The Collatz Conjecture and the Quantum Mechanical Harmonic Oscillator

Ramon Carbo-Dorca
Institute of Computational Chemistry and Catalysis
Universitat de Girona
Girona, Spain
ramoncarbodorca@gmail.com
Carlos Castro Perelman
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, GA. 30314
Ronin Institute, 127 Haddon Pl., NJ. 07043
perelmanc@hotmail.com

March 21, 2020

Abstract

By establishing a dictionary between the QM harmonic oscillator and the Collatz process, it reveals very important clues as to why the Collatz conjecture most likely is true. The dictionary requires expanding any integer n into a binary basis (bits) $n = \sum a_{nl} 2^l$ (l ranges from 0 to $N-1$) that allows to find the correspondence between every integer n and the state $|\Psi_n\rangle$, obtained by a superposition of bit states $|l\rangle >$, and which are related to the energy eigenstates of the QM harmonic oscillator. In doing so, one can then construct the one-to-one correspondence between the Collatz iterations of numbers $n \to \frac{n}{2}$ (n even); $n \to 3n+1$ (n odd) and the operators $\mathbf{L}_{\frac{n}{2}}; \mathbf{L}_{3n+1}$, which map Ψ_n to $\Psi_{\frac{n}{2}}$, or to Ψ_{3n+1}, respectively, and which are constructed explicitly in terms of the creation $\mathbf{a}$, annihilation $\mathbf{a}^\dagger$, and unit operator $\mathbf{1}$ of the QM harmonic oscillator. A rigorous analysis reveals that the Collatz conjecture is most likely true, if the composition of a chain of $\mathbf{L}_{\frac{n}{2}}; \mathbf{L}_{3n+1}$ operators (written as L_* in condensed notation) obey the conditions $\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* = \mathcal{P}$, where $\mathcal{P}$ is the operator that *projects* any state Ψ_n into the ground state $\Psi_1 \equiv |0\rangle >$ representing the zero bit state $|0\rangle$ (since $2^0 = 1$). In essence, one has a realization of the integer/state correspondence typical of QM such that the Collatz paths from n to 1 are encoded in terms of quantum transitions among the states Ψ_n, and leading effectively to an overall downward cascade to Ψ_1.

1 Introduction

In the Wikipedia page it states that the Collatz conjecture is named after Lothar Collatz, who introduced the idea in 1937, two years after receiving his doctorate. [1]. It is also known as the $3n + 1$ problem, the $3n + 1$ conjecture, the Ulam conjecture (after Stanislaw Ulam), Kakutani's problem (after Shizuo Kakutani), the Thwaites conjecture (after Sir Bryan Thwaites), Hasse's algorithm (after Helmut Hasse), or the Syracuse problem. The sequence of numbers involved is sometimes referred to as the hailstone sequence or hailstone numbers (because the values are usually subject to multiple descents and ascents like hailstones in a cloud. As of 2020, the conjecture has been checked by computer for all starting values up to $2^{68} \sim 2.95 \times 10^{20}$ [3].

The conjecture is based on a very simple iteration process. For any positive even integer greater than 1, the rule is $n \to \frac{n}{2}$. And for n odd, one takes $n \to 3n + 1$, leading to an even number. The Collatz conjecture states that starting from any integer $n > 1$, the Collatz iteration process **always** ends up at 1, after a finite number of steps. This is not the case for negative integers since there are nontrivial cycles. For instance $\{-5, -14, -7, -20, -10, -5, \ldots\}$ and one does not reach -1 . The Collatz process has the trivial cycle $\{1, 4, 2, 1, 4, 2, \ldots\}$ if one continues iterating after reaching 1. A computer program would never end in this case, so once we reach the final destination at 1, the process should stop.

Inspired by the work of one of us (RCD) [4], [5] on Boolean Hypercubes, Natural Vector Spaces and the Collatz conjecture, we shall show next how one can establish a dictionary between operators associated with the QM harmonic oscillator and the Collatz process which reveals very important clues as to why the Collatz conjecture most likely is true. More precisely, we show that it is the very special decomposition of $3n + 1 = n + 2n + 1$ which leads to the operator representation for the process $\Psi_n \to \Psi_{3n+1}$ to have the key **diagonal** form $(\mathbf{I} + \mathcal{L}^+ + \mathcal{P})_{n=odd}$, where $\mathbf{I}$ is the identity operator, $\mathcal{L}^+$ is a ladder operator that increases the bit number by one, and $\mathcal{P}_{n=odd}$ is a projection operator that maps any state Ψ_n into the ground state $\Psi_1 \equiv |0\rangle >$ representing the zero bit state $|0\rangle$ (since $2^0 = 1$).

