Sinusoidal and Isochronous Periodic Oscillations of Lienard type Equations without Restoring Force

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Abstract

We present in this paper some interesting Lienard type equations without restoring force. We show that these equations can exhibit sinusoidal periodic solutions that can be exploited to represent harmonic and isochronous periodic oscillations in nonlinear damped dynamical systems.

Keywords: Lienard type equations, exact periodic solutions, harmonic and isochronous oscillations, nonlinear damping force.

Introduction

A rich variety of solution methods like approximate techniques, transformation theory of equations, first integral approach and dynamical systems theory are exploited in the literature to study the general second-order differential equation of Lienard type

$$\ddot{x} + \sigma(x,\dot{x})\dot{x} + h(x) = 0 \tag{1}$$

where overdot means derivative with respect to time, $\sigma(x,\dot{x})$ denotes a function of x and $\dot{x}$, and $h(x)$ is a function of x. A lot of theories of existence and uniqueness of periodic solutions of the equation (1) from Poincaré-Bendixsonprinciple can be found in the literature [1]. The existence of limit cycles of special cases of the equation (1) has been widely investigated in the literature [1]. The linear damped harmonic oscillator

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0 \tag{2}$$

where γ and ω are arbitrary constants, is the basic case of the equation (1), where $\sigma(x,\dot{x}) = \gamma$, and $h(x) = \omega^2 x$. In this context the term $-\sigma(x,\dot{x})\dot{x}$ denotes the damping force, and $-h(x)$ is the restoring force in the equation (1). It is known, when $\sigma(x,\dot{x}) = 0$, that the equation (1) can exhibit conservative periodic

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oscillations so that the equation (1) is said conservative system. There is a vast literature in this type of equations where $\sigma(x,\dot{x})=0$. The cubic Duffing equation

$$\ddot{x}+\alpha x+\beta x^{3}=0 \tag{3}$$

where α and β are arbitrary constants, belongs to this type of Lienard equations. The equation (3) has deeply been examined in the literature and used to describe several nonlinear phenomena in physics [2-4]. The exact solution of the cubic Duffing equation is well known to be each of the Jacobi elliptic functions. However, in some recent papers [3, 4] the authors have successfully shown that this equation can exhibit unbounded tangent periodic solutions such that it could not be a nonlinear conservative oscillator as claimed in the literature. An interesting class of equations of the type (1)can read

$$\ddot{x}+\vartheta(x)\dot{x}+h(x)=0 \tag{4}$$

The well-studied Van der Pol equation

$$\ddot{x}+\mu(x^{2}-1)\dot{x}+x=0 \tag{5}$$

in the literature, where $\mu>0$ is an arbitrary constant, belongs to the class of equations (4). The existence of limit cycles of the equation (5) is broadly documented in the literature [1]. Another equation of physical importance that belongs to the class (4) is the generalized and modified Emden type equation [5,6].

$$\ddot{x}+\alpha_{1}x\dot{x}+\alpha_{2}x^{3}+\lambda x=0 \tag{6}$$

where α_1, $\alpha_2$and λ are arbitrary parameters. In [5] the authors succeeded to calculate exact harmonic and isochronous periodic solution of the equation (6) for the first time and claimed to find an unusual Lienard type nonlinear oscillator. However, in a recent paper, Doutètien et al. [6], have successfully shown the existence of unbounded periodic solution for the equation (6). Thus the authors in [6] concluded that the equation (6) is in fact a pseudo-oscillator. The quadratic Lienard type equation

$$\ddot{x}+u(x)\dot{x}^{2}+h(x)=0 \tag{7}$$

where $u(x)$ is a function of x, belongs also to the general class of equations (1). In this case $\sigma(x,\dot{x})=u(x)\dot{x}$. A special case of the equation (7) presented in [7] has acquired a high consideration in the literature and has been intensively treated from physical as well as mathematical point of view. In [7] the authors claimed to find a unique oscillator of type (7) exhibiting sinusoidal periodic oscillations but with amplitude-dependent frequency. However, Monsia and his group succeeded to show the existence of many equations of type (7) exhibiting sinusoidal periodic solutions [8-10] with amplitude-dependent frequency. Recently, Akande et al. [11] have shown successfully that the so-called Mathews-Lakshmanan oscillator presented in [7] is in fact a pseudo-oscillator since this equation can exhibit non-oscillatory behavior. In the literature the mixed Lienard type equation

