An Extension of Vajda's Identity for Fibonacci and Lucas Numbers

Suaib Lateef
lateefholuyinka@gmail.com

Abstract
In this paper, we present two identities involving Fibonacci numbers and Lucas numbers. The first identity generalizes Vajda's identity, which in turn generalizes Catalan's identity, while the second identity is a corresponding result involving Lucas numbers. Binet's formulas for generating the nth term of Fibonacci numbers and Lucas numbers will be used in proving the identities.

Keywords: Fibonacci numbers, Lucas numbers, Cassini's identity, Catalan's identity, Binet's formulas, Vajda's and D'Ocagne's identities.

1 Introduction

The Fibonacci numbers, commonly known as F_n, form a sequence, called the Fibonacci sequence, such that each number is a sum of the two preceeding ones, starting from 0 and 1. The Fibonacci sequence is denoted as

$$F_n = F_{n-1} + F_{n-2}, n \geq 2, F_0 = 0, F_1 = 1.$$

The first few Fibonacci numbers are given by:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...$$

The Lucas numbers, just like the Fibonacci numbers, but commonly known as L_n, form a sequence, called the Lucas sequence, such that each number is a sum of the two preceeding ones, starting from 2 and 1. The Lucas sequence is denoted as

$$L_n = L_{n-1} + L_{n-2}, n \geq 2, L_0 = 2, L_1 = 1.$$

The first few Lucas numbers are given by:

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...$$

In 1680, Cassini discovered without proof that

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \tag{1.1}$$

which was later independently proved by Robert Simson in 1753. Jeffrey R. Chesnov([3], P.29) proved (1.1) using the matrix method. In 1879, Eugene Charles Catalan discovered and proved that

$$F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2, \tag{1.2}$$

which generalizes Cassini's identity. Ashwini Panwar, Kiran Sisodiya, G. P. S. Rathore[1] and Alongkot Suvarnamani[2] proved (1.2) using Binet's formula for generating nth Fibonacci number. Another identity which generalizes Cassini's identity

$$F_mF_{n+1} - F_{m+1}F_n = (-1)^nF_{m-n}, \tag{1.3}$$

was discovered by Philbert Maurice D'Ocagne. In 1989, Steven Vajda[5] published a book on Fibonacci numbers which contains the identity

$$F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^nF_iF_j. \tag{1.4}$$

A variant of Vajda's identity which states that

$$F_mF_n - F_{m-r}F_{m+r} = (-1)^{m-r}F_rF_{n+r-m}, \tag{1.5}$$

was proved by Michael Z. Spivey[4] using the matrix method.
In section 2, we state our main results and give their proofs in section 3.

2 The Main Results

Identity 1: If n, i, j, k, x, y, z are integers, then

$$F_{n+i+x-z}F_{n+j+y+z} - F_{n+x+y-k}F_{n+i+j+k} = (-1)^{n+x+y-k}F_{i+k-y-z}F_{j+k+z-x} \tag{2.1}$$

From (2.1), we can see that setting $x = y = z = k = 0$, we get

$$F_{n+i+x-z}F_{n+j+y+z} - F_{n+x+y-k}F_{n+i+j+k} = (-1)^{n+x+y-k}F_{i+k-y-z}F_{j+k+z-x}$$
$$F_{n+i+0-0}F_{n+j+0+0} - F_{n+0+0-0}F_{n+i+j+0} = (-1)^{n+0-0-0}F_{i+0+0-0}F_{j+0+0-0}$$

$$F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^nF_iF_j \tag{2.2}$$

Now, we can see that (2.2) is equal to (1.4)

Also, we can see that setting $i = 1, j = m - n$ in (2.2), we get

$$F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^nF_iF_j$$

$$F_{n+1}F_{n+m-n} - F_nF_{n+1+m-n} = (-1)^nF_1F_{m-n}$$

$$F_{n+1}F_m - F_nF_{m+1} = (-1)^nF_{m-n} \tag{2.3}$$

Also, we can see that (2.3) is equal to (1.3)

Identity 2: If n, i, j, k, x, y, z are integers and

$$\alpha_1 = L_{n+i+x-z}L_{n+j+y+z}, \tag{2.4}$$

$$\beta_1 = L_{n+x+y-k}L_{n+i+j+k}, \tag{2.5}$$

$$\gamma_1 = ((-1)^{n+j+y+z})L_{i+x-j-y-2z} + ((-1)^{n+i+j+k+1})L_{x+y-i-j-2k}, \tag{2.6}$$

then

$$\alpha_1 - \beta_1 = \gamma_1. \tag{2.7}$$

3 The Proofs

Proof of identity 1: We know from Binet's formula for generating nth Fibonacci number that if

$$\phi = \frac{1+\sqrt{5}}{2}, \varphi = \frac{1-\sqrt{5}}{2}$$

then

$$F_n = \frac{\phi^n - \varphi^n}{\sqrt{5}}$$

where F_n is the nth Fibonacci number.

