The statement is the real part of the non-trivial zero points of the Riemann Zeta function is 1/2. Very famous and difficult this conjecture has not been solved by many mathematicians for many years. In this paper, I try to solve the proposition about the Mobius function equivalent to the Riemann Hypothesis. First, the non-trivial formula for Mobius function is proved in theorem**??**. In theorem**??**, I think this formula into 2 parts. By calculation for the latter part, I get upper bound for the sum of the mobius functions (for meaning of R.H. See theorem**??**).

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Handles propositions equivalent to the Riemann Hypothesis. I express the Riemann Hypothesis as R.H, and the Mobius function as $\mu(n)$.

Next theorem is well-known

Theorem .

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon}) \Leftrightarrow R.H$$

I will prove Left hand formula.

Lemma 1.

$$\sum_{n|m} \mu(n) = 1(m = 1), \sum_{n|m} \mu(n) = 0(m \neq 1)$$

Proof. First, if $m = 1$, it is $\sum_{n|m} \mu(n) = \mu(1) = 1$. Second case. There is a little explanation for this. Let m's prime factorization be $m = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}$. Then it becomes $\sum_{n|m} \mu(n) =_k C_0 -_k C_1 +_k C_2 -_k C_3 + \cdots_k C_k = (1-1)^k = 0$. $\square$

Theorem 1.

$$\sum_{n \leq m} \mu(n)[\frac{m}{n}] = 1$$

Proof. $\sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = 1$ is from Lemma**??**

$$1 = \sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = (\mu(1)) + (\mu(1) + \mu(2)) + (\mu(1) + \mu(3))$$

$$+(\mu(1) + \mu(2) + \mu(4)) + \cdots$$

See $\mu(n)$ in this expression as a character. $\mu(1)$ appears m times in the expression. $\mu(2)$ appears $[\frac{m}{2}]$ times that is a multiple of 2 less than m. In general, the number of occurrences of $\mu(n)(n < m)$ in this expression is the number $\left[\frac{m}{n}\right]$ that is a multiple of n below m. I get $\sum_{n \leq m} \mu(n)[\frac{m}{n}] = 1$. $\square$

example

$m = 10$ case, $10 - 5 - 3 - 2 + 1 - 1 + 1 = 1$.$m = 13$ case, $13 - 6 - 4 - 2 + 2 - 1 + 1 - 1 - 1 = 1$ etc..

Theorem 2.

$$\sum_{n=1}^{x} \mu(n) \textit{ changes sign at } n_0 \in [m^{\frac{1}{2}(1-\epsilon')}, m^{\frac{1}{2}}](m > \exists m_\epsilon$$

$$\sum_{n=1}^{x} \frac{m}{n} \mu(n) \textit{ changes sign at } n' \in [m'^{(1-\epsilon')}, m'](m > m' > \exists m_{\epsilon'})$$

Proof. $\sum_{n=1}^{x} \mu(n)$ changes sign in the interval $[m^{\frac{1}{2}(1-\epsilon')}, m^{\frac{1}{2}}], m > \exists m_{\epsilon'}$ ([**?**]). $\sum_{n=1}^{x} \frac{m}{n} \mu(n)$ changes sign in the interval $[m'^{(1-\epsilon')}, m'], m > m' > \exists m_{\epsilon'}$ ([**?**]). $\square$

Theorem 3.

$$|\sum_{n=1}^{m}\mu(n)| < Km^{\frac{1}{2}+\epsilon}$$

R.H. is got.

Proof. From theorem**??**

$$\sum_{n\le n_0}\mu(n)[\frac{m}{n}] + \sum_{n_0<n\le m}\mu(n)[\frac{m}{n}] = 1$$

By theorem**??** $m^{\frac{1}{2}(1-\epsilon')} < n_0 < m^{\frac{1}{2}}$ is the point satisfies $\sum_{n\le n_0}\mu(n) = 0$.

The following is obtained by calculation for $\sum_{n_0<n\le m}\mu(n)[\frac{m}{n}]$. This represents the terms coresponds to $[\sqrt{m}]$ are $\sqrt{m}$ to $m/(\sqrt{m}-1)$, the terms coresponds to $[\sqrt{m}]-1$ are $m/(\sqrt{m}-1)$ to $m/(\sqrt{m}-2)$ and the terms corresponds to 1 are $\frac{m}{2}$ to m.