2 Dictionary between the QM oscillator and the Collatz process

The following dictionary between the QM oscillator and the Collatz process reveals very important clues as to why the Collatz conjecture might be true. Given an arbitrary number n it admits the binary expansion $\sum_{l=0}^{N-1} a_{nl} 2^l$, and whose binary coefficients are $a_{nl} = \{0, 1\}$. The one-to-one correspondence between the number n and the state Ψ_n is

$$n \ = \ \sum_{l=0}^{N-1} \ a_{nl} \ 2^l \ \leftrightarrow \ |\Psi_n > \ \ \equiv \ \sum_{l=0}^{N-1} \ a_{nl} \ |l > \qquad (1)$$

where the superposition of bit states $|l>$, which are associated with the numbers 2^l, ranges from $l=0$ to $N-1$. The state $|\Psi_n>$ is a *superposition* of bit states $|l>$ and is a close relative of the coherent state $|z> \equiv e^{-\frac{|z|^2}{2}} \sum e^{za^\dagger}|0>$ with z a complex number. The state $|\Psi_n>$ can be represented by a *column* vector whose entries are $\{a_{n0}|0>, a_{n1}|1>, a_{n2}|2>, \ldots, a_{nN-1}|N-1>\}$.

When n is even, the first Collatz iteration yields $\frac{n}{2}$, therefore one has a transition from the state $|\Psi_n>$ to the state $|\Psi_{\frac{n}{2}}>$ given by

$$|\Psi_{\frac{n}{2}}> \;\; \equiv \;\; \sum_{l=0}^{N-1} \; a_{nl} \; |l-1> \;\; = \;\; \sum_{l=1}^{N-1} \; a_{nl} \; |l-1> \tag{2}$$

since dividing by two amounts to removing one bit $\frac{1}{2}2^l = 2^{l-1}$, and the first binary coefficient $a_{n0}=0$ is always *zero* when n is even.

When n is odd, the first Collatz iteration yields $3n+1$. The binary expansion of $3n+1$ (an even number) is

$$3n+1 \;=\; 1 \;+\; \sum_{l=0}^{N-1} \; a_{nl} \; 2^l \;+\; \sum_{l=0}^{N-1} \; a_{nl} \; 2^{l+1} \tag{3}$$

therefore one has a transition from the state $|\Psi_n>$ to the state $|\Psi_{3n+1}>$ given by

$$|\Psi_{3n+1}> \;\; \equiv \;\; |0> \;\; + \;\; \sum_{l=0}^{N-1} a_{nl} \; |l> \;\; + \;\; \sum_{l=0}^{N-1} a_{nl} \; |l+1> \tag{4}$$

since multiplying by two amounts to adding one bit $2\;2^l = 2^{l+1}$.

Given the binary expansion of any even number n in eq-(1), the first binary coefficient of an even number n is always $a_{n0}=0$ zero, and in general one will have a set of non-zero binary coefficients at the specific locations $l_1, l_2, l_3, \ldots$, Namely $a_{nl_1} = a_{nl_2} = a_{nl_3} = \ldots = 1$ and the rest of the binary coefficients are zero. One may have all the binary coefficients to be non-vanishing except the first one $a_{n0}=0$. Or one may have all the binary coefficients to be vanishing except the last one $a_{nN-1}=1$. And so forth. Using the well known relations involving the action of the creation and annihilation operators on the energy eigenstates $|l>$ $(l=0,1,2,.....)$ of a harmonic oscillator $\mathbf{a}^\dagger|l> = \sqrt{l+1}\;|l+1>$ and $\mathbf{a}|l> = \sqrt{l}\;|l-1>$, one learns that the operator $\mathbf{L}_{\frac{n}{2}}$ which maps the state $|\Psi_n>$ to $|\Psi_{\frac{n}{2}}>$ (when n is even) is given by

$$\mathbf{L}_{\frac{n}{2}} \;=\; Diag\left(\;\mathbf{1},\; \mathbf{1}, \ldots, \frac{\mathbf{a}}{\sqrt{l_1}},\; \mathbf{1},\; \mathbf{1}, \ldots,\; \frac{\mathbf{a}}{\sqrt{l_2}}, \mathbf{1},\; \mathbf{1}, \ldots,\; \frac{\mathbf{a}}{\sqrt{l_3}},\; \ldots\; \right) \tag{5}$$

which is a diagonal $N \times N$ matrix and whose entries are comprised of the annihilation operator $\mathbf{a}$ (up to judicious numerical coefficients) and the identity operator $\mathbf{1}$. The location of the unit operators $\mathbf{1}$ in eq-(5) correspond to the location of the *vanishing* binary coefficients. And the $\mathbf{a}$ operators will *reduce* the bits by one unit at each specific location where the binary coefficients are non-vanishing. For this reason we may rewrite $\mathbf{L}_{\frac{n}{2}} = \mathcal{L}^-$, where $\mathcal{L}^-$ plays the role of a ladder operator that reduces the bits by one unit; i.e. it reduces the size of the column vector by one.