From (2.1), let

$$\alpha = F_{n+i+x-z}F_{n+j+y+z} \tag{3.1}$$

$$\beta = F_{n+x+y-k}F_{n+i+j+k} \tag{3.2}$$

$$\gamma = (-1)^{n+x+y-k}F_{i+k-y-z}F_{j+k+z-x} \tag{3.3}$$

Therefore, we have

$$\alpha - \beta = \gamma \tag{3.4}$$

We can see that to prove (2.1), it suffices to show that (3.4) is true.

Also, let

$$P_1 = \frac{\phi^{2n+i+x+j+y}}{5},\ P_2 = \frac{\varphi^{2n+i+x+j+y}}{5},\ P_3 = \frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5},$$

$$P_4 = \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5},\ P_5 = \frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5},\ P_6 = \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5},$$

$$P_7 = \phi^{i+k-y-z}\varphi^{j+z+k-x},\ P_8 = \phi^{j+z+k-x}\varphi^{i+k-y-z},\ P_9 = \phi^{2k+i+j-x-y},$$

$$P_{10} = \varphi^{2k+i+j-x-y},\ P_{11} = F_{i+k-y-z}F_{j+k+z-x}.$$

From (3.1), we see that

$$\alpha = F_{n+i+x-z}F_{n+j+y+z}$$

$$\alpha = \left(\frac{\phi^{n+i+x-z} - \varphi^{n+i+x-z}}{\sqrt{5}}\right)\left(\frac{\phi^{n+j+y+z} - \varphi^{n+j+y+z}}{\sqrt{5}}\right)$$

$$\alpha = \frac{\phi^{2n+i+x+j+y}}{5} - \frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5} - \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5} + \frac{\varphi^{2n+i+x+j+y}}{5}$$

$$\alpha = P_1 - P_3 - P_4 + P_2 \tag{3.5}$$

From (3.2), we see that

$$\beta = F_{n+x+y-k}F_{n+i+j+k}$$

$$\beta = \left(\frac{\phi^{n+x+y-k} - \varphi^{n+x+y-k}}{\sqrt{5}}\right)\left(\frac{\phi^{n+i+j+k} - \varphi^{n+i+j+k}}{\sqrt{5}}\right)$$

$$\beta = \frac{\phi^{2n+i+x+j+y}}{5} - \frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5} - \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5} + \frac{\varphi^{2n+i+x+j+y}}{5}$$

$$\beta = P_1 - P_5 - P_6 + P_2 \tag{3.6}$$

Deducting (3.6) from (3.5) gives

$$\alpha - \beta = (P_5 + P_6) - (P_3 + P_4) \tag{3.7}$$

From (3.7), let

$$V_1 = -(P_3 + P_4)$$

$$V_2 = (P_5 + P_6)$$

then

$$V_1 = -\left(\frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5} + \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5}\right)$$

$$V_1 = -\frac{(\phi\varphi)^{n+x+y-k}}{(\phi\varphi)^{n+x+y-k}}\left(\frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5} + \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5}\right)$$

$$V_1 = -\frac{1}{5}((\phi\varphi)^{n+x+y-k})\left(\frac{\phi^{n+i+x-z}}{\phi^{n+x+y-k}}\frac{\varphi^{n+j+y+z}}{\varphi^{n+x+y-k}} + \frac{\phi^{n+j+y+z}}{\phi^{n+x+y-k}}\frac{\varphi^{n+i+x-z}}{\varphi^{n+x+y-k}}\right)$$

$$V_1 = -\frac{1}{5}((\phi\varphi)^{n+x+y-k})(\phi^{i+k-y-z}\varphi^{j+z+k-x} + \phi^{j+z+k-x}\varphi^{i+k-y-z})$$

