THE GENERALISED FERMAT EQUATION $Pa^x + Qb^y = Rc^z$ AND RELATED PROBLEMS

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ABSTRACT. The focus of this paper is the generalised Fermat equation, $Pa^x + Qb^y = Rc^z$, considered by Henri Darmon and Andrew Granville. It is closely related to a family of theorems and conjectures including the Fermat-Catalan Conjecture, the Darmon-Granville Theorem, the Beal Conjecture (also known as the Tijdeman-Zagier Conjecture) and Fermat's Last Theorem. We will consider these briefly before offering a proof that no solutions exist even for $P, Q, R > 1$, for cases $x, y, z > 2$, using a new binomial identity for $a^x + b^y$ to an indeterminate power, z. The proof extends to its corollaries the Beal Conjecture and Fermat's Last Theorem.

Introduction

The generalised Fermat equation, $Pa^x + Qb^y = Rc^z$, is part of a family of related theorems and conjectures, where a, b, c, P, Q, R are square-free integers, $\gcd(a, b, c, P, Q, R) = 1$, and $x, y, z \in \mathbb{Z}$. We list them here, briefly.

The Fermat-Catalan Conjecture

The Fermat-Catalan Conjecture states[1] that *if* $P, Q, R = 1$ *and* $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$, (the hyperbolic case), *the equation*

$$a^x + b^y = c^z$$

has only finite solutions.

Only the following ten solutions are currently known:

$$1^7 + 2^3 = 3^2,$$

$$2^5 + 7^2 = 3^4,$$

$$7^3 + 13^2 = 2^9,$$

$$2^7 + 17^3 = 71^2,$$

$$3^5 + 11^4 = 122^2,$$

$$17^7 + 76271^3 = 21063928^2,$$

$$1414^3 + 2213459^2 = 65^7$$

$$9262^3 + 15312283^2 = 113^7,$$

$$43^8 + 96222^3 = 30042907^2,$$

$$33^8 + 1549034^2 = 15613^3.$$

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[1] "Computational Number Theory", in Gowers, Timothy; Barrow-Green, June; Leader, Imre (eds.), The Princeton Companion to Mathematics, Princeton University Press, p.360

The Darmon and Granville Theorem

In 1994, continuing with the hyperbolic case, Henri Darmon and Andrew Granville using Faltings' Theorem proved that *if* P, Q, R, a, b, c *are fixed positive integers with*

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1,$$

then the equation

$$Pa^x + Qb^y = Rc^z$$

has at most finitely many solutions in coprime non-zero integers a*,* b *and* c*.*

The proof is considered very elegant and is outlined briefly by Bennett, Mihailescu and Siksek[2]. However, in this paper we go beyond the parameters of the hyperbolic case. We conjecture and then prove that no integer solutions exist even for values $x, y, z > 2$.

The Beal Conjecture (Tijdeman-Zagier Conjecture)

In the mid 1990's, a Texan banker called Andrew Beal noted that the smallest exponent in all ten solutions for the hyperbolic case (where $P, Q, R = 1$) was 2. He therefore conjectured that *for the equation*

$$a^x + b^y = c^z$$

no whole number solutions exist for cases $x, y, z > 2$*.*

Alternatively stated, *if* $a^x + b^y = c^z$*, where* a, b, c, x, y, z *are fixed positive integers and* $x, y, z > 2$*, then* a, b, c *must have a common prime factor.*
This has become known as the Beal Conjecture (also known as the Tijdeman-Zagier Conjecture)[3]. This conjecture remains unsolved.

Fermat's Last Theorem

The famous corollary of this conjecture is Fermat's Last Theorem, for the case $x, y, z = n$. It took about 350 years before a proof was finally discovered by Sir Andrew Wiles in 1993,[4] proving that *for the equation*

$$a^n + b^n = c^n$$

no whole number solutions exist for cases $n > 2$*.*

An elementary approach
In their paper Bennett, Mihailescu and Siksek survey different approaches to the this family of equations, including, among others, cyclotomic fields, elliptic curves

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[2] *The Generalized Fermat Equation* Michael Bennett, Preda Mihailescu and Samir Siksek, https://homepages.warwick.ac.uk/ maseap/papers/bealconj.pdf, p24

[3] See www.bealconjecture.com. Last accessed 14.12.17.