The operator $\mathbf{L}_{3n+1}$ that maps $|\Psi_n>$ to $|\Psi_{3n+1}>$ (when n is odd) is given by

$$\mathbf{L}_{3n+1} \;=\; Diag\,(\,\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ldots,\mathbf{1})\;+$$

$$Diag\left(\mathbf{a}^\dagger,\ \mathbf{1},\ \mathbf{1},\ \ldots,\ \frac{\mathbf{a}^\dagger}{\sqrt{l_1+1}},\ \mathbf{1},\ \mathbf{1},\ \ldots,\frac{\mathbf{a}^\dagger}{\sqrt{l_2+1}},\ \mathbf{1},\ \mathbf{1},\ \ldots,\ \frac{\mathbf{a}^\dagger}{\sqrt{l_3+1}},\ \ldots\ \right)+$$

$$Diag\,(\ \mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \ldots\mathbf{a}^{l_1+1},\ \mathbf{1},\ \mathbf{1},\ \ldots\ \mathbf{a}^{l_2+1},\mathbf{1},\ \mathbf{1},\ \ldots\ \mathbf{a}^{l_3+1},\ \ldots\ ) \qquad (6)$$

which is a diagonal $N\times N$ matrix whose entries are comprised of creation $\mathbf{a}^\dagger$ and annihilation operators $\mathbf{a}$, in addition to the identity operators $\mathbf{1}$. We may rewrite $\mathbf{L}_{3n+1}$ in condensed notation as $\mathbf{I}+\mathcal{L}^+ + \mathcal{P}_{odd}$, respectively. $\mathbf{I}$ is the identity operator comprised of ones along the diagonal. $\mathcal{L}^+$ is the ladder operator that increases the bits by one unit (increases the size of the column vector by one). And $\mathcal{P}_{odd}$ is the **projection operator** that maps the state Ψ_n (for n odd) to the ground state $\mathcal{P}_{odd}\Psi_n = \Psi_1 = |0>$.

Let us explain in detail the origins of the diagonal entries of $\mathcal{P}_{odd}$. For n odd, the binary coefficient $a_{n0} = 1$ is not zero, so the first unit $\mathbf{1}$ operator appearing in $\mathcal{P}_{odd}$ will act on the ground state $|0>$ giving $|0>$, as desired. The following $\mathbf{1}$'s in $\mathcal{P}_{odd}$ appear at the locations of the vanishing binary coefficients, until one hits the first non-vanishing binary coefficient at the l_1-th entry. The next non-vanishing locations are situated at the l_2-th, l_3-th, ... entries, respectively. Whereas, the location of the remaining $\mathbf{1}$'s in between correspond to the respective vanishing binary coefficients. Finally, the action of the operators $\mathbf{a}^{l_1+1};\mathbf{a}^{l_2+1};\mathbf{a}^{l_3+1},\ldots$ on the bit states $|l_1\rangle;|l_2\rangle;|l_3\rangle;\ldots$, respectively, is going to be zero. Therefore, the overall effect of the projection operator $\mathcal{P}_{odd}$ on Ψ_{odd} gives $|\Psi_1\rangle = |0\rangle$, as expected, due to the fact the $a_{n0} = 1 \neq 0$.

Therefore, one has in condensed notation

$$\mathbf{L}_{\frac{n}{2}} \;\equiv\; \mathcal{L}^-,\quad \mathbf{L}_{3n+1} \;\equiv\; \mathbf{I}+\mathcal{L}^+ + \mathcal{P}_{odd} \qquad (7)$$

For n even, the **projection operator** that maps the state Ψ_n to the ground state $\mathcal{P}_{even}\Psi_n = \Psi_1 = |0>$ is

$$\mathcal{P}_{even} \;\equiv\; Diag\left(\ \mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \ldots\frac{\mathbf{a}^{l_1}}{\sqrt{l_1!}},\ \mathbf{1},\ \mathbf{1},\ \ldots\ \mathbf{a}^{l_2+1},\mathbf{1},\ \mathbf{1},\ \ldots\ \mathbf{a}^{l_3+1},\ \ldots\ \right) \qquad (8a)$$

The action of $\frac{\mathbf{a}^{l_1}}{\sqrt{l_1!}}\,|l_1\rangle = |0\rangle$ is what projects Ψ_n (n even) into the ground state; while the action of $\mathbf{a}^{l_2+1}|l_2\rangle = 0$; $\mathbf{a}^{l_3+1}|l_3\rangle = 0,\ldots$ is what projects out the other bit states. The action of the unit operators $\mathbf{1}$ in eq-(8a) is **null** because the location of the unit operators corresponds to the location of the vanishing binary coefficients.

An important remark is in order. The expression for the projection operator $\mathcal{P}_{even}$ in eq-(8a) is **not** unique. One could have chosen instead of eq-(8a) the following projection operator

$$\mathcal{P}'_{even} \equiv Diag\left(\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \ldots \mathbf{a}^{l_1+1},\ \mathbf{1},\ \mathbf{1},\ \ldots\ \frac{\mathbf{a}^{l_2}}{\sqrt{l_2!}}, \mathbf{1},\ \mathbf{1},\ \ldots\ \mathbf{a}^{l_3+1},\ \ldots \right) \tag{8b}$$

that maps the state $|\Psi_n\rangle$ to the ground state $\mathcal{P}'_{even}|\Psi_n\rangle = |\Psi_1\rangle = |0\rangle$. And one could continue to find yet another projection operator

$$\mathcal{P}''_{even} \equiv Diag\left(\mathbf{1},\ \mathbf{1},\ \mathbf{1},\ \ldots \mathbf{a}^{l_1+1},\ \mathbf{1},\ \mathbf{1},\ \ldots\ \mathbf{a}^{l_2+1}, \mathbf{1},\ \mathbf{1},\ \ldots\ \frac{\mathbf{a}^{l_3}}{\sqrt{l_3!}},\ \ldots \right) \tag{8c}$$

and so forth. For this reason one must choose a selection criteria which avoids any ambiguities in the expression for the projections $\mathcal{P}_{even}$ in the even n case. We shall choose the expression provided by eq-(8a) which is based on the location of the **first** non-vanishing binary coefficient. In this way, the expression for the $\mathcal{P}_{odd}$ projection operator in the odd n case is the one displayed by the last term in eq-(6). The reason being that for n odd, the first non-vanishing binary coefficient is $a_{n0} = 1$.

