On sinusoidal and isochronous periodic solutions of Lienard type equations with only damping

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Abstract

We present in this contribution some exceptional Lienard type equations with only damping. We exhibit sinusoidal periodic solutions for these equations. In consequence such equations can be used to model harmonic and isochronous periodic oscillations of nonlinear damped dynamical systems.

Keywords: Lienard type equations, exact periodic solution, harmonic and isochronous oscillations, damping.

Introduction

The generalized second-order differential equation of Lienard type can be written as

$$\ddot{x}+\sigma(x,\dot{x})\dot{x}+h(x)=0 \qquad (1)$$

where overdot means differentiation with respect to time, σ is a function of x and its first derivative $\dot{x}$, and $h(x)$ is a function of x. The equation (1) contains several well known differential equations as special cases. The linear damped harmonic oscillator as the typical second-order differential equation is obtained for $\sigma(x,\dot{x})=\lambda$, and $h(x)=\omega^2 x$, where λ and ω are constants. λ denotes the damping term which is responsible of energy dissipation in heat of the system. The linear damped harmonic oscillator cannot exhibit periodic oscillations in opposition to the conservative linear harmonic oscillator where $\lambda=0$. The equation (1) includes also the quadratic Lienard type differential equation

$$\ddot{x}+u(x)\dot{x}^2+h(x)=0 \qquad (2)$$

where $\sigma(x,\dot{x})=\dot{x}u(x)$, and $u(x)$ is a function of x. The equation (2) has gained high importance in physics as well as in pure and applied mathematics since the discovery of the equation called Mathews-Lakshmanan equation in 1974 [1].

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Such an equation is famous because it reproduces the solution of the linear harmonic oscillator but with amplitude-dependent frequency for the first time. The equation (1) contains another dissipative Lienard type equation

$$\ddot{x} + \vartheta(x)\dot{x} + h(x) = 0 \tag{3}$$

where $\sigma(x,\dot{x}) = \vartheta(x)$, and $\vartheta(x)$ is a function of x , of physical importance. The famous modified Emden type equation used to model several physical phenomena belongs to this class of equation [2]. The celebrated Van der Pol equation [3] belongs also to this class of dissipative Lineard type equation. Another important equation of the class (3) is the so-called generalized and modified Emden type equation solved in 2005 by Chandrasekar et al. [4] to exhibit for the first time exact and explicit harmonic and isochronous periodic oscillations. However, in a recent paper [5] Doutètien and coworkers investigated this equation solved in [4]. Their result is without objections: The claimed Lienard type nonlinear oscillator presented in [4] is a pseudo-oscillator. Indeed the authors in [5] succeeded to exhibit with ease and clarity unbounded periodic solution for this equation. As an oscillator can only have bounded periodic solutions, the authors [5] concluded that the feature of conservative nonlinear oscillator claimed by Chandrasekar and coworkers in [4] is not consistent. The same discovery has been also noticed for the Mathews-Lakshmanan oscillator [1] by Akande et al. [6]. The authors in [6] found that the celebrated Mathews-Lakshmanan equation claimed to be a unique oscillator is in fact a pseudo-oscillator. The authors in [6] have successfully exhibited exact non-oscillatory solutions to this equation so that it could not be a conservative nonlinear oscillator. The quadratic Lienard type equations have been the object of intensive studies in the literature during the last decades. In this regard, the Lie point symmetries of such equations have been investigated in [7]. In [8] the authors used the generalized Sundman transformation theory to identify a class of quadratic Lienard type equations that can exhibit trigonometric periodic solutions. Later such a generalized Sundman transformation theory is used to find periodic solutions of other quadratic Lienard type equations [9, 10]. Recently, in several papers [11-13] Monsia and his group presented successfully some dissipative Lienard type nonlinear oscillators of the form (3) with harmonic and isochronous periodic solutions. The mixed Lienard type equations