[4] Wiles, Andrew (1995). "Modular elliptic curves and Fermat's Last Theorem". Annals of Mathematics. 141 (3): 443–551.

and modular forms, and Galois representations.[5] Here we return to a more elementary approach.

Proving that no whole number solutions exist is problematic using elementary methods, since there are an infinite number of cases especially when multiple exponents, x, y, z, are involved. It may be easier to find counterexamples, but becomes harder to prove. Using a 'horizontal' approach, on a case-by-case basis as Fermat and his successors began to do for cases of n, would take forever. A 'vertical' approach, like infinite descent, would seem much better suited. But a 'vertical' approach also has its problems. As Peter Schorer warns, one of the inherent problems of proving the theorem using a 'vertical' approach appears to be that when one assumes that a counterexample exists and then tries to derive a contradiction, the very properties that created the contradiction in the first place appear to belong also to the *non*-counterexample.[6]

However, using binomial theorem we can circumvent this problem by expressing both sides of the equation $Pa^x + Qb^y = Rc^z$ in terms of a, b and z, effectively isolating z from x, y and opening up a way for a simple proof by contradiction for $z > 2$.

Theorem 0.1. *To demonstrate that for the Fermat equation $Pa^x + Qb^y = Rc^z$, where a, b, c, P, Q, R are square-free integers (of which one of Pa, Qb, Rc at most must be even), and $\gcd(a, b, c, P, Q, R) = 1$, no integer solutions exist for the values of $x, y, z_{\geq 3} \in \mathbb{N}$.*

We first observe the following identity for $Pa^x + Qb^y$ as a binomial expansion (where the upper index n is an indeterminate integer):

$$Pa^x + Qb^y = \sum_{k=0}^{n} \binom{n}{k} (a+b)^{n-k}(-ab)^k(Pa^{x-n-k} + Qb^{y-n-k}). \tag{0.1}$$

Note how this new identity includes *standard* factors for a binomial expansion, i.e. $(a+b)^{n-k}(-ab)^k$, but also a *non-standard* factor, i.e. $(Pa^{x-n-k} + Qb^{y-n-k})$.

Note, further, that regardless of the value of n, the right hand side always equals $Pa^x + Qb^y$. This allows us to fix n to any value we choose. So let $n = z$, such that:

$$Pa^x + Qb^y = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k(Pa^{x-z-k} + Qb^{y-z-k}). \tag{0.2}$$

Proof. We now assume that a solution exists for the equation $Pa^x + Qb^y = Rc^z$ for values of $x, y, z > 2$.

[5] *The Generalized Fermat Equation* Michael Bennett, Preda Mihailescu and Samir Siksek, https://homepages.warwick.ac.uk/ maseap/papers/bealconj.pdf, p24

[6] Peter Schorer "Is There a "Simple" Proof of Fermat's Last Theorem? Part (1) Introduction and Several New Approaches", 2014, www.occampress.com/fermat.pdf.

Now let s, t be dependent variables, where $s, t \in \mathbb{Z}$, $gcd(s, ab) = 1$, and $s \neq 0$, such that $[(a+b)s - abt] = c$. From this it follows that:

$$\sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k (Pa^{x-z-k} + Qb^{y-z-k}) = R[(a+b)s - abt]^z. \tag{0.3}$$

Using the binomial theorem we divide both sides of (0.3) by R and expand as:

$$\frac{1}{R}\sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k (Pa^{x-z-k} + Qb^{y-z-k}) = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k s^{z-k} t^k. \tag{0.4}$$