To sum up, the following operators

$$\mathbf{I},\ \mathcal{P}_{even},\ \mathcal{P}_{odd},\ \mathcal{L}^+,\ \mathcal{L}^-,\ \mathbf{L}_{3n+1} = \mathbf{I} + \mathcal{L}^+ + \mathcal{P}_{odd},\ \mathbf{L}_{\frac{n}{2}} = \mathcal{L}^- \tag{9}$$

are the ones involved in the whole Collatz process. Except for the identity operator, all the other operators are state-dependent. Note that despite that these operators are state dependent they are still **linear** involving powers of $\mathbf{a}, \mathbf{a}^\dagger$, because the latter powers are realized as higher order linear differential operators acting on the states Ψ_n, and which in turn, are given by a superposition of energy eigenstates of the harmonic oscillator.

The full Collatz symbolic process $C_{s(n)}[n] = 1$ of an integer n, where $s(n)$ is the total number of steps required for n to reach 1, is represented (reformulated) as a concatenation of $\mathbf{L}_{(1)}, \mathbf{L}_{(2)}, \ldots \mathbf{L}_{(s(n))}$ operators acting on Ψ_n, and whose net final effect is to end at the ground state. For n even, one has

$$\mathbf{L}_{(s(n))}\ \mathbf{L}_{(s(n)-1)}\ \ldots\ \mathbf{L}_{(2)}\ \mathbf{L}_{(1)}\ |\Psi_n\rangle = \mathcal{P}_{even}\ |\Psi_n\rangle = \Psi_1 \equiv |0\rangle \tag{10a}$$

And for n odd

$$\mathbf{L}_{(s(n))}\ \mathbf{L}_{(s(n)-1)}\ \ldots\ \mathbf{L}_{(2)}\ \mathbf{L}_{(1)}\ |\Psi_n\rangle = \mathcal{P}_{odd}\ |\Psi_n\rangle = \Psi_1 \equiv |0\rangle \tag{10b}$$

The column vector representing Ψ_1 is comprised of a single entry $|0\rangle$.

Each step in the concatenation process will select also **state dependent** operators written as

$$\mathbf{L}_{(1)},\ \mathbf{L}_{(2)},\ \ldots, \mathbf{L}_{(s(n))} \tag{11}$$

for simplicity due to the complexity of the Collatz iteration process. For example, given a positive integer n, the iteration process generates the sequence of numbers (a Collatz path)

$$n \ \rightarrow \ \{n \equiv n_0, \ n_1, \ n_2, \ n_3, \ \ldots, 8, \ 4, \ 2, \ 1\} \tag{12a}$$

and the corresponding eigenstates are

$$\{\Psi_n, \ \Psi_{n_1}, \ \Psi_{n_2}, \ \Psi_{n_3}, \ \ldots, \Psi_8, \ \Psi_4, \ \Psi_2, \ \Psi_1 = |0\rangle\} \tag{12b}$$

Therefore when one writes $L_{(i)}$ it represents the operator acting on the state $\Psi_{n_i} \neq \Psi_i$, because $n_i \neq i$. Extreme caution must be taken with the notation and the indices.

The key point now is to show how one performs the sequence of operations involving the string of operators in eqs-(10a, 10b). One must, firstly, start with the operation

$$\mathbf{L}_{(1)} \ |\Psi_n\rangle \ = \ \mathbf{L}[\Psi_n] \ |\Psi_n\rangle \ = \ |\Psi_{n_1}\rangle \tag{12c}$$

Then continue

$$\mathbf{L}_{(2)} \ |\Psi_{n_1}\rangle \ = \ \mathbf{L}[\Psi_{n_1}] \ |\Psi_{n_1}\rangle \ = \ |\Psi_{n_2}\rangle \tag{12d}$$

and proceed

$$\mathbf{L}_{(3)} \ |\Psi_{n_2}\rangle \ = \ \mathbf{L}[\Psi_{n_2}] \ |\Psi_{n_2}\rangle \ = \ |\Psi_{n_3}\rangle \tag{12d}$$

and so forth until reaching $|\Psi_1\rangle = |0\rangle$. As stated above, the string of linear operators are state-dependent and this explains the above notation $\mathbf{L}[\Psi_n]; \mathbf{L}[\Psi_{n_1}]; \mathbf{L}[\Psi_{n_2}]; \ldots$. These operators are represented by operator-valued entries of diagonal matrices of **different** sizes because the binary representation of the sequence of integers (12a) stops at **different** powers of two : $2^{N-1}, 2^{N_1-1}, 2^{N_2-1}, 2^{N_3-1}, \ldots$ and corresponding to the binary expansion of $n, n_1, n_2, n_3, \ldots$. $N, N_1, N_2, N_3, \ldots$ are the dimensions of the corresponding Boolean hypercubes.