$[\sqrt{m}]$ term is sum of all terms satisfy $[\frac{m}{n}] = [\sqrt{m}]-1$, $m/\sqrt{m} = \sqrt{m} \ge [\sqrt{m}]$ and $m/(m/(\sqrt{m}-1)) = \sqrt{m}-1 \ge [\sqrt{m}-1]$, $(m/(m/(\sqrt{m}-1)+1) = m(\sqrt{m}-1)/(m+\sqrt{m}-1) < \sqrt{m}-1,)$ so the range is $\sqrt{m}$ to $m/(\sqrt{m}-1)$. Next term is sum of all terms satisfy $[\frac{m}{n}] = [\sqrt{m}]-2$, $m/(m/(\sqrt{m}-2)) \ge [\sqrt{m}-2]$. The range is $m/(\sqrt{m}-1)$ to $m/(\sqrt{m}-2)$. The last term satisfy $[\frac{m}{n}] = 1$, that is $\frac{m}{2}$ to m.

$$\sum_{n_0<n\le m}\mu(n)[\frac{m}{n}] = ([m/(n_0)]-1)\times \sum_{m/([m/(n_0)])<n\le m/([m/(n_0)]-1)}\mu(n) + \cdots +$$

$$([\sqrt{m}])\times \sum_{m/(\sqrt{m}+1)<n\le m/\sqrt{m}}\mu(n) + ([\sqrt{m}]-1)\times \sum_{\sqrt{m}<n\le m/(\sqrt{m}-1)}\mu(n) + ([\sqrt{m}]-2)\times$$

$$\sum_{m/(\sqrt{m}-1)<n\le m/(\sqrt{m}-2)}\mu(n) + \cdots + 1\times \sum_{m/2<n\le m}\mu(n)$$

By induction for Riemann Hypothesis, $|\sum_{n\le m/N}\mu(n)| < K(\frac{m}{N})^{\frac{1}{2}+\epsilon}$. I get $|\sum_{m/(N+1)<n\le m/N}\mu(n)| < K(\frac{m}{N+1})^{\frac{1}{2}+\epsilon} + K(\frac{m}{N})^{\frac{1}{2}+\epsilon}$,

I want to calculate some terms. $|\sum_{m/2\le n\le m}\mu(n)| < K(m-1)^{\frac{1}{2}+\epsilon} + 1 + K(\frac{m}{2})^{\frac{1}{2}+\epsilon}$, $|\sum_{m/3\le n\le m/2}\mu(n)|$is less than $K(\frac{m}{3})^{\frac{1}{2}+\epsilon} + K(\frac{m}{2})^{\frac{1}{2}+\epsilon}$.
$|\sum_{\sqrt{m}<n\le m/(\sqrt{m}-1)}\mu(n)|$ is less than $K(\frac{m}{[\sqrt{m}]})^{\frac{1}{2}+\epsilon} + K(\frac{m}{[\sqrt{m}]-1})^{\frac{1}{2}+\epsilon}$

Later, I calculate in the real examples.
example:$m = 100$ case.

$$-6 = 10 - 9 + 0 + 0 + 6 - 5\times 2 - 4 - 3 + 2 + 1\times 4$$

$1\times 4-4+2-3$, $4-1+1-1=3$ give the almost value of $\sum_{[100/n_0]<n\leq 100}\mu(n)$. Actualy, $\sum_{9<n\leq 100}\mu(n)=2$. This gives $|\sum_{[100/n_0]<n\leq 100}\mu(n)|<K[100/n_0]=K\times 10$,
example:$m=10000$ case.

$$-95=\cdots+3\times 18-2\times 15-1\times 25$$

$-1\times 25+3\times 8-2\times 15+3\times 10$, $-25+8-15+10=-22$ gives the almost value of $\sum_{93<n\leq 10000}\mu(n)=-23$. This gives $|\sum_{[10000/n_0]<n\leq 10000}\mu(n)|<K[10000/n_0]=K\times 107$.

Why does the value coinside? By theorem2, $\sum_{n_0\leq n\leq n_1}\mu(n)=0\Leftrightarrow\sum_{n_0\leq n\leq n_1}[\frac{m}{n}]\mu(n)\approx 0, n_1<<m$.

example
m=10000 case.

$$-95=107+106+105+0-103+0+0+0-99-98-97+0-95+94-93+0-91$$
$$+0+0-88-87+86+0+0+84\times 2+0+0+81\times 2+-0+0-78+77$$
$$-76\times 2+75+74+0-72\times 2-71+70\times 2+69+68\times 2-67-66+0+0+0$$
$$+62\times 2-61+0-59-58-57\times 2+56\times 2-55\times 2+54+0-52\times 2-51-50\times 2$$
$$+49\times 3+48\times 2+47+46\times 4+45\times 2-44-43\times 3+0-41\times 2+40+0-38$$
$$+0-36\times 2-35\times 3-34+33\times 6-32-31+30\times 4+29\times 3-28\times 4-27+0$$
$$+25\times 3+0-23\times 6+22-21\times 2+0+19\times 4+18\times 7+17-16\times 9-15\times 9+14\times 9$$
$$+0+0+11\times 2+10\times 3-9\times 15+8\times 10+7\times 12-6\times 20+5\times 16-4\times 6+3\times 18-2\times 15$$
$$-1\times 25$$