We know that the right hand side is a power to z since all the components have the correct exponential form for a standard binomial expansion to power z; the left hand side may or may not be. Without the independent variables, $(a+b)$ and $(-ab)$, there could be other circumstances when the left hand side of (0.4) is a power to z. But since s and t are dependent on and inseparably tied to the independent variables, $(a+b)$ and $(-ab)$, and must therefore conform to the standard binomial exponential form, the *only* circumstances when the left hand side can be a power to z are when the following equation holds true for every k^{th} term, for any given value of z, where $0 \leq k \leq z$:

$$\frac{1}{R}(Pa^{x-z-k} + Qb^{y-z-k}) = s^{z-k}t^k. \tag{0.5}$$

On this basis, we can now complete the proof. Since it is true that:

$$\left(\frac{s^{z-1}t}{s.t^{z-1}}\right)^z = \left(\frac{s^z}{t^z}\right)^{z-2}, \tag{0.6}$$

we can deduce values from (0.5), using $k = 0, 1, z-1, z$, and cancelling R, such that:

$$\left(\frac{Pa^{x-z-1} + Qb^{y-z-1}}{Pa^{x-2z+1} + Qb^{y-2z+1}}\right)^z = \left(\frac{Pa^{x-z} + Qb^{y-z}}{Pa^{x-2z} + Qb^{y-2z}}\right)^{(z-2)}. \tag{0.7}$$

In Theorem 0.2 we prove that both these fractions are uniquely determined. This being the case, we prove in Theorem 0.3 that the since the denominators (to their respective outer exponents) cannot be equal, there can be no solutions to (0.7).

Theorem 0.2. *To prove that the fractions in (0.7) are uniquely determined.*

We can prove this by showing that if $(Pa^{x-2z+1} + Qb^{y-2z+1})$ divides $(Pa^{x-z-1} + Qb^{y-z-1})$ then $(Pa^{x-2z} + Qb^{y-2z})$ cannot also divide $(Pa^{x-z} + Qb^{y-z})$.

Proof. Using proof by contradiction, we assume that if $(Pa^{x-2z+1} + Qb^{y-2z+1}) | (Pa^{x-z-1} + Qb^{y-z-1})$ then $(Pa^{x-2z} + Qb^{y-2z}) | (Pa^{x-z} + Qb^{y-z})$. To simplify, temporarily let $(x-z-1) = d, (y-z-1) = e, (x-2z+1) = f, (y-2z+1) = g$ such that:

$$\left(\frac{Pa^d + Qb^e}{Pa^f + Qb^g}\right) \text{and} \left(\frac{Pa^{d+1} + Qb^{e+1}}{Pa^{f-1} + Qb^{g-1}}\right).$$

We are assuming, then, that if $(Pa^f + Qb^g) | (Pa^d + Qb^e)$ then $(Pa^{f-1} + Qb^{g-1}) | (Pa^{d+1} + b^{e+1})$.

First, if $(Pa^f + Qb^g)|(Pa^d + Qb^e)$, then let j be an integer such that:

$$j(Pa^f + Qb^g) = (Pa^d + Qb^e), \tag{0.8}$$

and if $(Pa^{f-1} + Qb^{g-1})|(Pa^{d+1} + Qb^{e+1})$, then let k be an integer such that:

$$k(Pa^{f-1} + Qb^{g-1}) = (Pa^{d+1} + Qb^{e+1}). \tag{0.9}$$

We also note the following identities:

$$(Pa^{d+1} + Qb^{e+1}) = b(Pa^d + Qb^e) - Pa^d(b-a) \tag{0.10}$$

and

$$(Pa^{f-1} + Qb^{g-1}) = \frac{(Pa^f + Qb^g) + Pa^{f-1}(b-a)}{b}. \tag{0.11}$$

Substituting terms from (0.8)-(0.11) and rearranging, it follows that:

$$k[(Pa^f + Qb^g) + Pa^{f-1}(b-a)] = b[bj(Pa^f + Qb^g) - Pa^d(b-a)]. \tag{0.12}$$