$$\ddot{x} + [u(x)\dot{x} + \vartheta(x)]\dot{x} + h(x) = 0 \tag{8}$$

where $\sigma(x,\dot{x}) = \dot{x}u(x) + \vartheta(x),$ has been also the object of interesting studies [12-16]. The authors in [14,15] presented for the first time some equations of type (8) exhibiting exact sinusoidal and isochronous periodic solutions. Although the equation (8),that is the equation (1) has been well studied within the framework of existence and uniquenesstheorems obtained from the Poincaré-Bendixson principle [1], the determination of exact and explicit periodic solutions continues to be a challenge. In this way, Monsia and his group investigated recently the quadratically dissipative equation [16]

$$\ddot{x} + \frac{x}{\mu^2 - x^2}\dot{x}^2 = 0 \tag{9}$$

where $\mu > 0$ is an arbitrary parameter. The equation (9) is of Lienard type (1) without restoring force, that is to say with $h(x) = 0.$ The equation (9) is an exceptional quadratically damped equation, since it does not satisfy the theorem of existence of at least one periodic solution formulated in [1] (see Theorem 11.2), but the authors [16] succeeded to calculate exact and explicit general periodic solution of the equation (9). More interesting, this periodic solution is sinusoidal and isochronous, that is this solution is identical to the solution of the linear harmonic oscillator equation

$$\ddot{x} + b^2 x = 0 \tag{10}$$

with amplitude of oscillations $\mu > 0$ and angular frequency $b > 0$ Thus, for the first time a periodic solution has been obtained for an equation of the form

$$\ddot{x} + \sigma(x,\dot{x})\,\dot{x} = 0 \tag{11}$$

that does not obey the theorem of existence of periodic solutions [1]. More recently, Adjaï et al. [17] have successfully revealed for this purpose the existence of several others equations of type (11) exhibiting also sinusoidal and isochronous periodic solutions. In the present paper, the problem is to ask whether there are equations of type (11) that can exhibit identical sinusoidal and isochronous periodic solutions to the equations solved in [17]. The current work predicts the existence of such equations of type (11). In this perspective, we briefly present [17] an overview of the required theory (section 2) and exhibit the identical solutions of equations (section 3). We present finally a conclusion for the contribution.

2.Overview of the theory

According to [17] one can verify that

$$b = g(x)\dot{x} + a f(x)x^{\ell} \tag{12}$$

is a first integral of the general second-order differential equation [17]

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a\ell x^{\ell-1}\frac{f(x)}{g(x)}\dot{x} + a x^{\ell}\frac{f'(x)}{g(x)}\dot{x} = 0 \tag{13}$$

Using the relation (12), the equation (13) becomes

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + a x^{\ell}\frac{f'(x)}{g(x)}\dot{x} - a^2\ell x^{2\ell-1}\frac{f^2(x)}{g^2(x)} + ab\ell x^{\ell-1}\frac{f(x)}{g^2(x)} = 0 \tag{14}$$

Substituting $g(x) = f^2(x)$, into the equation (14), allows one to obtain

$$\ddot{x} + \frac{2f'(x)}{f(x)}\dot{x}^2 + a x^{\ell}\frac{f'(x)}{f^2(x)}\dot{x} - a^2\ell\frac{x^{2\ell-1}}{f^2(x)} + ab\ell\frac{x^{\ell-1}}{f^3(x)} = 0 \tag{15}$$

By application of $\ell = 0$, the equation (15) reduces to

$$\ddot{x} + \frac{2f'(x)}{f(x)}\dot{x}^2 + a\frac{f'(x)}{f^2(x)}\dot{x} = 0 \tag{16}$$

which can be written in the form

$$\ddot{x} + \left[\frac{2f'(x)}{f(x)}\dot{x} + a\frac{f'(x)}{f^2(x)}\right]\dot{x} = 0 \tag{17}$$

The equation (17) has the form of the equation (11), that is of the equation (1) where the restoring force $h(x)=0$. In this way the equation (17) does not also satisfy the Theorem 11.2 of [1]. However, the objective in the following section is to show that the equation (17) without restoring force can exhibit sinusoidal and isochronous periodic solutions identical to the solutions of equations exhibited in [17].