Note that

$$\phi\varphi = -1$$

So

$$V_1 = -\frac{1}{5}((-1)^{n+x+y-k})(P_7 + P_8)$$

Also,

$$V_2 = \left(\frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5} + \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5}\right)$$

$$V_2 = \frac{(\phi\varphi)^{n+x+y-k}}{(\phi\varphi)^{n+x+y-k}}\left(\frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5} + \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5}\right)$$

$$V_2 = \frac{1}{5}((\phi\varphi)^{n+x+y-k})\left(\frac{\phi^{n+x+y-k}}{\phi^{n+x+y-k}}\frac{\varphi^{n+i+j+k}}{\varphi^{n+x+y-k}} + \frac{\phi^{n+i+j+k}}{\phi^{n+x+y-k}}\frac{\varphi^{n+x+y-k}}{\varphi^{n+x+y-k}}\right)$$

$$V_2 = \frac{1}{5}((\phi\varphi)^{n+x+y-k})(\phi^0\varphi^{2k+i+j-x-y} + \phi^{2k+i+j-x-y}\varphi^0)$$

$$V_2 = \frac{1}{5}((\phi\varphi)^{n+x+y-k})(\phi^{2k+i+j-x-y} + \varphi^{2k+i+j-x-y})$$

Note that

$$\phi\varphi = -1$$

So

$$V_2 = \frac{1}{5}((-1)^{n+x+y-k})(P_9 + P_{10})$$

Now, we see from (3.7) that

$$\alpha - \beta = (P_5 + P_6) - (P_3 + P_4)$$

$$\alpha - \beta = V_1 + V_2$$

$$\alpha - \beta = \frac{1}{5}((-1)^{n+x+y-k})(P_9 - P_7 - P_8 + P_{10}) \tag{3.8}$$

From (3.3), we see that

$$\gamma = (-1)^{n+x+y-k} F_{i+k-y-z} F_{j+k+z-x}$$

$$\gamma = (-1)^{n+x+y-k}(P_{11}) \tag{3.9}$$

But

$$P_{11} = F_{i+k-y-z} F_{j+k+z-x}$$

$$P_{11} = \left(\frac{\phi^{i+k-y-z} - \varphi^{i+k-y-z}}{\sqrt{5}}\right)\left(\frac{\phi^{j+k+z-x} - \varphi^{j+k+z-x}}{\sqrt{5}}\right)$$

$$P_{11} = \frac{1}{5}(\phi^{i+k-y-z} - \varphi^{i+k-y-z})(\phi^{j+k+z-x} - \varphi^{j+k+z-x})$$

$$P_{11} = \frac{1}{5}(\phi^{2k+i+j-x-y} - \phi^{i+k-y-z}\varphi^{j+z+k-x} - \phi^{j+z+k-x}\varphi^{i+k-y-z} + \varphi^{2k+i+j-x-y})$$

$$P_{11} = \frac{1}{5}(P_9 - P_7 - P_8 + P_{10}) \tag{3.10}$$

So, putting (3.10) in (3.9) gives

$$\gamma = \frac{1}{5}((-1)^{n+x+y-k})(P_9 - P_7 - P_8 + P_{10}) \tag{3.11}$$

Since (3.8) equals (3.11) then, (3.4) is true which completes the proof.

Proof of identity 2: We know that the formula for generating nth Lucas number is, if

$$\phi = \frac{1+\sqrt{5}}{2}, \varphi = \frac{1-\sqrt{5}}{2}$$

then

$$L_n = \phi^n + \varphi^n$$

where L_n is the nth Lucas number.

Let

$$Q_1 = \phi^{2n+i+x+j+y},\ Q_2 = \varphi^{2n+i+x+j+y},\ Q_3 = \phi^{n+i+x-z}\varphi^{n+j+y+z},$$
$$Q_4 = \phi^{n+j+y+z}\varphi^{n+i+x-z},\ Q_5 = \phi^{n+x+y-k}\varphi^{n+i+j+k},\ Q_6 = \phi^{n+i+j+k}\varphi^{n+x+y-k}.$$

From (2.4), we see that

$$\alpha_1 = L_{n+i+x-z} L_{n+j+y+z}$$

$$\alpha_1 = (\phi^{n+i+x-z} + \varphi^{n+i+x-z})(\phi^{n+j+y+z} + \varphi^{n+j+y+z})$$

$$\alpha_1 = \phi^{2n+i+x+j+y} + \phi^{n+i+x-z}\varphi^{n+j+y+z} + \phi^{n+j+y+z}\varphi^{n+i+x-z} + \varphi^{2n+i+x+j+y}$$

$$\alpha_1 = Q_1 + Q_3 + Q_4 + Q_2 \tag{3.12}$$

From (2.5), we see that

$$\beta_1 = L_{n+x+y-k}L_{n+i+j+k}$$
$$\beta_1 = (\phi^{n+x+y-k} + \varphi^{n+x+y-k})(\phi^{n+i+j+k} + \varphi^{n+i+j+k})$$

$$\beta_1 = \phi^{2n+i+x+j+y} + \phi^{n+x+y-k}\varphi^{n+i+j+k} + \phi^{n+i+j+k}\varphi^{n+x+y-k} + \varphi^{2n+i+x+j+y}$$