If we multiply out and rearrange we get:

$$kPa^f + kQb^g + kPa^{f-1}b - kPa^f = jb^2Pa^f + jQb^{g+2} - b^2Pa^d + bPa^{d+1}, \tag{0.13}$$

$$\Rightarrow a(kPa^{f-1} + kPa^{f-2}b - kPa^{f-1} - jb^2Pa^{f-1} + b^2Pa^{d-1} - bPa^d) = b(jQb^{g+1} - kQb^{g-1}), \tag{0.14}$$

$$\Rightarrow a(kPa^{f-2}b - jb^2Pa^{f-1} + b^2Pa^{d-1} - bPa^d) = b(jQb^{g+1} - kQb^{g-1}). \tag{0.15}$$

Since $\gcd(a,b) = 1$ it follows that:

$$a = Qb^{g-1}(jb^2 - k), \tag{0.16}$$

and

$$b = (kPa^{f-2}b - jb^2Pa^{f-1} + b^2Pa^{d-1} - bPa^d). \tag{0.17}$$

We can ignore (0.17). Rearranging (0.16) we get:

$$\frac{a}{Qb^{g-1}} = (jb^2 - k). \tag{0.18}$$

However, since $\gcd(a,Q) = 1$ the left hand side is an irreducible fraction. But since j, b, k are integers it follows that:

$$\frac{a}{Qb^{g-1}} \neq (jb^2 - k). \tag{0.19}$$

With this contradiction, it implies that our initial assumption is false. The two fractions in (0.7) cannot simultaneously be integers and must be uniquely determined. □

Since the two fractions are uniquely determined, solutions must exist to (0.7) if the numerators (to their respective outer exponents) on both sides are equal, *and* simultaneously if the denominators (to their respective outer exponents) on both sides are equal.

Theorem 0.3. *To prove that the denominators (to their respective outer exponents) on both sides cannot be equal.*

Proof. Since $(Pa^{x-2z+1} + Qb^{y-2z+1})^z > (Pa^{x-2z} + Qb^{y-2z})^{z-2}$ the denominators cannot be equal. There is no need to consider the numerators. □

From Theorems 0.2 and 0.3, it follows that no solutions exist for the equation in (0.7). Therefore, for any given value of $z > 2$, and all values of k:

$$s^{z-k}t^k \neq (Pa^{x-z-k} + Qb^{y-z-k}). \tag{0.20}$$

However, this contradicts our equation in (0.5). In turn, therefore, the left hand side of the equation in (0.4) cannot be a perfect power (as we assumed it was). And so our initial assumption that solutions exist for the equation $Rc^z = Pa^x + Qb^y$ for values of $x, y, z > 2$ is false. Therefore the conjecture is true. □

What happens for the cases for $z = 1, 2$? Well, from (0.7), when $z = 1$ it follows that:

$$\left(\frac{Pa^{x-2} + Qb^{y-2}}{Pa^{x-1} + Qb^{y-1}}\right)^1 = \left(\frac{Pa^{x-1} + Qb^{y-1}}{Pa^{x-2} + Qb^{y-2}}\right)^{-1}, \tag{0.21}$$

$$\Rightarrow \left(\frac{Pa^{x-2} + Qb^{y-2}}{Pa^{x-1} + Qb^{y-1}}\right) = \left(\frac{Pa^{x-2} + Qb^{y-2}}{Pa^{x-1} + Qb^{y-1}}\right). \tag{0.22}$$

No contradiction.

And again from (0.7), when $z = 2$, it follows that:

$$\left(\frac{Pa^{x-3} + Qb^{y-3}}{Pa^{x-3} + Qb^{y-3}}\right)^2 = \left(\frac{Pa^{x-2} + Qb^{y-2}}{Pa^{x-4} + Qb^{y-4}}\right)^0, \tag{0.23}$$

$$\Rightarrow 1 = 1. \tag{0.24}$$

Again, no contradiction.

So in both cases, when $z = 1$ and when $z = 2$, there is no contradiction. Our non-standard binomial factor, $(Pa^{x-z-k} + Qb^{y-z-k})$ is equal to $s^{z-k}t^k$ for every value of k (when $z = 1, 2$).

This, in turn, proves both the corollaries, the Beal Conjecture and Fermat's Last Theorem.

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