An Extension of Vajda's and D'Ocagne's Identities

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Abstract

In this paper, we present an identity which generalizes Vajda's and D'ocagne's identities involving Fibonacci sequence. Vajda's and D'ocagne's identities are known to contain 3 and 2 variables respectively, but the identity to be presented contains 7 variables. Binet's formula for generating the nth term of Fibonacci sequence will be used in proving the identity.

Keywords: Fibonacci sequence, Binet's formula, Vajda's and D'Ocagne's identities.

1   Introduction

It is widely known that Fibonacci sequence, along with Lucas sequence, are two of the most commonly used sequences for establishing recurrence relations. The recurrence relations concerning both Fibonacci and Lucas sequences give their next numbers as the sum of the preceeding two numbers. The classic Fibonacci sequence is denoted as

$$F_k = F_{k-1} + F_{k-2}, k \geq 2, F_0 = 0, F_1 = 1.$$

The first few Fibonacci numbers are given by:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...$$

Also, the classic Lucas sequence is denoted as

$$L_k = L_{k-1} + L_{k-2}, k \geq 2, L_0 = 2, L_1 = 1.$$

The first few Lucas numbers are given by:

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...$$

Cennet Bolat, Ahmet Ipek, and Hasan Kose[4] noted that in recent years, many interesting properties of classic Fibonacci numbers, classic Lucas numbers and their generalizations have been shown by researchers and applied to almost every aspect of science and art. From the richness and related applications of these numbers, one can refer to the nature and different areas of science. Francisco Regis Vieira Alves[5] noted that the role of the

Fibonacci's sequence is usually discussed by most of mathematics history books. Despite its presentation in a form of mathematical problem,concerning the birth of rabbits' pairs, still occurs a powerful mathematical model that became the object of research, especially with the French mathematician François Édouard Anatole Lucas(1842–1891). From his work, a profusion of mathematical properties became known in the pure mathematical research, specially, from the sixties and the seventies. Many identities which relate to both Fibonacci and Lucas sequences have been discovered over time. For instance, in 1680, Cassini discovered without proof that

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \tag{1}$$

which was later independently proven by Robert Simson in 1753.
In 1879, Eugene Charles Catalan discovered and proved that

$$F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2 \tag{2}$$

which generalizes cassini's identity. Another identity which generalizes Cassini's identity was discovered by Philbert Maurice D'Ocagne and the identity states that

$$F_m F_{n+1} - F_{m+1}F_n = (-1)^n F_{m-n} \tag{3}$$

In 1989, Steven Vajda[8] published a book on fibonacci numbers which contains the identity

$$F_{n+i}F_{n+j} - F_n F_{n+i+j} = (-1)^n F_i F_j \tag{4}$$

which carries his name and thus generalizes both Cassini's and Catalan's Identities. Various methods have been employed to prove (1), (2), (3), and (4), some of which are proof by induction, Binet's formula for generating the nth term of Fibonacci sequence, proof by matrix method, combinatorial method.

2 Main Result

Identity : If n, i, j, k, x, y, z are integers, then

$$F_{n+i+x-z}F_{n+j+y+z} - F_{n+x+y-k}F_{n+i+j+k} = (-1)^{n+x+y-k}F_{i+k-y-z}F_{j+k+z-x} \tag{5}$$

From (3), we can see that setting $x = y = z = k = 0$, we get

$$F_{n+i+x-z}F_{n+j+y+z} - F_{n+x+y-k}F_{n+i+j+k} = (-1)^{n+x+y-k}F_{i+k-y-z}F_{j+k+z-x}$$
$$F_{n+i+0-0}F_{n+j+0+0} - F_{n+0+0-0}F_{n+i+j+0} = (-1)^{n+0-0-0}F_{i+0+0-0}F_{j+0+0-0}$$

$$F_{n+i}F_{n+j} - F_n F_{n+i+j} = (-1)^n F_i F_j \tag{6}$$

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Now, we can see that (6) is equal to (4)

Also, we can see that setting $i = 1, j = m - n$ in (6), we get

$$F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^n F_i F_j$$

$$F_{n+1}F_{n+m-n} - F_nF_{n+1+m-n} = (-1)^n F_1 F_{m-n}$$

$$F_{n+1}F_m - F_nF_{m+1} = (-1)^n F_{m-n} \tag{7}$$

Also, we can see that (7) is equal to (3)

3 Proof

We know from Binet's formula for generating nth Fibonacci number that
if
$$\phi = \frac{1 + \sqrt{5}}{2}, \varphi = \frac{1 - \sqrt{5}}{2}$$

then
$$F_n = \frac{\phi^n - \varphi^n}{\sqrt{5}}$$

where F_n is the nth Fibonacci number

From (5), let
$$\alpha = F_{n+i+x-z}F_{n+j+y+z} \tag{8}$$

$$\beta = F_{n+x+y-k}F_{n+i+j+k} \tag{9}$$

$$\gamma = (-1)^{n+x+y-k}F_{i+k-y-z}F_{j+k+z-x} \tag{10}$$

such that
$$\alpha - \beta = \gamma \tag{11}$$

We can see that to prove (5), it suffices to show that (11) is true

Also, let
$$P_1 = \frac{\phi^{2n+i+x+j+y}}{5},$$

$$P_2 = \frac{\varphi^{2n+i+x+j+y}}{5},$$

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$$P_3 = \frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5},$$