For this reason we have to explain how to **compose** a string of operators associated with diagonal matrices of **different** sizes. In the Appendix we provide some examples of how to perform such composition of operators by **adding** a judicious string of unit operators $\mathbf{1}$'s to the left, and/or to the right, of the diagonal entries such that **all** the operators are now represented by diagonal matrices of the same size. The common size of all the diagonal matrices is provided by the value of the maximum dimension N_{max} of the Boolean hypercube associated with the binary expansion of the maximum number $n_{max} = \sum_{l=0}^{N_{max}-1} \ a_{n_{max} l} \ 2^l$ attained in the Collatz chain (path, sequence) (12a). By the same token, the projection operators $\mathcal{P}_n$ in the right-hand side of eqs-(10a, 10b) must also be represented by diagonal matrices of the same size N_{max} to match the size of the matrices in the left hand side. It is also necessary to **embed** all the column vectors into a column vector of maximal size N_{max} as well by adding extra zeros.

After doing so, one may write eqs-(10a,10b) in the following **symbolic** form which reflect the state-dependence of the operators,

$$\mathbf{L}[\Psi_{n_{s(n)}}]\ \mathbf{L}[\Psi_{n_{s(n)-1}}]\ \ldots\ \mathbf{L}[\Psi_{n_3}]\ \mathbf{L}[\Psi_{n_2}]\ \mathbf{L}[\Psi_{n_1}]\ |\Psi_n\rangle\ =\ \mathcal{P}_n\ |\Psi_n\rangle\ =\ \Psi_1\ \equiv |0\rangle \tag{13}$$

and where $\mathcal{P}$ is either the projection operator for the $n =$ even, or the $n =$ odd number case. From eq-(13) one infers the relations for all values of n

$$\Big(\ \mathbf{L}[\Psi_{n_{s(n)}}]\ \mathbf{L}[\Psi_{n_{s(n)-1}}]\ \ldots\ \mathbf{L}[\Psi_{n_3}]\ \mathbf{L}[\Psi_{n_2}]\ \mathbf{L}[\Psi_{n_1}]\ -\ \mathcal{P}_n\ \Big)\ \Psi_n\ =\ 0 \tag{14}$$

The main result of this work is that if, and only if, the Collatz conjecture is true the concatenation process of the operators must obey the relations (14) for all values of n even, odd, respectively. To verify these relations for *all* values of n is **extremely** difficult. By **construction**, the projection operator acting on Ψ_n **always** obeys $\mathcal{P}[\Psi_n]\Psi_n = \Psi_1 = |0\rangle$, for all values of n. The reason we rewrite (13) in the form (14) will be explained below.

Let us provide an specific example. Upon writing the operators in symbolic notation $\mathbf{L}[\Psi_{odd}] = \mathbf{I} + \mathcal{L}^+ + \mathcal{P}_{odd}$, and $\mathbf{L}[\Psi_{even}] = \mathcal{L}^-$, the Collatz sequence $\{3, 10, 5, 16, 8, 4, 2, 1\}$ yields the following relation

$$\Big(\ \mathcal{L}^-_2\ \mathcal{L}^-_4\ \mathcal{L}^-_8\ \mathcal{L}^-_{16}\ (\ \mathbf{I}\ +\ \mathcal{L}^+_5\ +\ \mathcal{P}_5\ )\ \mathcal{L}^-_{10}\ (\ \mathbf{I}\ +\ \mathcal{L}^+_3\ +\ \mathcal{P}_3\ )\ -\ \mathcal{P}_3\ \Big)\ \Psi_3\ =\ 0 \tag{15}$$

The subscripts in (15) denote on what specific states Ψ_{n_i} the operators are acting.

Rather than focusing on the infinite number of relations (14), it might be "easier" to ask if there are cycles (loops) besides the trivial one $\{1, 4, 2, 1, 4, 2, \ldots\}$. One can discard it be setting $\mathbf{L}[\Psi_1]\Psi_1 = \Psi_1 \Rightarrow \mathbf{L}[\Psi_1] \equiv \mathbf{I}$. If such nontrivial cycles exist then the Collatz conjecture is false. And if there are **no** cycles (loops) in the Collatz process, the Collatz conjecture is true (if there are no divergences) and after a certain number of steps we reach 1.

If the Collatz conjecture is **false** then one does **not** have the equalities displayed by eqs-(10a, 10b), for **all** values of $n > 1$, and it is possible that that there is at **least one** nontrivial cycle (loop) for a given value of n such that one ends **back** at Ψ_n. In this exceptional case the left-hand sides of eqs-(10a, 10b) become

$$\mathbf{L}_{(s(n))}\ \mathbf{L}_{(s(n)-1)}\ \ldots\ \mathbf{L}_{(2)}\ \mathbf{L}_{(1)}\ |\Psi_n >\ =\ |\Psi_n >,\quad n > 1 \tag{16}$$

instead of being equal to Ψ_1. Consequently, if eq-(16) holds, there are **two** cases to examine :

(i) The composition of the string of operators $\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_*$ in eq-(14) yields the identity operator $\mathbf{I}$ instead of $\mathcal{P}_n$. Since the last two operators displayed in eq-(9) **encode** precisely the whole Collatz iteration process in terms of the operators $\mathbf{L}_{\frac{n}{2}};\ \mathbf{L}_{3n+1}$, the Collatz conjecture could be recast by stating that it is not plausible for the product of the string of operators $\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_*$ (involving explicitly the last two operators displayed in eq-(9)) to yield the identity operator

$\mathbf{I}$, and which is realized as a diagonal matrix comprised of the unit operators $\mathbf{1}$'s in **all** of its entries. Because this is such an extremely restrictive requirement, it is highly unlikely that the products of operators furnish the identity operator.