By calculation,

$$-1\times 25-2\times 15+3\times 18-4\times 6+5\times 5$$

I get $-25-15+18-6+5=-23$. $+-+=0$ is well taken as

$$5\times 11-6\times 20+7\times 9$$

This part's the sum of mobius function is $11-20+9=0$. Next, $+-+=0$ is well taken as

$$7\times 3+8\times 10-9\times 15+10\times 3$$

This part's the sum of mobius function is $3+10-15+3=1$. Next, $+--=0$ is well taken as

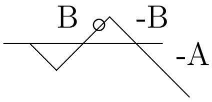
$$11\times 2+14\times 9-15\times 9-16\times 1$$

This part's the sum of mobius function is $2+9-9-1=1$ Mobius function's partial sum is gradually small. Mobius function's partial sum is $-23+1+1=-21$.

By theorem**??**, at enough after (at least 4) terms, $+-+=0$ or $-+-=0$ is taken like in $m=10000$ example. The partial sum of the Mobius function is near 0. The sum takes big coefficient. So the partial sum of the Mobius function is near 0. These value is very smaller than first 3 (or more) terms' sum.

By calculation result, 4 or more latter terms' influence are gradually small. First some (≥ 3) terms decide all value.

By theorem**??**, $\sum_{n=1}^{x}\frac{m}{n}\mu(n)$ takes 0 frequently. This suport this calculation. I calculate general (4 or more terms') case. The absolute value of 4,5,6 term's sum is less than the absolute value of 1,2,3 term's sum about. The absolute value of 7,8,9 term's sum is less than the absolute value of 4,5,6 term's sum about. The rest terms are similarlly. Next picture corresponds.



In this picture, $m=10000$ case,

$$-1\times 25-2\times 15+3\times 18$$

$-A=-22, B=18$.

If 4,5,6 terms takes same sign, for example $m=10000$ case, then "it is contained in the first sum". For example, $K((\frac{m}{4})^{\frac{1}{2}+\epsilon}+(\frac{m}{6})^{\frac{1}{2}+\epsilon})(\frac{\gamma}{\gamma+\delta}(1-\frac{4}{5})+\frac{\delta}{\gamma+\delta}(1-\frac{5}{6}))$ is contained in first sum. The latter term, for example, later 7 term. Like 3 terms's case (later shown), I "retake" the range of sum. This term's sum is less than theorical value. (The possible value is less than 70% about.) The same sign terms (with near defferent sign terms) are contained in before sum. If later same sign terms exist, then I take positive and negative sum's 70%. The defferent sign terms do cancel only.

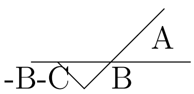
$m=10000$ case, the terms are

$$-4\times 6+5\times 5$$

The influence is $-6+5=-1$. $|-1|<|-22-1|$.

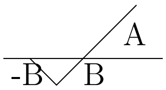
3 terms case is the only to prove part. It gives all the possible cases.

I calculate 3 terms case. At first,



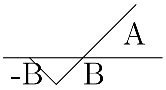
$(C>0, A<K(m-1)^{\frac{1}{2}+\epsilon}+1, B<(\frac{m}{2})^{\frac{1}{2}+\epsilon})$ case. I get $A<K(m-1)^{\frac{1}{2}+\epsilon}+1-C<Km^{\frac{1}{2}+\epsilon}$. $C\leq 0$case, I retake $B+C$ as B. Next 3 cases and 1 case can be occur.

For example, $26\times 1-10\times 2-2\times 3$ case. $+--- = 0$ case.



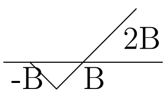
$(A<K(m-1)^{\frac{1}{2}+\epsilon}+1, B<(\frac{m}{2})^{\frac{1}{2}+\epsilon})$
$A+B=\frac{(2\alpha+3\beta)}{\alpha+\beta}B$ holds. This is $A=\frac{(\alpha+2\beta)}{\alpha+\beta}B$. I supose $\alpha\neq 0$ and $A<\sqrt{3}\frac{\beta}{\alpha+\beta}B, (\frac{\alpha}{\alpha+\beta}<\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}})$. $A=|\sum_{n_0<n\leq m}\mu(n)|<<Km^{\frac{1}{2}+\epsilon}$. Later, I see $\alpha=0$ case.

For example, $17\times 1+5\times 2-9\times 3$ case. $++--=0$ case.

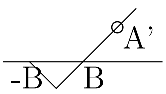


$(A+B)\frac{(\alpha+2\beta)}{\alpha+\beta}=3B$ holds. This is $A=B(3\frac{(\alpha+\beta)}{\alpha+2\beta}-1)$. I supose $\beta\neq 0$ and $A<\sqrt{3}B$. $A=|\sum_{n_0<n\leq m}\mu(n)|<<Km^{\frac{1}{2}+\epsilon}$.