3. Equations with sinusoidal periodic solutions

In the situation where $g(x)=f^2(x)$, the first integral (12) can ensure the general solution of the equation (17) as

$$-a(t+K)=\int f(x)dx \tag{18}$$

where K, is a constant of integration. Now we can investigate the first special case of the equation (18).

3.1 $f(x)=\dfrac{1}{\sqrt{c_1x^2+c_2}}$

In this case the equation (17) is written as

$$\ddot{x}-\left[\frac{2c_1x}{c_1x^2+c_2}\dot{x}+\frac{ac_1x}{\sqrt{c_1x^2+c_2}}\right]\dot{x}=0 \tag{19}$$

where c_1 and c_2 are arbitrary parameters. Putting the expression of $f(x)$ in the equation (18), yields

$$-a(t+K)=\int\frac{dx}{\sqrt{c_1x^2+c_2}} \tag{20}$$

such that, after integration, one can get

$$\sin^{-1}\left(x\sqrt{-\frac{c_1}{c_2}}\right)=-a\sqrt{-c_1}\,(t+K) \tag{21}$$

which allows one to obtain the solution of the equation (19) as

$$x(t)=\sqrt{-\frac{c_2}{c_1}}\sin\left(-a\sqrt{-c_1}(t+K)\right) \tag{22}$$

The formula (22) represents a sinusoidal solution of the equation (19) but with amplitude-dependent frequency, where $c_1 < 0$, $c_2 > 0$ and $a < 0$. Applying $c_1 = -1$, transforms the solution (22) into

$$x(t) = \sqrt{c_2}\sin[-a(t+K)] \tag{23}$$

which becomes sinusoidal and isochronous periodic solution. The solutions (22) and (23) are identical to the solutions (19) and (21) of the Reference [17] respectively, where $c_2 = \mu^2$ and $\mu > 0$, is an arbitrary constant, while the equation (19) is quite different from the equation (16) of the Reference [17].

3.2 $f(x) = \dfrac{1}{\sqrt{c_1 x^2 + c_2 x}}$

This case leads to

$$\ddot{x} - \left[\frac{2c_1 x + c_2}{c_1 x^2 + c_2 x}\dot{x} + \frac{a(2c_1 x + c_2)}{2\sqrt{c_1 x^2 + c_2 x}}\right]\dot{x} = 0 \tag{24}$$

Using the relation (18), the solution of the equation (24) is given by

$$-a(t+K_1) = \int \frac{dx}{\sqrt{c_1 x^2 + c_2 x}} \tag{25}$$

which, after integration, reduces to

$$\sin^{-1}\left(\frac{2c_1 x + c_2}{c_2}\right) = a\sqrt{-c_1}\,(t+K_1) \tag{26}$$

From the equation (26), one can get the general solution of the equation (24) in the form

$$x(t) = \frac{c_2}{2c_1}\left[-1 + \sin\left[a\sqrt{-c_1}\,(t+K)\right]\right] \tag{27}$$

where $c_1 < 0$, $c_2 > 0$ and $a > 0$. The solution (27) is sinusoidal but with amplitude-dependent frequency. However,, the solution (27) can be made isochronous by setting $c_1 = -1$. In this context, the solution (27) becomes

$$x(t) = \frac{c_2}{2}[1 - \sin[a(t+K)]] \tag{28}$$

where $c_2 > 0$, and $a > 0$. The solutions (27) and (28) are identical to the solutions (26) and (27) of the Reference [17] respectively, while the equation (24) is quite different from the equation (23) of the Reference [17]. So with that a conclusion can be performed for the work.

Conclusion

In this contribution, some equations of Lienard type have been studied. We have successfully shown that these equations without restoring force can exhibit sinusoidal and isochronous periodic oscillations.

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