$$\beta_1 = Q_1 + Q_5 + Q_6 + Q_2 \tag{3.13}$$

Deducting (3.13) from (3.12) gives

$$\alpha_1 - \beta_1 = (Q_3 + Q_4) - (Q_5 + Q_6) \tag{3.14}$$

From (3.14), let

$$U_1 = (Q_3 + Q_4)$$
$$U_2 = -(Q_5 + Q_6)$$

then

$$U_1 = \phi^{n+i+x-z}\varphi^{n+j+y+z} + \phi^{n+j+y+z}\varphi^{n+i+x-z}$$
$$U_1 = \left(\frac{(\phi\varphi)^{n+j+y+z}}{(\phi\varphi)^{n+j+y+z}}\right)(\phi^{n+i+x-z}\varphi^{n+j+y+z} + \phi^{n+j+y+z}\varphi^{n+i+x-z})$$
$$U_1 = ((\phi\varphi)^{n+j+y+z})\left(\frac{\phi^{n+i+x-z}}{\phi^{n+j+y+z}}\frac{\varphi^{n+j+y+z}}{\varphi^{n+j+y+z}} + \frac{\phi^{n+j+y+z}}{\phi^{n+j+y+z}}\frac{\varphi^{n+i+x-z}}{\varphi^{n+j+y+z}}\right)$$

$$U_1 = ((\phi\varphi)^{n+j+y+z})(\phi^{i+x-j-y-2z} + \varphi^{i+x-j-y-2z})$$

Note that

$$\phi\varphi = -1$$

So

$$U_1 = ((-1)^{n+j+y+z})L_{i+x-j-y-2z}$$

Also,

$$U_2 = -(\phi^{n+i+x-z}\varphi^{n+j+y+z} + \phi^{n+j+y+z}\varphi^{n+i+x-z})$$

$$U_2 = -\left(\frac{(\phi\varphi)^{n+i+j+k}}{(\phi\varphi)^{n+i+j+k}}\right)(\phi^{n+x+y-k}\varphi^{n+i+j+k} + \phi^{n+i+j+k}\varphi^{n+x+y-k})$$

$$U_2 = -((\phi\varphi)^{n+i+j+k})\left(\frac{\phi^{n+x+y-k}}{\phi^{n+i+j+k}}\frac{\varphi^{n+i+j+k}}{\varphi^{n+i+j+k}} + \frac{\phi^{n+i+j+k}}{\phi^{n+i+j+k}}\frac{\varphi^{n+x+y-k}}{\varphi^{n+i+j+k}}\right)$$

$$U_2 = -((\phi\varphi)^{n+i+j+k})(\phi^{x+y-i-j-2k} + \varphi^{x+y-i-j-2k})$$

Note that

$$\phi\varphi = -1$$

So

$$U_2 = -((-1)^{n+i+j+k})L_{x+y-i-j-2k}$$

$$U_2 = ((-1)^{n+i+j+k+1})L_{x+y-i-j-2k}$$

Now, we see from (3.14) that

$$\alpha_1 - \beta_1 = (Q_3 + Q_4) - (Q_5 + Q_6)$$

$$\alpha_1 - \beta_1 = U_1 + U_2$$

$$\alpha_1 - \beta_1 = ((-1)^{n+j+y+z})L_{i+x-j-y-2z} + ((-1)^{n+i+j+k+1})L_{x+y-i-j-2k} \tag{3.15}$$

Since (2.6) equals (3.15) then, (2.7) is true which completes the proof.

4 Conclusion

In this paper, we have stated and proved two identities which involve Fibonacci numbers and Lucas numbers. Binet's formulas for generating nth Fibonacci number and nth Lucas number were used in proving the identities.

References

[1] Ashwini Panwar, Kiran Sisodiya, G. P. S., Rathore, *On Some identities for Generalized(k,r) Fibonacci Numbers*, International Journal of Mathematics Trends and Technology (IJMTT), **51** (2017), 146–148.

[2] Alongkot Suvarnamani, *Some properties of the Generalized (p,r)-Fibonacci-Like number*, MATEC Web of Conferences, **189** (2018), 1–6.

[3] Jeffrey R. Chasnov, *Fibonacci Numbers and the Golden Ratio*, The Hong kong University of science and technology, Hong Kong,(2016), 5.

[4] Michael Z. Spivey, *Fibonacci identities via the Determinant Sum Property*, The College Mathematics Journal, **37** (2006), 286—289.

[5] Steven Vajda, *Fibonacci and Lucas Numbers, and the Golden Section:Theory and Applications*, Dover Publications, Inc, New York, 1989.

AMS Mathematical Number: 11B39