$$\ddot{x} + u(x)\dot{x}^2 + \vartheta(x)\dot{x} + h(x) = 0 \tag{4}$$

where $u(x)$, $\vartheta(x)$, and $h(x)$ are arbitrary functions of x, have been also studied in the literature from mathematical as well as physical points of view [14-17]. Finding periodic solutions for an equation of type (4) is of course more difficult than in the cases studied previously due to the presence of several types of nonlinearities. However, in [16,17] Monsia and his group succeeded to exhibit sinusoidal and isochronous periodic solutions of some equations of type (4) for the first time. The equation (4) belongs to the class of equations (1) where $\sigma(x,\dot{x}) = \dot{x}u(x) + \vartheta(x)$. In [18] it was for the first time a sinusoidal and isochronous periodic solution is obtained for a nonlinear differential equation that is to say, a nonlinear differential equation and the linear harmonic oscillator have identical exact periodic solutions. This exclusivity of the linear harmonic oscillator to have sinusoidal and isochronous periodic solution was also canceled in [19]. Studying the quadratically damped Lienard type equation

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 = 0 \qquad (5)$$

where $\sigma(x,\dot{x}) = \frac{g'(x)}{g(x)}\dot{x}$, is the damping term, Nonti and coworkers [19] have successfully shown that the equation (5) can exhibit sinusoidal and isochronous periodic solution for $g(x) = \left(\mu^2 - x^2\right)^{-1/2}$, that is the equation (5) and the linear harmonic oscillator

$$\ddot{x} + b^2 x = 0 \qquad (6)$$

have identical sinusoidal periodic solution with amplitude of oscillations $\mu > 0$, and angular frequency $b > 0$. It was for the first time such a discovery has been made for a differential equation of the form

$$\ddot{x} + \sigma(x,\dot{x})\dot{x} = 0 \qquad (7)$$

As can be seen the problem of finding periodic or sinusoidal and isochronous periodic solution of equations of type (7) has never been addressed previously in the literature. In fact the equations of type (7) are damped systems known to exhibit usually damped oscillations. It was therefore a revolutionary discovery that was made in [19]. Now the question is to ask whether such an equation is unique. In other words, one can ask whether there are functions $\sigma(x,\dot{x})$ that ensure exact periodic or sinusoidal and isochronous periodic solutions to the damped equation (7). We suppose the existence of such functions $\sigma(x,\dot{x})$ in this paper. To do so we review briefly the theory of mixed Lienard type differential equations introduced recently in the literature [20,21] by Monsia and his group (section 2) and provide examples of functions $\sigma(x,\dot{x})$ ensuring the existence of sinusoidal periodic solutions of the equation (7) (section 3). We present finally a conclusion for the work.

2. General Theory

Let us consider the mixed Lienard type differential equations [20, 21]

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+a\ell x^{\ell-1}\frac{f(x)}{g(x)}\dot{x}+a\,x^{\ell}\frac{f'(x)}{g(x)}\dot{x}=0 \tag{8}$$

with the corresponding first integral

$$b=g(x)\dot{x}+a\,f(x)x^{\ell} \tag{9}$$

where prime stands for differentiation with respect to x, and a, b, and ℓ are arbitrary constants. The functions $f(x)$ and $g(x)$ are arbitrary functions of x. Substituting the equation (9) into the equation (8) yields the mixed Lienard type equation

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+a\,x^{\ell}\frac{f'(x)}{g(x)}\dot{x}-a^2\ell x^{2\ell-1}\frac{f^2(x)}{g^2(x)}+ab\ell\,x^{\ell-1}\frac{f(x)}{g^2(x)}=0 \tag{10}$$

Applying $f(x)=g^2(x)$, turns the equation (10) into

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+2a\,x^{\ell}g'(x)\dot{x}-a^2\ell x^{2\ell-1}g^2(x)+ab\ell\,x^{\ell-1}=0 \tag{11}$$

The choice $\ell=0$, reduces the equation (11) to

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+2a\,g'(x)\dot{x}=0 \tag{12}$$

with can be rearranged in the form

$$\ddot{x}+\left[\frac{g'(x)}{g(x)}\dot{x}+2a\,g'(x)\right]\dot{x}=0 \tag{13}$$

where $\sigma(x,\dot{x})=\frac{g'(x)}{g(x)}\dot{x}+2ag'(x)$. In this situation the solution of the equation (13) is ensured by the quadrature

$$(t+K)=\int\frac{g(x)}{b-af(x)x^{\ell}}dx \tag{14}$$

which becomes

$$-a(t+K)=\int\frac{dx}{g(x)} \tag{15}$$

where K is a constant of integration and $b=0$. We are able to present now the desired equations with sinusoidal periodic solutions.