(ii) The second case is when Ψ_n is the null eigenfunction of $(\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathbf{I})\Psi_n = 0$, with zero eigenvalue. As stated earlier, despite that the operators $\mathbf{L}$ are are state-dependent, they are still **linear**. The composition of the $\mathbf{L}$ operators leads to higher order linear differential equations of the functions Ψ_n by recalling the definitions of the operators

$$\mathbf{a} \ = \ \sqrt{\frac{m\omega}{2\hbar}} \ (\hat{x} \ + \frac{i}{m\omega} \ \hat{p}), \ \ \mathbf{a}^\dagger \ = \ \sqrt{\frac{m\omega}{2\hbar}} \ (\hat{x} \ - \frac{i}{m\omega} \ \hat{p}) \tag{17}$$

in the QM harmonic oscillator, obeying $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{1}$, where m is the particle's mass, ω is the angular frequency, and $\hbar$ is the reduced Planck's constant ($\frac{h}{2\pi}$).

The Heisenberg uncertainty relations reveal that the momentum operator $\hat{p}$ in (17) can be realized as a differential operator $\hat{p} \leftrightarrow -i\hbar\frac{d}{dx}$. And vice versa, the $\hat{x}$ operator can be realized in terms of momentum derivatives $\frac{d}{dp}$. Because the $\mathbf{L}, \mathcal{P}$ operators are explicitly constructed in terms of the $\mathbf{1}, \mathbf{a}, \mathbf{a}^\dagger$ operators, and which in turn, can be realized in terms of x and $\frac{d}{dx}$, the full operator $(\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathbf{I})$ turns into a (higher order) linear differential operator acting on Ψ_n. Consequently, in case (ii) it is appropriate to say that Ψ_n is a null eigenfunction of the full operator encompassed inside the parenthesis, with zero eigenvalue. However, even if there exists at least one value of n such that Ψ_n satisfies $(\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathbf{I})\Psi_n = 0$, with zero eigenvalue, it will still lead to **extraneous** differential constraints imposed on the harmonic oscillator energy eigenstates, since Ψ_n is a superposition of the QM harmonic oscillator eigenstates (bits).

On the other hand, if the Collatz conjecture is true, this is equivalent to stating $(\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P})\Psi_n = 0$ as mentioned earlier in eq-(14). In the first case, this leads to the **operator** relations (iii) $\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P} = 0$. And in the second case, this implies (iv) that the Ψ_n's (for all n) are the null eigenfunctions of the infinite set of (higher order) linear differential equations $(\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P})\Psi_n = 0$, with zero eigenvalue. Since the Ψ_n's themselves are a superposition of the QM harmonic oscillator eigenstates (bits), the latter linear differential equations can be decomposed themselves into an even larger set of linear differential equations involving the harmonic oscillator energy eigenstates $\phi_l(x), (l = 0, 1, 2, \ldots)$, which are given by the product of Gaussian exponentials e^{-x^2} times the Hermite polynomials $H_l(x)$. Because case (iv) will introduce a very large number of **extraneous** differential constraints on the harmonic oscillator energy eigenstates that cannot be satisfied, it is ruled out.

To sum up, case (i) is extremely unlikely; cases (ii) and (iv) lead to **extraneous** differential constraints imposed on the harmonic oscillator energy eigenstates. And, consequently, one is left with the case (iii) $\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P} = 0$. In the Appendix we provide explicit examples which show that one has $\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P} = 0$. In other words, one has an operator identity $\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* = \mathcal{P}$

Concluding, one finds that the Collatz conjecture can be encoded in the infinite number of operator relations written symbolically as $\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P} = 0$, associated to every positive integer, even or odd, and involving explicitly the infinite number of operators displayed by eq-(9). Because one has an infinite number of operators at our disposal, in principle, one can impose an infinite number of operator relations $\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P} = 0$. If all of these relations are mutually consistent, then the Collatz conjecture is most likely to be true. It is important to remark that there has to be a cancellation of the $\mathbf{a}^\dagger$ terms, which are always present whenever the operators $\mathbf{I} + \mathcal{L}^+ + \mathcal{P}$ are part of the composition $\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_*$, because there are no $\mathbf{a}^\dagger$ terms in $\mathcal{P}$. Namely, this is yet another reflection of the operator identities.