$$P_4 = \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5},$$

$$P_5 = \frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5},$$

$$P_6 = \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5}$$

$$P_7 = \phi^{i+k-y-z}\varphi^{j+z+k-x},$$

$$P_8 = \phi^{j+z+k-x}\varphi^{i+k-y-z},$$

$$P_9 = \phi^{2k+i+j-x-y},$$

$$P_{10} = \varphi^{2k+i+j-x-y}$$

$$P_{11} = F_{i+k-y-z}F_{j+k+z-x}$$

From (8), we see that

$$\alpha = F_{n+i+x-z}F_{n+j+y+z}$$

$$\alpha = \left(\frac{\phi^{n+i+x-z} - \varphi^{n+i+x-z}}{\sqrt{5}}\right)\left(\frac{\phi^{n+j+y+z} - \varphi^{n+j+y+z}}{\sqrt{5}}\right)$$

$$\alpha = \frac{\phi^{2n+i+x+j+y}}{5} - \frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5} - \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5} + \frac{\varphi^{2n+i+x+j+y}}{5}$$

$$\alpha = P_1 - P_3 - P_4 + P_2 \tag{12}$$

From (9), we see that

$$\beta = F_{n+x+y-k}F_{n+i+j+k}$$

$$\beta = \left(\frac{\phi^{n+x+y-k} - \varphi^{n+x+y-k}}{\sqrt{5}}\right)\left(\frac{\phi^{n+i+j+k} - \varphi^{n+i+j+k}}{\sqrt{5}}\right)$$

$$\beta = \frac{\phi^{2n+i+x+j+y}}{5} - \frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5} - \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5} + \frac{\varphi^{2n+i+x+j+y}}{5}$$

$$\beta = P_1 - P_5 - P_6 + P_2 \tag{13}$$

Deducting (13) from (12) gives

$$\alpha - \beta = (P_5 + P_6) - (P_3 + P_4) \tag{14}$$

From (14), let

$$V_1 = -(P_3 + P_4)$$

$$V_2 = (P_5 + P_6)$$

then

$$V_1 = -\left(\frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5} + \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5}\right)$$

$$V_1 = -\frac{(\phi\varphi)^{n+x+y-k}}{(\phi\varphi)^{n+x+y-k}}\left(\frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5} + \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5}\right)$$

$$V_1 = -\frac{1}{5}((\phi\varphi)^{n+x+y-k})\left(\frac{\phi^{n+i+x-z}}{\phi^{n+x+y-k}}\frac{\varphi^{n+j+y+z}}{\varphi^{n+x+y-k}} + \frac{\phi^{n+j+y+z}}{\phi^{n+x+y-k}}\frac{\varphi^{n+i+x-z}}{\varphi^{n+x+y-k}}\right)$$

$$V_1 = -\frac{1}{5}((\phi\varphi)^{n+x+y-k})(\phi^{i+k-y-z}\varphi^{j+z+k-x} + \phi^{j+z+k-x}\varphi^{i+k-y-z})$$

Note that

$$\phi\varphi = -1$$

$$V_1 = -\frac{1}{5}((-1)^{n+x+y-k})(P_7 + P_8)$$

Also,

$$V_2 = \left(\frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5} + \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5}\right)$$

$$V_2 = \frac{(\phi\varphi)^{n+x+y-k}}{(\phi\varphi)^{n+x+y-k}}\left(\frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5} + \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5}\right)$$

$$V_2 = \frac{1}{5}((\phi\varphi)^{n+x+y-k})\left(\frac{\phi^{n+x+y-k}}{\phi^{n+x+y-k}}\frac{\varphi^{n+i+j+k}}{\varphi^{n+x+y-k}} + \frac{\phi^{n+i+j+k}}{\phi^{n+x+y-k}}\frac{\varphi^{n+x+y-k}}{\varphi^{n+x+y-k}}\right)$$

$$V_2 = \frac{1}{5}((\phi\varphi)^{n+x+y-k})(\phi^0\varphi^{2k+i+j-x-y} + \phi^{2k+i+j-x-y}\varphi^0)$$

$$V_2 = \frac{1}{5}((\phi\varphi)^{n+x+y-k})(\phi^{2k+i+j-x-y} + \varphi^{2k+i+j-x-y})$$

Note that

$$\phi\varphi = -1$$

$$V_2 = \frac{1}{5}((-1)^{n+x+y-k})(P_9 + P_{10})$$

Now, we see from (14) that

$$\alpha - \beta = (P_5 + P_6) - (P_3 + P_4)$$
$$\alpha - \beta = V_1 + V_2$$
$$\alpha - \beta = \frac{1}{5}((-1)^{n+x+y-k})(P_9 - P_7 - P_8 + P_{10}) \qquad (15)$$

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From (10), we see that

$$\gamma = (-1)^{n+x+y-k} F_{i+k-y-z} F_{j+k+z-x}$$

$$\gamma = (-1)^{n+x+y-k}(P_{11}) \tag{16}$$

But

$$P_{11} = F_{i+k-y-z} F_{j+k+z-x}$$

$$P_{11} = \left(\frac{\phi^{i+k-y-z} - \varphi^{i+k-y-z}}{\sqrt{5}}\right)\left(\frac{\phi^{j+k+z-x} - \varphi^{j+k+z-x}}{\sqrt{5}}\right)$$

$$P_{11} = \frac{1}{5}(\phi^{i+k-y-z} - \varphi^{i+k-y-z})(\phi^{j+k+z-x} - \varphi^{j+k+z-x}) \tag{17}$$

We can see that the expansion of (17) gives

$$P_{11} = \frac{1}{5}(P_9 - P_7 - P_8 + P_{10}) \tag{18}$$

So, putting (18) in (16) gives

$$\gamma = \frac{1}{5}(-1)^{n+x+y-k}(P_9 - P_7 - P_8 + P_{10}) \tag{19}$$

Since (15) equals (19) then, (11) is true which completes the proof.

4   Conclusion

In this paper, we have stated and proven an identity which generalizes both Vajda's and D'Ocagne's identities (involving Fibonacci sequence) using Binet's formula for generating nth Fibonacci number.

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