$\beta=0$ or the former $\alpha=0$ case.



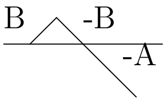
$|\sum_{n\leq m}\mu(n)|<2K(\frac{m}{3})^{\frac{1}{2}+\epsilon}$ at best. $+---=0$ case is almost same as other cases. I treat 3 cases similarlly. In this case, for example, I prove temporaly $|\sum_{n\leq\frac{3}{4}m}\mu(n)|<K(\frac{3}{4}m)^{\frac{1}{2}+\epsilon}$. I take first point before $|\sum_{n\leq m}\mu(n)|\approx 2K(\frac{m}{3})^{\frac{1}{2}+\epsilon}$. (The tilt of influence is less than 1.) If this point is near $n=\frac{3}{4}m$, then $|\sum_{n\leq\frac{3}{4}m}\mu(n)|$ is about less than $K\frac{3}{2}(\frac{m}{3})^{\frac{1}{2}+\epsilon}$. Next picture corresponds.



Generally, I think the case $|\sum_{n\leq km}\mu(n)|<K(km)^{\frac{1}{2}+\epsilon}, (\frac{1}{2}<<k<<1)$. $|\sum_{n\leq km}\mu(n)|$ is about less than $K\sqrt{3k}(\frac{m}{3})^{\frac{1}{2}+\epsilon}$. This condition holds for enough small k. The proof for m (in the before 3 cases is got by using large $M>m$. I can take $m<M<2m$. I only use the term less than m. It does not contradict to induction for m. For M, I think 3 terms' case. For $m<M$, I can lead $|\sum_{n\leq m}\mu(n)|<Km^{\frac{1}{2}+\epsilon}$. Don't use actual value, I use $|\sum_{n\leq M}\mu(n)|\approx 2K(\frac{M}{3})^{\frac{1}{2}+\epsilon}$. I draw the picture for M for actual first 3 terms.

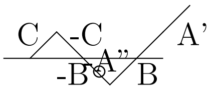
later 4 terms, I use for M case. I write down just in case.

Genarally, the Mobius function of the conbolution of 1 and 2 are 1 and -1. More, the Mobius functions of the conbolutions of 1 to p_n $(< p_n)$ take flutlly distribution. Surely large number case, the distribution are wide. The Mobius function's sum is changed also naturaly.
More, near $\frac{M}{2}$, if $|\sum_{n\leq m}\mu(n)| \approx 2K(\frac{m}{3})^{\frac{1}{2}+\epsilon}$ holds, then



$|\sum_{n\leq m}\mu(n)|$ is not extremely small. And it is not minus.

For example, $12\times 1 - 12\times 2 + 4\times 3$ case. $+-+=0$ case.



$(A' < K(m-1)^{\frac{1}{2}+\epsilon}+1, B < K(\frac{m}{2})^{\frac{1}{2}+\epsilon}, C < K(\frac{m}{3})^{\frac{1}{2}+\epsilon})$
By $A'+B=2B$, $A'=B$. $A' < K(\frac{m}{2})^{\frac{1}{2}+\epsilon}$,$A'-A''=A$, $A < K(\frac{m}{2})^{\frac{1}{2}+\epsilon} << Km^{\frac{1}{2}+\epsilon}$

In 3 terms' case, I get $|\sum_{[\frac{m}{4}]<n\leq m}\mu(n)| < Km^{\frac{1}{2}+\epsilon}$. 4 or more terms' case, evaluation is more less.

$$|\sum_{n\leq m}\mu(n)| < 0 + Km^{\frac{1}{2}+\epsilon}$$

R.H. is got. □

appendix
example:$m = 1000000$ case.

$$\cdots 42\times 11 - 63\times 10 + 50\times 9 + 30\times 8 - 104\times 7 + 104\times 6 - 32\times 5 - 38\times 4 - 70\times 3$$

$$+103\times 2 + 218\times 1$$

$\sum_{n<1000000}\mu(n) = 212$

$$218\times 1 + 103\times 2 - 141\times 3$$

$218 + 103 - 141 = 180$

$$71\times 3 - 38\times 4 - 12\times 5$$

$71 - 38 - 12 = 21$

$$-20\times 5 + 104\times 6 - 74\times 7$$

$-20 + 104 - 74 = 10$

$$-29 \times 7 + 30 \times 8 - 4 \times 9$$

$-29 + 30 - 4 = -3$

$$54 \times 9 - 63 \times 10 + 13 \times 11$$

$54 - 63 + 13 = 4$
$180 + 21 + 10 - 3 + 4 = 212$

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