If one replaced the $3n+1$ iteration (n odd) for $3n+b$, with b odd and greater than 1, the operator representation for the process $\Psi_n \to \Psi_{3n+b}$ will **no** longer have the **diagonal** form $(\mathbf{I} + \mathcal{L}^+ + \mathcal{P})_{n=odd}$, and which is intrinsically tied up to the very special decomposition of $3n+1 = n + 2n + 1$, but it is going to have another expression given by $\mathbf{L}_{3n+b} = (\mathbf{I} + \mathcal{L}^+ + \mathcal{P})_{n=odd} + \Delta_{n,b-1}$, where $\Delta_{n,b-1}$ is now an off-diagonal operator-valued matrix whose entries are comprised of $\mathbf{1}, \mathbf{a}, \mathbf{a}^\dagger$, and that maps Ψ_n to Ψ_{b-1}.

Consequently, due to the presence of this extra off-diagonal operator-valued matrix $\Delta_{n,b-1}$, it is now possible to find null eigenfunctions $(\mathbf{L}_*\mathbf{L}_* \ldots \mathbf{L}_* - \mathbf{I})\Psi_n = 0$ and obtain nontrivial cycles for very specific values of b and n. In this case, the Collatz conjecture would be false. For example, when $b = n$ one has the trivial cycle $\{b, 4b, 2b, b, 4b, 2b, b, \ldots\}$. Nowak [6] has found that when $a = 2000003$, any iteration process of $n = n_o$ that is **not** a multiple of 2000003 will end up in the cycle involving the Mersenne prime $127 = 2^7 - 1$. This cycle based on 127 has a length of 15126 steps and a maximum value (number) of 48382644622. Nowak uses the iteration $\frac{3n+a}{2}$ for n odd, and $\frac{n}{2}$ for n even as usual.

To finalize we should add that an heuristic explanation as to why there are no divergences in the Collatz process can be found if one proceeds with the Syracuse iteration (proposed by Hasse) $n \to \frac{3n+1}{2}$, when n is odd, and $n \to \frac{n}{2}$ when n is even. The former leads to a dilation with a factor of $\frac{3}{2}$. The latter leads to a contraction by a factor of $\frac{1}{2}$. Because there are an equal number of even and odd numbers, on average, the overall scaling factor is $\frac{3}{2} \times \frac{1}{2} = \frac{3}{4} < 1$, leading to a contraction, and the Collatz process does not diverge. If one uses, for example, the iteration $\frac{5n+1}{2}$ for n odd, the overall scaling factor would be $\frac{5}{2} \times \frac{1}{2} = \frac{5}{4} > 1$, and the process should diverge.

APPENDIX

In the Appendix we provide two examples of how to perform the composition of operators by **adding** a judicious string of unit operators $\mathbf{1}$'s to the left, and/or to the right, of the diagonal entries such that **all** the operators are now represented by diagonal matrices of the same size. It is also necessary to **embed** all the column vectors into a column vector of maximal size N_{max} as well by adding extra zeros.

Let us consider the Collatz process $\{8, 4, 2, 1\}$ associated with the states $\{\Psi_8, \Psi_4, \Psi_2, \Psi_1 = |0\rangle\}$. The projection operator $\mathcal{P}[\Psi_8] = Diag(\mathbf{1}, \mathbf{1}, \mathbf{1}, \frac{\mathbf{a}^3}{\sqrt{3!}})$ maps Ψ_8 into Ψ_1, where Ψ_8 is represented by the column vector of maximal size whose entries are $\{0, 0, 0, |3\rangle\}$.

The lowering operators $\mathcal{L}^-$ that reduce the bits by one unit are

$$\mathbf{L}[\Psi_8] \ = \ Diag(\mathbf{1}, \mathbf{1}, \mathbf{1}, \frac{\mathbf{a}}{\sqrt{3}}); \ \ \mathbf{L}[\Psi_4] \ = \ Diag(\mathbf{1}, \mathbf{1}, \frac{\mathbf{a}}{\sqrt{2}}); \ \ \mathbf{L}[\Psi_2] \ = \ Diag(\mathbf{1}, \mathbf{a}) \tag{A.1}$$

Since the last two diagonal operators do not have the same size as the first diagonal operator of maximal size, we **add** the sufficient number of $\mathbf{1}$'s entries to their **left** and arrive at

$$\mathbf{L}[\Psi_4] \ \rightarrow \ Diag(\mathbf{1}, \mathbf{1}, \mathbf{1}, \frac{\mathbf{a}}{\sqrt{2}}), \ \ \mathbf{L}[\Psi_2] \ \rightarrow \ Diag(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{a}) \tag{A.2}$$

Now one can verify that the products obey the desired relation

$$Diag(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{a}) \, Diag(\mathbf{1}, \mathbf{1}, \mathbf{1}, \frac{\mathbf{a}}{\sqrt{2}}) \, Diag(\mathbf{1}, \mathbf{1}, \mathbf{1}, \frac{\mathbf{a}}{\sqrt{3}}) \ = \ Diag(\mathbf{1}, \mathbf{1}, \mathbf{1}, \frac{\mathbf{a}^3}{\sqrt{3!}}) \ = \ \mathcal{P}[\Psi_8] \tag{A.3}$$

and, therefore one has the desired operator relation $\mathbf{L}_* \mathbf{L}_* \ldots \mathbf{L}_* - \mathcal{P} = 0$. It is also necessary to **embed** the column vector Ψ_4 with entries $\{0, 0, |2\rangle\}$ into a column vector with entries $\{0, 0, 0, |2\rangle\}$; and the column vector Ψ_2 with entries $\{0, |1\rangle)$ into the one with entries $\{0, 0, 0, |1\rangle\}$. In this way the latter column vectors will have the same number of entries as Ψ_8 whose entries are $\{0, 0, 0, |3\rangle\}$.