3. Equations with sinusoidal periodic solutions

3.1 $g(x)=\sqrt{c_1x^2+c_2}$

In this case the equation (13) takes the form

$$\ddot{x}+\frac{c_1x}{c_1x^2+c_2}\dot{x}^2+\frac{2a\,c_1x}{\sqrt{c_1x^2+c_2}}\dot{x}=0 \tag{16}$$

where c_1 and c_2 are arbitrary constants. The solution of the equation (16) is given , according to (15), by

$$-a(t+K)=\int\frac{dx}{\sqrt{c_1x^2+c_2}} \tag{17}$$

By integration, one can get

$$\sin^{-1}\left(x\sqrt{-\frac{c_1}{c_2}}\right)=-a\sqrt{-c_1}\,(t+K) \tag{18}$$

which secures the solution of the equation (16) as

$$x(t)=\sqrt{-\frac{c_2}{c_1}}\sin\left(-a\sqrt{-c_1}(t+K)\right) \tag{19}$$

where $a<0$, and $c_1<0$. The equation (19) is sinusoidal solution but with amplitude-dependent frequency characterizing nonlinear conservative Lienard type oscillators. However, if $c_1=-1,$ and $c_2=\mu^2$, then $g(x)=\sqrt{\mu^2-x^2}$, and the equation (16) reduces to

$$\ddot{x}-\frac{x}{\mu^2-x^2}\dot{x}^2-\frac{2a\,x}{\sqrt{\mu^2-x^2}}\dot{x}=0 \tag{20}$$

where μ is an arbitrary parameter. The interesting fact is that the exact and explicit general solution of (20), viz

$$x(t)=\mu\sin\left(-a\,(t+K)\right) \tag{21}$$

is identical with the solution of the linear harmonic oscillator equation

$$\ddot{x}+a^2x=0 \tag{22}$$

and can exhibit harmonic and isochronous periodic oscillations of amplitude $\mu > 0$ and angular frequency $-a > 0$.

3.2 $g(x) = \sqrt{c_1 x^2 + c_2 x}$

The equation (12), or (13) can reduce in this context to

$$\ddot{x} + \frac{2c_1 x + c_2}{2\left(c_1 x^2 + c_2 x\right)} \dot{x}^2 + \frac{a\left(2c_1 x + c_2\right)}{\sqrt{c_1 x^2 + c_2 x}} \dot{x} = 0 \tag{23}$$

Using the equation (15) the solution of the equation (23) is secured by the quadrature

$$-a\left(t + K_1\right) = \int \frac{dx}{\sqrt{c_1 x^2 + c_2 x}} \tag{24}$$

The integration of the right hand side member leads to

$$\sin^{-1}\left(\frac{2c_1 x + c_2}{c_2}\right) = a\sqrt{-c_1}\left(t + K_1\right) \tag{25}$$

such that one can obtain

$$x(t) = \frac{c_2}{2c_1}\left[-1 + \sin\left[a\sqrt{-c_1}\left(t + K_1\right)\right]\right] \tag{26}$$

where K_1 is a constant of integration, and $a > 0$, $c_2 > 0$ and $c_1 < 0$. The isochronous periodic oscillations can be ensured for $c_1 = -1$, so that the solution (26) takes the form

$$x(t) = \frac{c_2}{2}\left[1 - \sin\left[a\left(t + K_1\right)\right]\right] \tag{27}$$

where $c_2 > 0$, and $a > 0$.

Conclusion

The Lienard type differential equations with only damping are investigated in this paper. We have been able to exhibit sinusoidal periodic solutions of these equations. In this context, these equations can be used to describe harmonic and isochronous oscillations in damped dynamical systems.

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