In the second example we shall see the case where we **add** the sufficient number of $\mathbf{1}$'s entries to their **right**, instead. Let us consider the Collatz process $\{3, 10, 5, 16, 8, 4, 2, 1\}$ associated with the states $\{\Psi_3, \Psi_{10}, \Psi_5, \Psi_{16}, \Psi_8, \Psi_4, \Psi_2, \Psi_1 = |0\rangle\}$. One can read-off the maximum value given by 16 in the Collatz chain. The binary representation of 16 is 2^4, thus the diagonal matrix of maximal size will have $4 + 1 = 5$ entries since Ψ_{16} is represented by a column vector whose entries are $\{0, 0, 0, 0, |4\rangle\}$.

Because Ψ_3 is represented by a column vector whose entries are $\{|0\rangle, |1\rangle\}$ one needs to add 3 extra zeros $\{|0\rangle, |1\rangle\}, 0, 0, 0\}$ in order to match the same number of entries as Ψ_{16} which will appear in the Collatz chain process.

The expression for the operator $\mathbf{L}[\Psi_3]$ that maps Ψ_3 into Ψ_{10} has the $\mathbf{I} + \mathcal{L}^+ + \mathcal{P}$ form given by

$$\mathbf{L}[\Psi_3] \ = \ Diag(\mathbf{1}, \mathbf{1}) \ + \ Diag(\mathbf{a}^\dagger, \ \frac{\mathbf{a}^\dagger}{\sqrt{2}}) \ + \ Diag(\mathbf{1}, \mathbf{a}^2) \tag{A.4}$$

A careful inspection of the Collatz chain process reveals that now we must **add** three $\mathbf{1}$'s entries to their **right** , leading to diagonal operators of the maximal size

$$\mathbf{L}[\Psi_3] \ \rightarrow \ Diag(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) \ + \ Diag(\mathbf{a}^\dagger, \ \frac{\mathbf{a}\dagger}{\sqrt{2}}, \mathbf{1},\mathbf{1},\mathbf{1}) \ + \ Diag(\mathbf{1},\mathbf{a}^2,\mathbf{1},\mathbf{1},\mathbf{1}) \tag{A.5}$$

and that matches the number of entries of the column vector of maximal size Ψ_{16}, whose entries are $\{0,0,0,0,|4\rangle\}$. Finally one can show that $\mathbf{L}[\Psi_3]\Psi_3 = \Psi_{10}$ after using the binary composition rules

$$|1\rangle + |1\rangle \ = \ |2\rangle; \ \ |2\rangle + |2\rangle \ = \ |3\rangle, \ \ |l\rangle + |l\rangle \ = \ |l{+}1\rangle, \ \ |l{+}1\rangle + |l{+}1\rangle \ = \ |l{+}2\rangle, \ \ldots$$

with

$$\Psi_3 \equiv \{|0\rangle, |1\rangle, 0, 0, 0\}, \quad \Psi_{10} \equiv \{0, |1\rangle, 0, |3\rangle, 0\}, \tag{A.6}$$

In other words, one has a map from $\Psi_3 = \{|0\rangle, |1\rangle, 0, 0, 0\}$ into $\Psi_{10} = \{0, |1\rangle, 0, |3\rangle, 0\}$ corresponding to the first Collatz iteration $\mathcal{C}(3) = 10$. Despite this very tedious process, one can show that the composition of all the 7 string of operators acting on Ψ_3 will lead finally, and effectively, to the projection operator acting on $\Psi_3 : \mathcal{P}[\Psi_3]\Psi_3 = \Psi_1 = |0\rangle$.

Acknowledgements

We (CCP) thank M. Bowers for invaluable assistance.

References

[1] The Collatz Conjecture. Wikipedia, https://en.wikipedia.org/wiki/Collatz_conjecture

[2] Lagarias, Jeffrey C., ed. (2010). The ultimate challenge: the 3x + 1 problem. Providence, R.I.: American Mathematical Society. p. 4.

Lagarias, Jeffrey C. (1985). "The 3x + 1 problem and its generalizations". The American Mathematical Monthly. **92** (1): 323.

[3] Barina, David (2020). "Convergence verification of the Collatz problem". The Journal of Supercomputing. doi:10.1007/s11227-020-03368-

[4] R. Carbo-Dorca, "Boolean Hypercubes, Mersenne numbers and the Collatz conjecture", J. Math. Sic. Mod. **3** (2020) 120-129.

[5] R. Carbo-Dorca, "Natural Vector Spaces : Inward Power and Minkowski Norm of a Natural Vector, Natural Boolean Hypercubes and Fermat's Last Theorem", J. Math. Chem. **55** (2017) 914-940.

R. Carbo-Dorca, "Boolean Hypercubes and the Structure of Vector Spaces:, J. Math. Sci. Mod **1** (2018) 1-14.

[6] H. Nowak, "Collatz Conjecture and Emergent Properties" https://www.youtube.com/watch?v=QrzcHhBQ2b0